SIAM J. MATH. ANAL.
Vol. 49, No. 2, pp. 954–983

ON A DEGENERATE NONLOCAL PARABOLIC PROBLEM DESCRIBING INFINITE DIMENSIONAL REPLICATOR DYNAMICS*

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Abstract. We establish the existence of locally positive weak solutions to the homogeneous Dirichlet problem for $u_t = u\Delta u + u\int_\Omega |\nabla u|^2$ in bounded domains $\Omega \subset \mathbb{R}^n$ which arises in game theory. We prove that solutions converge to 0 if the initial mass is small, whereas they undergo blow-up in finite time if the initial mass is large. In particular, it is shown that in this case the blow-up set coincides with $\overline{\Omega}$; i.e., the finite-time blow-up is global.

Key words. degenerate diffusion, nonlocal nonlinearity, blow-up, evolutionary games, infinite dimensional replicator dynamics

AMS subject classifications. 35K55, 35K65, 35B44, 91A22

DOI. 10.1137/15M1053840

1. Introduction. In a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, we consider nonnegative solutions to the quasi-linear degenerate and nonlocal parabolic problem

$$\begin{cases} u_t = u\Delta u + u\displaystyle\int_\Omega |\nabla u|^2, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

which arises in a game theoretical description of replicator dynamics in the case of a Bomze-type infinite dimensional setting [7] when pursuing the modeling procedure introduced in [21, 22, 36] and which actually assumes steep payoff-kernels of Gaussian type. For completeness in this direction we include a concise derivation of problem (1.1) in Appendix A.

Strongly degenerate diffusion meets nonlocal gradient sources. From a mathematical perspective, the evolution in (1.1) is governed by two characteristic mechanisms, each of which already gives rise to considerable challenges on its own. First, diffusion in (1.1) is strongly degenerate at small densities in the sense that near points where $u = 0$, typical diffusive effects are substantially inhibited. Indeed, already in the unforced counterpart of (1.1) with general power-type degeneracy, as given by

$$u_t = u^p \Delta u \tag{1.2}$$

with $p > 0$, it is known that the particular value $p = 1$, corresponding to the choice in (1.1), marks a borderline between somewhat mild degeneracies and strongly degenerate diffusion: In the case when $p < 1$, namely, (1.2) allows for a transformation into

*Received by the editors December 21, 2015; accepted for publication (in revised form) October 31, 2016; published electronically March 28, 2017.
http://www.siam.org/journals/sima/49-2/M105384.html
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the porous medium equation $v_t = \Delta v^m$ with $m := \frac{1}{1-p} > 1$, thus meaning that in this case, unique global continuous weak solutions to the associated Dirichlet problem exist for all reasonably regular nonnegative initial data [2] and that these eventually become positive and smooth, and hence classical, inside Ω [5]. If $p \geq 1$, then nonnegative global weak solutions can still be constructed for any nonnegative continuous initial data, but they need no longer be continuous [4] nor uniquely determined by the initial data [31], and moreover their spatial support will not increase with time [6, 31, 55].

Even in the case when one resorts to continuous initial data which are strictly positive throughout Ω in which, in fact, unique classical solutions exist for any $p > 0$, the value $p = 1$ corresponds to a critical strength of degeneracy. In particular, for $p < 1$, after an appropriate waiting time, all solutions will enter the cone $\mathscr{K} := \{\varphi : \Omega \to \mathbb{R} \mid \varphi(x) \geq c\, dist(x, \partial\Omega)$ for all $x \in \Omega$ and some $c > 0\}$ [5], which reflects a diffusion-driven effect generalizing the Hopf boundary point property in nondegenerate diffusion processes. On the other hand, in the case $p \geq 1$, solutions to (1.2) emanating from initial data which are suitably small near $\partial\Omega$ will never enter $\mathscr{K}$ [54].

Now in (1.1), this degenerate diffusion process interacts with a spatially nonlocal source which is such that unlike in large bodies of the literature on related nonlocal parabolic equations [39], even basic questions concerning local solvability appear to be far from trivial. Indeed, in light of an expected loss of appropriate solution regularity due to strongly degenerate diffusion, even for smooth initial data it seems a priori unclear whether solutions can be constructed which allow for a meaningful definition of the Dirichlet integral $\int_\Omega |\nabla u|^2$ for positive times. This is in stark contrast to most nonlocal parabolic problems previously studied, in which either diffusion is nondegenerate, and hence such first-order expressions are controllable by L^∞ bounds for solutions at least for small times, such as, e.g., in the semilinear problem

$$u_t = \Delta u + u^m \left(\int_\Omega |\nabla u|^2 \, dx \right)^r$$

studied for $m \geq 1$, $r > 0$ in [8, 45], or the nonlocal terms involve only zero-order expressions which thus in a natural manner also in cases of degeneracies as in (1.2) allow for local theories based on extensibility criteria in $L^\infty(\Omega)$ only (see [9, 43] and also the book [39]).

Main results. Previous mathematical studies on the PDE arising in (1.1) have concentrated on analyzing self-similar solutions only. In [21], the authors constructed self-similar solutions in the case $\Omega = \mathbb{R}$, and in [36] the same could be achieved in the multidimensional case $\Omega = \mathbb{R}^N$ with $N \geq 2$. More recently, the authors in [37] investigated the existence of self-similar solutions in the one-dimensional case in a closely related problem in which the Laplacian is perturbed by a time-dependent term containing the first derivative as well; all these self-similar solutions are shown to be regular and to approach Dirac-type distributions as $t \searrow 0^+$. An analogous study in higher dimensions is provided in [38].

The goals of the present work consist of developing a fundamental theory of local solvability for (1.1) and of providing a first step toward an understanding of the qualitative solution behavior. In order to formulate our results, let us concretize the specific setting within which (1.1) will be studied by requiring that throughout this paper, Ω denotes a bounded domain in $\mathbb{R}^N, N \geq 1$, with smooth boundary, and by introducing the solution concept that we shall pursue as follows.

Definition 1.1. *Let $T \in (0,\infty]$. By a* weak solution *of* (1.1) *in $\Omega \times (0,T)$ we mean a nonnegative function*

$$u \in L^\infty_{loc}(\bar\Omega \times [0,T)) \cap L^2_{loc}([0,T); W_0^{1,2}(\Omega)) \qquad \text{with} \qquad u_t \in L^2_{loc}(\bar\Omega \times [0,T)),$$

which satisfies

$$-\int_0^T\!\!\int_\Omega u\varphi_t \,dxdt + \int_0^T\!\!\int_\Omega \nabla u \cdot \nabla(u\varphi)\,dxdt = \int_\Omega u_0\varphi(\cdot,0)\,dx + \int_0^T \left(\int_\Omega u\varphi\,dx\right)\cdot\left(\int_\Omega |\nabla u|^2\,dx\right)dt \tag{1.3}$$

for all $\varphi \in C_0^\infty(\Omega \times [0,T))$.

A weak solution u of (1.1) *in $\Omega \times (0,T)$ will be called* locally positive *if $\frac{1}{u} \in L^\infty_{loc}(\Omega \times [0,T])$.*

Remark 1.2. Since $u \in L^2_{loc}([0,T); W_0^{1,2}(\Omega))$ and $u_t \in L^2_{loc}(\bar\Omega \times [0,T))$ imply that $u \in C^0([0,T); L^2(\Omega))$, (1.3) is equivalent to requiring that $u(\cdot,0) = u_0$ and that

$$\int_0^T \int_\Omega u_t\varphi\,dx\,dt + \int_0^T \int_\Omega \nabla u \cdot \nabla(u\varphi)\,dx\,dt = \int_0^T \left(\int_\Omega u\varphi\,dx\right)\cdot\left(\int_\Omega |\nabla u|^2\,dx\right)dt \tag{1.4}$$

holds for any $\varphi \in C_0^\infty(\overline{\Omega} \times (0,T))$.

In order to construct such locally positive weak solutions, we shall assume that the initial data satisfy the following:

(H1) $u_0 \in L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$,
(H2) $u_0 \geq 0$ and $\frac{1}{u_0} \in L^\infty_{loc}(\Omega)$, and
(H3) there exists $L > 0$ such that $\|u_0\|_{\Phi,\infty} \leq L$.

Here and below, for a measurable function $v\colon \Omega \to \mathbb{R}$ we have set

$$\|v\|_{\Phi,\infty} := \operatorname*{ess\,sup}_{x\in\Omega} \left|\frac{v}{\Phi}\right|,$$

where $\Phi \in C^2(\overline{\Omega})$ denotes the solution to

$$-\Delta\Phi = 1 \quad \text{in } \Omega, \qquad \Phi|_{\partial\Omega} = 0. \tag{1.5}$$

Note that according to the Hopf boundary point lemma, requiring $\|u_0\|_{\Phi,\infty}$ to be finite is an equivalent way of asking for the possibility to estimate u_0 by a multiple of the function measuring the distance of a point to $\partial\Omega$.

In this framework, the first of our main results indeed asserts local existence of locally positive weak solutions, along with a favorable extensibility criterion only involving the norm of the solution in $L^\infty(\Omega)$.

Theorem 1.3. *Let u_0 satisfy* (H1)–(H3). *Then there exist $T_{max} \in (0,\infty]$ and a locally positive weak solution u to* (1.1) *in $\Omega \times (0,T_{max})$ which satisfies*

$$\text{either } T_{max} = \infty \quad \text{or} \quad \limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty, \tag{1.6}$$

and which is such that for each smoothly bounded subdomain $\Omega' \subset\subset \Omega$ there exists $C_{\Omega'} > 0$ with

$$(1.7)\quad \int_\Omega |\nabla u(\cdot,t)|^2 \le \int_\Omega |\nabla u_0|^2 \cdot \exp\left[\frac{1}{2C_{\Omega'}}\left(\sup_{\tau\in(0,t)}\int_\Omega u(\cdot,\tau)\right)\left(\int_{\Omega'}\phi\ln u(\cdot,t) - \int_{\Omega'}\phi\ln u_0 + \int_0^t\int_{\Omega'} u\right)\right],$$

where ϕ *denotes the solution to* $-\Delta\phi = 1$ *in* Ω', $\phi|_{\partial\Omega'} = 0$, *as well as*

$$\|u(\cdot,t)\|_{\Phi,\infty} \le \max\left\{\|u_0\|_{\Phi,\infty}\,,\ \sup_{\tau\in(0,t)}\int_\Omega |\nabla u(x,\tau)|^2\,dx\right\} \tag{1.8}$$

for a.e. $t \in (0, T_{max})$.

Remark 1.4. Here, we have to leave open the question of uniqueness of solutions. In view of precedent nonuniqueness results for weak solutions of $u_t = u\Delta u$ even with merely local ingredients [31], however, we do not expect the uniqueness property to hold in the considered generalized solution framework. The reader can find a uniqueness proof for positive classical solutions to the latter equation in [51]. Since we do not know whether the solutions provided by Theorem 1.3 are classical, the argument used there apparently cannot be carried over to the present situation.

We emphasize that the extensibility criterion (1.6) particularly excludes any *gradient blow-up* phenomenon in the sense of finite-time blow-up of ∇u despite boundedness of u itself. Indeed, the occurrence of unbounded gradients of bounded solutions appears to be a characteristic qualitative implication of various types of interplay between diffusion—possibly degenerate—and gradient-dependent nonlinearities [1, 3, 29, 48].

A natural succeeding topic appears to consist of deriving conditions on the initial data which ensure that the solutions found above either exist for all times or blow up in finite time. Here, in view of the essentially cubic character of the production term in (1.1) it is not surprising that this may dominate the smoothing effect of the merely quadratic-type diffusion term when the initial data are suitably large in an adequate sense; precedent works indicate that indeed, such intuitive considerations are appropriate in related nondegenerate and degenerate parabolic equations with local reaction terms [39, 41, 47, 52].

A remarkable feature of the precise structure of this interplay in (1.1) is that actually a complete classification of all initial data in this respect is possible, exclusively involving the size of the total initial mass $m := \int_\Omega u_0$ as the decisive quantity: In fact, the second of our main results identifies the value $m = 1$ to be critical with regard to global solvability and, moreover, gives some basic information on the asymptotic behavior of solutions.

THEOREM 1.5. *Let* u_0 *satisfy* (H1)–(H3), *and let* u *and* T_{max} *denote the corresponding locally positive weak solution of* (1.1) *as well as its maximal time of existence, provided by Theorem* 1.3.

(i) *If* $\int_\Omega u_0 < 1$, *then* $T_{max} = \infty$ *and*

$$\int_\Omega u(x,t)\,dx \to 0 \quad \text{as } t \to \infty.$$

(ii) *Suppose that* $\int_\Omega u_0 = 1$. *Then* $T_{max} = \infty$ *and*

$$\int_\Omega u(x,t)\,dx = 1 \quad \text{for all } t > 0.$$

(iii) *In the case* $\int_\Omega u_0\,dx > 1$, *we have* $T_{max} < \infty$ *and*

$$\limsup_{t \nearrow T_{max}} \int_\Omega u(x,t)\,dx = \infty.$$

Remark 1.6. Statement (ii) of Theorem 1.5 says that if the initial data u_0 is a probability measure, then we have conservation of probability in time. This is actually a desired feature of the replicator dynamics model described by (1.1), since $u(\cdot,t)$ stands for a probability distribution of the state of some population of players; see also Appendix A.

In the situation of Theorem 1.5 (iii) when finite-time blow-up occurs, understanding the solution behavior near the respective blow-up time necessarily requires us to describe the set of all points where the solution becomes unbounded. Accordingly, next we shall be concerned with the blow-up set

$$\mathscr{B} = \Big\{ x \in \overline{\Omega} \,\Big|\, \text{there exists a sequence } (x_k,t_k)_{k\in\mathbb{N}} \subset \Omega \times (0,T_{max}) \text{ such that } x_k \to x, t_k \to T_{max} \text{ and } u(x_k,t_k) \to \infty \text{ as } k \to \infty \Big\}$$

of exploding solutions. In numerous related equations, involving either linear or degenerate diffusion, blow-up driven by local superlinear production terms is known to occur only in thin spatial sets, which in radial settings typically reduce to single points [13, 16, 41]. Only a few exceptional situations detected in the literature lead to *regional* or even *global* blow-up, thus referring to cases in which $|\mathscr{B}| > 0$ or even $\mathscr{B} = \overline{\Omega}$ (cf., for instance, [14, 15, 25, 47, 53]). In cases of sources which at least partially consist of nonlocal terms, blow-up in sets of positive measure may occur if the relative size of a possibly contained local contribution at large densities is predominant, as compared to the strength of the respective diffusion term [11, 28, 30, 44, 46, 50].

Our main result in this direction will reveal that any of our nonglobal solutions in fact blow up globally in space, thus indicating a certain balance in the competition of diffusion and nonlocal production in (1.1).

THEOREM 1.7. *Suppose that* $\int_\Omega u_0\,dx > 1$, *and let* u *denote the locally positive weak solution of* (1.1) *from Theorem* 1.3. *Then* u *blows up globally in the sense that its blow-up set satisfies* $\mathscr{B} = \overline{\Omega}$.

The outline of the paper is as follows. In section 2 we introduce an approximate sequence of nondegenerate problems and derive some estimates for their solutions u_ε. Here, one key step toward the existence proof will consist of deriving the associated approximate variant of (2.35) (Lemma 2.6), which will rely on an energy-type argument combined with an analysis of the functional $\int_{\Omega'} \phi \ln u_\varepsilon(\cdot,t)$ for $\Omega' \subset\subset \Omega$, $t > 0$, and appropriate ϕ. Another important observation, based on an integral estimate involving certain singular weights (cf. Lemma 2.5 and in particular (2.31)) will reveal that the functions ∇u_ε enjoy a favorable strong compactness property with respect to spatio-temporal L^2-norms (cf. (2.44)) rather than merely the respective weak precompactness feature obtained from corresponding boundedness results. In section 3 we study an ODE problem associated with the evolution of the total mass of the solution, and depending on whether this total mass initially is equal to, less than, or greater than 1, we prove global existence and conservation of the total mass, convergence to zero total mass, and finite-time blow-up, respectively. Further, in section 4 we concentrate on the latter case and examine the corresponding blow-up set of the solution, and we prove that any such blow-up actually occurs globally in space. Finally, Appendix A is devoted to the motivation and derivation of the mathematical model using an evolution game dynamics approach, while Appendix B deals with a more detailed proof of Lemma 2.1.

2. Weak solutions: Existence and approximation. Following an approach that is well established in the context of degenerate parabolic equations, we aim at constructing a solution u to (1.1) as the limit of solutions to certain regularized problems. For this purpose, let us fix a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and a sequence $(u_{0\varepsilon})_{\varepsilon=\varepsilon_j} \subset C^3(\bar{\Omega})$ with the properties

$$u_{0\varepsilon} \geq \varepsilon \text{ in } \Omega, \quad u_{0\varepsilon} = \varepsilon \text{ on } \partial\Omega, \quad \Delta u_{0\varepsilon} = -\int_\Omega |\nabla u_{0\varepsilon}|^2 \text{ on } \partial\Omega \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \tag{2.1}$$

and

$$\limsup_{\varepsilon=\varepsilon_j\searrow 0} \|u_{0\varepsilon} - \varepsilon\|_{\Phi,\infty} \leq L, \tag{2.2}$$

with $L > \max\{\int_\Omega |\nabla u_0|^2, \|u_0\|_{\Phi,\infty}\}$ (cf. (H3)), as well as the property that

$$\text{for any compact set } K \subset \Omega \text{ there is } C_K > 0 \text{ such that } \liminf_{\varepsilon\searrow 0} \inf_K u_{0\varepsilon} \geq C_K, \tag{2.3}$$

and such that, moreover,

$$u_{0\varepsilon} \to u_0 \quad \text{in } W^{1,2}(\Omega) \quad \text{as } \varepsilon = \varepsilon_j \searrow 0 \tag{2.4}$$

and

$$\int u_{0\varepsilon} = \int u_0 \quad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}. \tag{2.5}$$

A necessary first observation is that such an approximation actually is possible.

LEMMA 2.1. *Let u_0 satisfy* (H1)–(H3). *Then there is a sequence $(u_{0\varepsilon})_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}} \subset C^3(\bar{\Omega})$ having the properties* (2.1)–(2.5).

Proof. Here we restrict ourselves to giving an outline, and for a slightly more detailed version of the proof we refer the reader to Appendix B. By modification of the usual mollification procedure (cf. [56, section 3]) commonly employed to obtain (2.4), it is possible to obtain the other properties as well. More precisely, we set

$$u_{0\varepsilon} = \varepsilon + C(1-\rho)\Phi + \rho(\varphi + \alpha\vartheta),$$

where $\varphi \in C_0^\infty(\Omega)$ is a mollified version of u_0 (after "locally shifting u_0 toward the interior of the domain"), $\rho \in C_0^\infty(\Omega)$, $0 \leq \rho \leq 1$, such that the supports of $\nabla\rho$ and φ are disjoint, $0 \leq \vartheta \in C_0^\infty$ with $\int_\Omega \vartheta = 1$ (in order to adjust (2.5)), Φ is the solution to $-\Delta\Phi = 1$ in Ω, $\Phi = 0$ on $\partial\Omega$ (for achieving the third property in (2.1)), and C and α are appropriately adjusted constants, depending on ε as well as on several different integrals containing the functions Φ, ρ, ϑ, their gradients, and u_0. ∎

For $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, we consider the regularized problem

$$\begin{cases} u_{\varepsilon t} = u_\varepsilon \Delta u_\varepsilon + u_\varepsilon \cdot \rho_\varepsilon\big(\int_\Omega |\nabla u_\varepsilon|^2\big), & x \in \Omega,\ t > 0, \\ u_\varepsilon(x,t) = \varepsilon, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases} \tag{2.6}$$

where

$$\rho_\varepsilon(z) := \min\left\{z,\ \frac{1}{\varepsilon}\right\} \quad \text{for } z \ge 0.$$

LEMMA 2.2. *For all sufficiently small $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, problem* (2.6) *has a unique classical global-in-time solution $u_\varepsilon \in C^{2,1}(\overline{\Omega}\times[0,\infty))$.*

Proof. To prove the uniqueness statement for all ε, we assume that both u_1 and u_2 are classical solutions of (2.6) from the indicated class in $\Omega\times(0,T)$ for some $T>0$. Then $w := u_1 - u_2$ satisfies $w = 0$ on $\partial\Omega$ and at $t = 0$, and

$$w_t = u_1\Delta w + \Delta u_2\cdot w + \rho_\varepsilon\left(\int_\Omega |\nabla u_2|^2\right)\cdot w + u_1\cdot\left[\rho_\varepsilon\left(\int_\Omega |\nabla u_1|^2\right) - \rho_\varepsilon\left(\int_\Omega |\nabla u_2|^2\right)\right] \tag{2.7}$$

for $t\in(0,T)$. Now given $T'\in(0,T)$, we can find a constant $M>0$ such that $u_1, |\nabla u_1|, u_2$, and $|\nabla u_2|$ are bounded above by M in $\Omega\times(0,T')$, since u_1, u_2 are classical solutions. Thus, by Hölder's inequality and the pointwise estimate $\big||\nabla u_1| - |\nabla u_2|\big| \le |\nabla(u_1-u_2)|$, we obtain

$$\begin{aligned}
\left|\rho_\varepsilon\left(\int_\Omega |\nabla u_1|^2\right) - \rho_\varepsilon\left(\int_\Omega |\nabla u_2|^2\right)\right| &\le \|\rho_\varepsilon{}'\|_{L^\infty((0,\infty))}\cdot\left|\int_\Omega\Big(|\nabla u_1|^2 - |\nabla u_2|^2\Big)\right| \\
&\le \int_\Omega \Big||\nabla u_1| - |\nabla u_2|\Big|\cdot\big(|\nabla u_1| + |\nabla u_2|\big) \\
&\le 2M\int_\Omega |\nabla w| \\
&\le 2M|\Omega|^{\frac12}\cdot\left(\int_\Omega |\nabla w|^2\right)^{\frac12} \qquad (2.8)
\end{aligned}$$

for all $t\in(0,T')$, because $\|\rho_\varepsilon{}'\|_{L^\infty((0,\infty))}\le 1$. Upon multiplying (2.7) by w and integrating over Ω, we see that for $t\in(0,T')$,

$$\begin{aligned}
\frac12\frac{d}{dt}\int_\Omega w^2 &= \int_\Omega u_1\Delta w w + \int_\Omega w^2\Delta u_2 + \int_\Omega w^2\rho_\varepsilon\left(\int_\Omega |\nabla u_2|^2\right) \\
&\quad + \int_\Omega w u_1\left[\rho_\varepsilon\left(\int_\Omega |\nabla u_1|^2\right) - \rho_\varepsilon\left(\int_\Omega |\nabla u_2|^2\right)\right] \\
&\le -\int_\Omega u_1|\nabla w|^2 - \int_\Omega \nabla u_1\nabla w w - 2\int_\Omega w\nabla w\nabla u_2 \\
&\quad + \int_\Omega w^2\rho_\varepsilon\left(\int_\Omega |\nabla u_2|^2\right) + \int_\Omega |w|u_1\left|\rho_\varepsilon\left(\int_\Omega |\nabla u_1|^2\right) - \rho_\varepsilon\left(\int_\Omega |\nabla u_2|^2\right)\right|.
\end{aligned} \tag{2.9}$$

Together with Young's inequality, (2.8), and the facts that $u_1\ge\varepsilon$ (which, thanks to the actual nondegeneracy of problem (2.6) for positive ε, is an immediate consequence of the maximum principle) and $\rho_\varepsilon(s)\le\frac{1}{\varepsilon}$ for all $s>0$, this entails

$$\begin{aligned}
\frac12\frac{d}{dt}\int_\Omega w^2 &\le -\varepsilon\int_\Omega |\nabla w|^2 + \frac{\varepsilon}{4}\int_\Omega |\nabla w|^2 + \frac{1}{\varepsilon}\int_\Omega w^2|\nabla u_1|^2 + \frac{\varepsilon}{2}\int_\Omega |\nabla w|^2 + \frac{8}{\varepsilon}\int_\Omega w^2|\nabla u_2|^2 \\
&\quad + \frac{1}{\varepsilon}\int_\Omega w^2 + 2M|\Omega|^{\frac12}\left(\int_\Omega |\nabla w|^2\right)^{\frac12}\int_\Omega |w|u_1
\end{aligned}$$

for $t \in (0, T')$. The choice of M now ensures that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 \le & -\frac{\varepsilon}{4}\int_\Omega |\nabla w|^2 + \frac{M^2}{\varepsilon}\int_\Omega w^2 + \frac{8M^2}{\varepsilon}\int_\Omega w^2 + \frac{1}{\varepsilon}\int_\Omega w^2 \\ & + 2M|\Omega|^{\frac{1}{2}}\left(\int_\Omega |\nabla w|^2\right)^{\frac{1}{2}}\left(\int_\Omega |w|^2\int_\Omega u_1^2\right)^{\frac{1}{2}} \\ \le & -\frac{\varepsilon}{4}\int_\Omega |\nabla w|^2 + \frac{9M^2+1}{\varepsilon}\int_\Omega w^2 + \frac{\varepsilon}{4}\int_\Omega |\nabla w|^2 + \frac{4M^4|\Omega|^2}{\varepsilon}\int_\Omega |w|^2 \end{aligned} \tag{2.10}$$

for $t \in (0, T')$, so that (2.10) finally turns into

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 \le \left(\frac{9M^2+1}{\varepsilon} + \frac{4M^4|\Omega|^2}{\varepsilon}\right)\cdot\int_\Omega w^2$$

for all $t \in (0, T')$.

Integrating this ODI (ordinary differential inequality) yields that $w \equiv 0$ in $\Omega \times (0, T')$ and hence also in $\Omega \times (0, T)$, because $T' < T$ was arbitrary.

It remains to be shown that for all $T > 0$, (2.6) is classically solvable in $\Omega \times (0, T)$ provided ε is sufficiently small. To this end, fix $T > 0$ and let $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ be so small that $\int_\Omega |\nabla u_{0\varepsilon}|^2 < \frac{1}{\varepsilon}$, which is possible due to (2.4). By [26, Thm. V.1.1], there are $K_1 > 0$ and $\theta > 0$ such that any classical solution w to the problem

$$w_t = w\Delta w + c(x,t) \text{ in } \Omega \times [0, T], \qquad w|_{\partial\Omega} = \varepsilon, \qquad w(\cdot, 0) = u_{0\varepsilon},$$

with $c \in L^\infty(\Omega \times (0, T))$ fulfilling $0 \le c \le \frac{1}{\varepsilon}\|u_{0\varepsilon}\|_{L^\infty(\Omega)} e^{\frac{T}{\varepsilon}}$, which in addition obeys the estimate $\varepsilon \le w \le \|u_{0\varepsilon}\|_\infty e^{\frac{T}{\varepsilon}}$, satisfies

$$\|w\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \le K_1. \tag{2.11}$$

Fix $\delta > 0$. Corresponding to θ, K_1, and δ, there is K_2 such that any solution w to

$$w_t = a(x,t)\Delta w + b(x,t) \text{ in } \Omega \times [0, T], \qquad w|_{\partial\Omega} = \varepsilon, \qquad w(\cdot, 0) = u_{0\varepsilon},$$

for some $a \in C^{\theta,\frac{\theta}{2}}(\overline{\Omega} \times [0, T])$ having the properties $a(x,t) = \varepsilon$ for $(x,t) \in \partial\Omega \times [0, T]$, $\varepsilon \le a \le \|u_{0\varepsilon}\|_{L^\infty} e^{\frac{T}{\varepsilon}}$, $\|a\|_{C^{\theta,\frac{\theta}{2}}(\overline{\Omega}\times[0,T])} \le K_1$ and continuous b with $b(x, 0) = b_0 \in \mathbb{R}$, $\|b\|_\infty \le \frac{K_1}{\varepsilon}$, by an application of [12, Thm. 7.4] to $w - u_{0\varepsilon} - tb_0$, fulfills

$$\|w\|_{C^{1+\delta,\frac{\delta}{2}}(\overline{\Omega}\times[0,T])} \le K_2. \tag{2.12}$$

With this in mind, in the space $X = C^{1+\frac{\delta}{2},\frac{\delta}{4}}(\overline{\Omega} \times [0, T])$ we consider the set

$$S := \left\{ v \in X \;\middle|\; v \ge \varepsilon \text{ in } \Omega \times (0, T), v(\cdot, 0) = u_{0\varepsilon} \text{ and } \|v\|_{C^{1+\delta,\frac{\delta}{2}}(\overline{\Omega}\times[0,T])} \le K_2 \right\},$$

which is evidently closed, bounded, convex, and compact in X. For each $v \in S$, the definition of ρ_ε implies that

$$f(t) := \rho_\varepsilon\!\left(\int_\Omega |\nabla v(\cdot, t)|^2\right), \qquad t \in [0, T], \tag{2.13}$$

defines a nonnegative $\frac{\delta}{2}$-Hölder continuous function f on $[0, T]$. The choices of f, S, and ε show that $f(0) = \int_\Omega |\nabla u_{0\varepsilon}|^2$, and thus (2.1) ensures that the compatibility condition of first order is satisfied. Therefore, the quasi-linear, actually nondegenerate parabolic problem

$$
\begin{cases}
u_{\varepsilon t} = u_\varepsilon \Delta u_\varepsilon + f(t) u_\varepsilon, & x \in \Omega,\ t > 0,\\
u_\varepsilon(x,t) = \varepsilon, & x \in \partial\Omega,\ t > 0,\\
u_\varepsilon(x,0) = u_{0\varepsilon}(x), & x \in \Omega,
\end{cases}
\tag{2.14}
$$

possesses a classical solution $u_\varepsilon \in C^{2,1}(\overline{\Omega} \times [0,T])$ by [26, Thm. V.6.1], which, by comparison, satisfies

$$
\varepsilon \le u_\varepsilon \le \|u_{0\varepsilon}\|_{L^\infty(\Omega)} \cdot e^{\frac{T}{\varepsilon}} \quad \text{in } \Omega \times (0,T),
\tag{2.15}
$$

because $\underline{u}(x,t) := \varepsilon$ and $\overline{u}(x,t) := \|u_{0\varepsilon}\|_{L^\infty(\Omega)} \cdot e^{\frac{t}{\varepsilon}}$ are easily seen to define a sub- and a supersolution of (2.14), respectively.

We now introduce a mapping $F\colon S \to X$ by setting $Fv := u_\varepsilon$, where u_ε solves (2.14) with (2.13).

Then defining $c(x,t) := u_\varepsilon(x,t) f(t)$, $x \in \Omega, t \in [0,T]$, this function satisfies $\|c\|_\infty \le \frac{1}{\varepsilon} \|u_{0\varepsilon}\|_\infty e^{\frac{T}{\varepsilon}}$ and, accordingly, as stated in (2.11) above, $\|Fv\|_{C^{\theta,\frac{\theta}{2}}} \le K_1$ for any $v \in S$.

Using $a(x,t) := (Fv)(x,t)$ and $b(x,t) := (Fv)(x,t) \cdot f(t)$, we see that, again, the above considerations are applicable, and $\|Fv\|_{C^{1+\delta,\frac{\delta}{2}}(\overline{\Omega}\times[0,T])} \le K_2$ for any $v \in S$ by (2.12). In particular, we observe that $FS \subset S$.

Furthermore invoking [26, IV.5.2], we can conclude the existence of $k > 0$ and $K_3 > 0$ such that

$$
\|Fv\|_{C^{2+\delta,1+\frac{\delta}{2}}(\overline{\Omega}\times[0,T])} \le k \left(\|Fv \cdot f\|_{C^{\delta,\frac{\delta}{2}}(\overline{\Omega}\times[0,T])} + \|u_{0\varepsilon}\|_{C^{2+\delta}(\overline{\Omega}\times[0,T])} + \varepsilon \right) \le K_3
\tag{2.16}
$$

for all $v \in S$. To see that F is continuous, we suppose that $(v_k)_{k\in\mathbb{N}} \subset S$ and $v \in S$ are such that $v_k \to v$ in X. Then $f_k(t) := \rho_\varepsilon\big(\int_\Omega |\nabla v_k(\cdot,t)|^2 \big)$ satisfies

$$
f_k \to f \quad \text{in } C^0([0,T])
\tag{2.17}
$$

as $k \to \infty$, with f as given by (2.13). By (2.16) and the theorem of Arzelà–Ascoli, $(Fv_k)_{k\in\mathbb{N}}$ is relatively compact in $C^{2,1}(\overline{\Omega}\times[0,T])$, and if $k_i \to \infty$ is any sequence such that $u_{k_i} := Fv_{k_i}$ converges in $C^{2,1}(\overline{\Omega}\times[0,T])$ to some w as $i \to \infty$, then in

$$
\partial_t u_{k_i} = u_{k_i} \Delta u_{k_i} + f_{k_i}(t) u_{k_i}, \qquad x \in \Omega,\ t \in (0,T),
$$

we may let $k_i \to \infty$ and use (2.17) to obtain that w is a classical solution of (2.14). Since classical solutions of (2.14) are unique due to the comparison principle, we must have $w = Fv$. We thereby derive that the whole sequence $(Fv_k)_{k\in\mathbb{N}}$ converges to Fv and hence conclude that F is continuous. Therefore, the Schauder fixed point theorem asserts the existence of at least one $u_\varepsilon \in S$ for which $u_\varepsilon = Fu_\varepsilon$ holds. Since such a fixed point obviously solves (2.6), the proof is complete. ∎

The basis of both our existence proof and our boundedness result is formed by the next two lemmata which provide useful a priori estimates for u_ε in terms of certain presupposed bounds. The first lemma essentially derives a uniform pointwise bound for u_ε from a space-time integral estimate for $|\nabla u_\varepsilon|^2$.

LEMMA 2.3. *For all $M > 0$ and $B > 0$ there exists $C(M,B) > 0$ with the following property: If*

$$u_{0\varepsilon} \le M \quad \text{in } \Omega \quad \text{and} \quad \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \le B \tag{2.18}$$

holds for some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and $T \in (0,\infty]$, then we have

$$u_\varepsilon \le C(M,B) \quad \text{in } \Omega \times [0,T). \tag{2.19}$$

Proof. Our plan is to use a separated function of the form

$$\overline{u}(x,t) := z(t) \cdot (M + \Phi(x)), \qquad x \in \bar{\Omega},\ t \in [0,T), \tag{2.20}$$

as a comparison function, where M is as in the hypothesis of the lemma, $\Phi \in C^2(\bar{\Omega})$ is the solution of (1.5), and z denotes the solution of

$$z' = -z^2 + \big(f(t)+1\big) \cdot z, \quad t \in (0,T), \qquad z(0) = 1, \tag{2.21}$$

with $f(t) := \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2$. In fact, it follows from (2.21) that $\zeta := \frac{1}{z}$ is a solution of $\zeta' = 1 - (f(t)+1)\zeta$, $\zeta(0) = 1$ and hence given by

$$\zeta(t) = e^{-\int_0^t f(s)ds - t} + \int_0^t e^{-\int_s^t f(\sigma)d\sigma - (t-s)} ds, \qquad t \in [0,T).$$

We claim that

$$1 \le z(t) \le e^{B+1} \quad \text{for all } t \in (0,T). \tag{2.22}$$

To see this, we note that if $t \in (0,T)$ satisfies $t < 1$, then (2.18) implies

$$\zeta(t) \ge e^{-\int_0^t f(s)ds - t} \ge e^{-B-t} \ge e^{-B-1},$$

whereas if $t \in [1,T)$, then again (2.18) shows

$$\begin{aligned} \zeta(t) &\ge \int_{t-1}^t e^{-\int_s^t f(\sigma)d\sigma - (t-s)} ds \ge \int_{t-1}^t e^{-B-(t-s)} ds \\ &\ge \int_{t-1}^t e^{-B-1} ds = e^{-B-1}. \end{aligned}$$

This yields the right inequality in (2.22), while the left immediately results from an ODE comparison of z with $\underline{z}(t) \equiv 1$, because $\underline{z}' + \underline{z}^2 - (f(t)+1)\underline{z} = -f(t) \le 0$. Consequently, since $\Phi \ge 0$ in Ω, the function $\overline{u}$ defined by (2.20) satisfies

$$\overline{u}(x,0) = M + \Phi(x) \ge M \ge u_\varepsilon(x,0) \quad \text{for all } x \in \Omega$$

due to (2.18), and on the lateral boundary we have

$$\overline{u}(x,t) = z(t) \cdot M \ge M \ge \varepsilon \quad \text{for all } x \in \partial\Omega \text{ and } t \in (0,T).$$

Moreover,

$$\begin{aligned}\overline{u}_t - \overline{u}\Delta\overline{u} - f(t)\cdot\overline{u} &= z'\cdot(M+\Phi) + z^2\cdot(M+\Phi) - f(t)\cdot z\cdot(M+\Phi)\\ &= z\cdot(M+\Phi)\\ &\geq 0 \quad \text{for all } x\in\Omega \text{ and } t\in(0,T),\end{aligned}$$

whence the comparison principle ensures that $u_\varepsilon \leq \overline{u}$ in $\Omega\times(0,T)$. In view of (2.22), this entails that

$$u_\varepsilon(x,t) \leq e^{B+1}\cdot\big(M + \|\Phi\|_{L^\infty(\Omega)}\big) \quad \text{for all } x\in\Omega \text{ and } t\in(0,t),$$

so that (2.19) is valid upon an obvious choice of $C = C(M,B)$. ∎

Next, the fact that solutions of (2.6) cannot blow up immediately can be turned into a quantitative local-in-time boundedness estimate in terms of the norm of the initial data in $L^\infty(\Omega)\cap W^{1,2}(\Omega)$. Moreover, our technique at the same time yields an estimate involving integrals of $u_{\varepsilon t}$ and ∇u_ε, as long as u_ε is appropriately bounded.

LEMMA 2.4. (i) *For all $M>0$ there exist $T_1(M)>0$ and $C_1(M)>0$ such that if*

$$u_{0\varepsilon} \leq M \quad \text{in } \Omega \qquad \text{and} \qquad \int_\Omega |\nabla u_{0\varepsilon}|^2 \leq M \tag{2.23}$$

hold for some $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$, then

$$u_\varepsilon \leq C_1(M) \quad \text{in } \Omega\times[0,T_1(M)). \tag{2.24}$$

(ii) *For each $M>0$ and $T>0$ there exist $T_2(M)\in(0,T]$ and $C_2(M)>0$ such that whenever $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$ is such that*

$$u_\varepsilon \leq M \quad \text{in } \Omega\times(0,T) \qquad \text{and} \qquad \int_\Omega |\nabla u_{0\varepsilon}|^2 \leq M \tag{2.25}$$

are satisfied, then

$$\int_0^{T_2(M)}\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \sup_{t\in(0,T_2(M))}\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \leq C_2(M). \tag{2.26}$$

Proof. (i) We multiply (2.6) by $\frac{u_{\varepsilon t}}{u_\varepsilon}$ and integrate by parts, use that $u_{\varepsilon t}=0$ on $\partial\Omega$, and apply Hölder's inequality, together with Young's inequality, to see that

$$\begin{aligned}\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_\varepsilon|^2 &= \left(\int_\Omega u_{\varepsilon t}\right)\cdot\rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right)\\ &\leq \left(\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon}\right)^{\frac12}\left(\int_\Omega u_\varepsilon\right)^{\frac12}\int_\Omega |\nabla u_\varepsilon|^2\\ &\leq \frac12\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \frac12\left(\int_\Omega u_\varepsilon\right)\left(\int_\Omega |\nabla u_\varepsilon|^2\right)^2\end{aligned} \tag{2.27}$$

for all $t>0$, because $\rho_\varepsilon(\xi)\leq\xi$ for all $\xi\geq 0$. Hence,

$$\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \frac{d}{dt}\int_\Omega |\nabla u_\varepsilon|^2 \leq \left(\int_\Omega u_\varepsilon\right)\left(\int_\Omega |\nabla u_\varepsilon|^2\right)^2. \tag{2.28}$$

Using the Poincaré inequality, we obtain

$$\int_\Omega u_\varepsilon(\cdot,t) \le c_1 \cdot \left(\left(\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \right)^{\frac{1}{2}} + 1 \right)$$

with a positive constant c_1 independent of $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}} \in (0,1)$ and $t > 0$. Therefore, (2.28) yields

$$\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \frac{d}{dt}\int_\Omega |\nabla u_\varepsilon|^2 \le c_1 \cdot \left(\left(\int_\Omega |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}} + 1 \right) \left(\int_\Omega |\nabla u_\varepsilon|^2 \right)^2, \tag{2.29}$$

which in particular implies that $z(t) := \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2$ satisfies

$$z'(t) \le c(\sqrt{z}+1)z^2 \quad \text{for all } t > 0, \qquad \text{and } z(0) \le M.$$

Hence, if we let ζ denote the local-in-time solution of

$$\begin{cases} \zeta'(t) = c(\sqrt{\zeta}+1)\zeta^2, & t > 0, \\ \zeta(0) = M, \end{cases}$$

with maximal existence time $T_\zeta > 0$, then due to (2.23) and an ODE comparison, we have $z \le \zeta$ in $(0, T_\zeta)$. Defining $T_1(M) := \frac{1}{2}T_\zeta$, for instance, we obtain from this that $\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le \zeta(T_1(M))$ for all $t \in [0, T_1(M))$, whereupon (2.24) now results from Lemma 2.3.

(ii) If the first inequality in (2.25) holds, then (2.28) entails that z as defined above even satisfies the nonlinear ODI

$$z'(t) \le M|\Omega| z^2 \quad \text{for all } t > 0,$$

whence we have $\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le \frac{1}{M^{-1} - M|\Omega| t}$ for all $t \in (0, T_2)$ with $T_2 := \min\{T, 1/(M^2|\Omega|)\}$, by the second inequality in (2.25). Inserting this into (2.29) again and integrating over $(0, T_2)$ proves (2.26). ∎

When constructing the solution u of (1.1) as the limit of solutions u_ε of (2.6), it will be comparatively easy to obtain the approximation property $\nabla u_\varepsilon \to \nabla u$ in the sense of $L^2_{loc}(\Omega \times [0,T))$-convergence. For handling the nonlocal term in the equation, however, it seems appropriate to make sure that also $\int_\Omega |\nabla u_\varepsilon|^2 \to \int_\Omega |\nabla u|^2$ in $L^1_{loc}([0,T))$.

In order to achieve the latter we exclude certain boundary concentration phenomena of ∇u_ε in the following sense.

LEMMA 2.5. *For any $T > 0$, $C > 0$, $M > 0$, and $\delta > 0$, there is $K = K(M,C,T,\delta) \subset\subset \Omega$ and $\eta > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ is such that $\varepsilon < \eta$ and*

$$\sup_{t\in[0,T]} \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le C \quad \textit{and} \quad u_\varepsilon \le M, \tag{2.30}$$

we have

$$\int_0^T \int_{\Omega\setminus K} |\nabla u_\varepsilon|^2 < \delta.$$

Proof. For $q \in (0,1)$, we multiply (2.6) by u_ε^{q-1} and integrate by parts to obtain

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u_\varepsilon^q = \int_{\partial\Omega} u_\varepsilon^q \partial_\nu u_\varepsilon - \int_\Omega q u_\varepsilon^{q-1}|\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^q\, \rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right),$$

where we can use $\partial_\nu u_\varepsilon \le 0$ on $\partial\Omega$ and integrate with respect to time to derive

$$q\int_0^T\int_\Omega u_\varepsilon^{q-1}|\nabla u_\varepsilon|^2 \le -\frac{1}{q}\int_\Omega u_\varepsilon^q(\cdot,T) + \frac{1}{q}\int_\Omega u_{0\varepsilon}^q + \int_0^T\left(\int_\Omega u_\varepsilon^q\int_\Omega|\nabla u_\varepsilon|^2\right) =: C(T) \tag{2.31}$$

for all $\varepsilon > 0$ satisfying (2.30), which gives control on $|\nabla u_\varepsilon|^2$ wherever u_ε is small—which is the case near the boundary, as we ensure next: In order to lay the groundwork for the corresponding comparison argument, note that by (2.30),

$$u_{\varepsilon t} = u_\varepsilon\Delta u_\varepsilon + u_\varepsilon\rho_\varepsilon\left(\int_\Omega|\nabla u_\varepsilon|^2\right) \le u_\varepsilon\Delta u_\varepsilon + Cu_\varepsilon, \qquad u_\varepsilon|_{\partial\Omega} = \varepsilon, \qquad u_\varepsilon(0) = u_{0\varepsilon}.$$

Fix $\eta > 0$ such that $\frac{(2\eta)^{1-q}C(T)}{q} < \delta$. Let Φ solve (1.5). Choose $A > C$ such that $A\Phi + \eta > u_{0\varepsilon}$ for all $0 < \varepsilon < \eta$, which is possible due to condition (2.2). Then $\overline{u} := A\Phi + \eta$ satisfies

$$\overline{u}_t = 0 \ge -(A\Phi+\eta)A + (A\Phi+\eta)C = \overline{u}A\Delta\Phi + C\overline{u} = \overline{u}\Delta\overline{u} + C\overline{u}. \tag{2.32}$$

As long as $\varepsilon < \eta$, also $\overline{u}|_{\partial\Omega} \ge u_\varepsilon|_{\partial\Omega}$ holds, and furthermore

$$\overline{u}(0) \ge u_{0\varepsilon}.$$

Therefore, by the comparison principle, we obtain $\overline{u} \ge u_\varepsilon$.

Now choose $K \subset\subset \Omega$ in such a way that

$$A\Phi \le \eta \quad \text{in } \Omega\setminus K.$$

This entails $u_\varepsilon \le \overline{u} = A\Phi + \eta \le 2\eta$ in $\Omega \setminus K$. Then

$$\begin{aligned}\int_0^T\int_{\Omega\setminus K}|\nabla u_\varepsilon|^2 &= \int_0^T\int_{\Omega\setminus K} u_\varepsilon^{q-1}|\nabla u_\varepsilon|^2 u_\varepsilon^{1-q}\\ &\le (2\eta)^{1-q}\int_0^T\int_{\Omega\setminus K}u_\varepsilon^{q-1}|\nabla u_\varepsilon|^2\\ &\le (2\eta)^{1-q}\int_0^T\int_\Omega u_\varepsilon^{q-1}|\nabla u_\varepsilon|^2 \le \frac{(2\eta)^{1-q}C(T)}{q},\end{aligned}$$

by virtue of (2.31). ∎

We are now ready to prove that the u_ε in fact approach a weak solution of (1.1) that is locally positive in the sense of Definition 1.1. Before we do so, however, we prepare the following estimate for u_ε that will be useful in proving assertions about the blow-up behavior of u.

LEMMA 2.6. *Let $\Omega' \subset\subset \Omega$ be a domain with smooth boundary. Assume also that ϕ denotes the solution to $-\Delta\phi = 1$ in Ω', $\phi|_{\partial\Omega'} = 0$. Then there exists $C_{\Omega'} > 0$ such that for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and any $t > 0$ the solution u_ε of (2.6) satisfies*

(2.33)

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2$$
$$\leq \int_\Omega |\nabla u_{0\varepsilon}|^2 \exp\left[\frac{1}{2C_{\Omega'}}\left(\sup_{\tau\in(0,t)}\int_\Omega u_\varepsilon(\cdot,\tau)\right)\left(\int_{\Omega'}\phi\ln u_\varepsilon(\cdot,t) - \int_{\Omega'}\phi\ln u_{0\varepsilon} + \int_0^t\int_{\Omega'} u_\varepsilon\right)\right].$$

Proof. As $u_{\varepsilon t} = 0$ on $\partial\Omega$, similarly to (2.27), multiplying (2.6) by $\frac{u_{\varepsilon t}}{u_\varepsilon}$ and integrating over Ω yields

$$\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} = \int_\Omega u_{\varepsilon t}\Delta u_\varepsilon + \int_\Omega u_{\varepsilon t}\rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right)$$
$$= -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u_\varepsilon|^2 + \int_\Omega u_{\varepsilon t}\rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right).$$

After rearranging, by Hölder's and Young's inequalities and the definition of ρ_ε this entails

$$\frac{d}{dt}\int_\Omega |\nabla u_\varepsilon|^2 \leq -2\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + 2\left[\left(\int_\Omega\left(\frac{u_{\varepsilon t}}{\sqrt{u_\varepsilon}}\right)^2\right)^{\frac{1}{2}}\left(\int_\Omega \sqrt{u_\varepsilon}^2\right)^{\frac{1}{2}}\right]\rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right)$$
$$\leq -2\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + 2\int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \frac{1}{2}\int_\Omega u_\varepsilon \rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right)^2$$
$$\leq \frac{1}{2}\int_\Omega u_\varepsilon\rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right)\int_\Omega |\nabla u_\varepsilon|^2 \quad \text{on } (0,\infty).$$

This looks like a quadratic differential inequality for $z(t) := \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2$ and at first does not seem helpful for obtaining an estimate for this quantity. Therefore, we shall split the respective quadratic term and apply Gronwall's lemma to $z'(t) \leq g(t)z(t)$, where

$$g(t) = \frac{1}{2}\int_\Omega u_\varepsilon(\cdot,t)\rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2\right),$$

which leads to

$$z(t) \leq z(0)\exp\int_0^t g(\tau)d\tau \quad \text{for all } t > 0. \tag{2.34}$$

In this situation, however, we are left with a term $\int_0^t \rho_\varepsilon\big(\int_\Omega |\nabla u_\varepsilon|^2\big)$ in the exponent, and we prepare an estimate for this in the following way: With ϕ as specified in the hypothesis, we let $C_{\Omega'} = \int_{\Omega'}\phi > 0$. Multiplication of (2.6) by $\frac{\phi}{u_\varepsilon}$ and integrating over Ω' then gives

$$\frac{d}{dt}\int_{\Omega'} \ln u_\varepsilon\phi = \int_{\Omega'} \Delta u_\varepsilon\phi + \int_{\Omega'}\phi\rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right)$$
$$= \int_{\Omega'} u_\varepsilon\Delta\phi + \int_{\partial\Omega'}\partial_\nu u_\varepsilon\phi - \int_{\partial\Omega'} u_\varepsilon\partial_\nu\phi + C_{\Omega'}\rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right) \text{ on } (0,\infty).$$

Taking into account the definition of ϕ and its consequence $\partial_\nu\phi|_{\partial\Omega'} \leq 0 = \phi|_{\partial\Omega'}$, we infer that

$$\frac{d}{dt}\int_{\Omega'}\phi\ln u_\varepsilon \geq -\int_{\Omega'} u_\varepsilon + C_{\Omega'}\rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right) \quad \text{on } (0,\infty).$$

Therefore,

$$\int_0^t \rho_\varepsilon\left(\int_\Omega |\nabla u_\varepsilon|^2\right) \le \frac{1}{C_{\Omega'}}\left[\int_0^t \int_{\Omega'} u_\varepsilon + \int_{\Omega'} \phi \ln u_\varepsilon(t) - \int_{\Omega'} \phi \ln u_{0\varepsilon}\right]$$

for any $t > 0$, and we can conclude from (2.34) that

$$\begin{aligned}&\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \\ &\quad\le \int_\Omega |\nabla u_{0\varepsilon}|^2 \cdot \exp\left[\frac{1}{2C_{\Omega'}} \sup_{\tau\in(0,\cdot,t)} \int_\Omega u(\tau)\left(\int_0^t \int_{\Omega'} u_\varepsilon + \int_{\Omega'} \phi \ln u_\varepsilon(t) - \int_{\Omega'} \phi \ln u_{0\varepsilon}\right)\right]\end{aligned}$$

for all $t > 0$. ∎

Another useful piece of information is that a condition like (H3) remains satisfied for any $t > 0$.

LEMMA 2.7. *Let $T > 0$, $M > 0$, and $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ be such that $\|u_{0\varepsilon} - \varepsilon\|_{\Phi,\infty} < \infty$. Then any solution u_ε of* (2.6) *which satisfies*

$$\int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le M \quad \textit{for any} \quad t \in [0,T]$$

already fulfills

$$\|u_\varepsilon - \varepsilon\|_{\Phi,\infty} \le \max\left\{M, \|u_{0\varepsilon} - \varepsilon\|_{\Phi,\infty}\right\}.$$

Proof. Let $C = \max\{M, \|u_{0\varepsilon} - \varepsilon\|_{\Phi,\infty}\}$ and consider $\overline{u} := C\Phi + \varepsilon$ with Φ as in (1.5). Then $\overline{u}_t = 0 \ge (M - C)(C\Phi + \varepsilon) = \overline{u}\Delta\overline{u} + M\overline{u}$, whereas $u_{\varepsilon t} = u_\varepsilon \Delta u_\varepsilon + u_\varepsilon \rho_\varepsilon\big(\int_\Omega |\nabla u_\varepsilon|^2\big) \le u_\varepsilon \Delta u_\varepsilon + M u_\varepsilon$. Additionally $\overline{u}|_{\partial\Omega} = \varepsilon = u_\varepsilon|_{\partial\Omega}$ and $\overline{u}(x,0) - \varepsilon = C\Phi(x) \ge \Phi(x)\,\|u_{0\varepsilon} - \varepsilon\|_{\Phi,\infty} \ge u_{0\varepsilon}(x) - \varepsilon$, and therefore the comparison principle [51] asserts that $u_\varepsilon \le \overline{u}$ and hence implies the claim. ∎

With this information at hand, we can proceed to the proof of convergence of the u_ε to a solution of (1.1) that still satisfies an inequality like (2.33).

LEMMA 2.8. *Suppose that u_0 satisfies* (H1)–(H3). *Then there exists $T > 0$ depending on bounds on $\|u_0\|_{L^\infty(\Omega)}$ and $\|\nabla u_0\|_{L^2(\Omega)}$ and a locally positive weak solution u of* (1.1) *in $\Omega \times (0,T)$. This solution can be obtained as the a.e. pointwise limit of a subsequence of the solutions u_ε of* (2.6) *as $\varepsilon = \varepsilon_j \searrow 0$, and for any smoothly bounded subdomain $\Omega' \subset\subset \Omega$ there is $C_{\Omega'} > 0$ such that*

(2.35)

$$\begin{aligned}&\int_\Omega |\nabla u(\cdot,t)|^2 \\ &\le \int_\Omega |\nabla u_0|^2 \exp\left[\frac{1}{2C_{\Omega'}}\left(\sup_{\tau\in(0,t)} \int_\Omega u(\cdot,\tau)\right)\left(\int_{\Omega'} \phi \ln u(\cdot,t) - \int_{\Omega'} \phi \ln u_0 + \int_0^t \int_{\Omega'} u\right)\right]\end{aligned}$$

as well as

$$\|u(\cdot,t)\|_{\Phi,\infty} \le \max\left\{\|u_0\|_{\Phi,\infty}, \operatorname*{ess\,sup}_{\tau\in(0,t)} \int_\Omega |\nabla u(\tau)|^2\right\} \tag{2.36}$$

for a.e. $t \in (0,T)$.

Proof. We set $M_1 := \max\{\|u_0\|_{L^\infty(\Omega)}+1, \int_\Omega |\nabla u_0|^2+1\}$ and let $T_1 = T_1(M_1)$ and $c_1 = C_1(M_1)$ be as in Lemma 2.4 (i). Then this lemma states that $u_\varepsilon \le c_1$ in $\Omega\times(0,T_1)$ for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Accordingly, corresponding to $M_2 = \max\{c_1, \int_\Omega |\nabla u_0|^2 + 1\}$, Lemma 2.4 (ii) provides $T = T_2(M_2) \in (0,T_1)$ and $c_2 = C_2(M_2) > 0$ such that

$$\int_0^T \int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon} + \sup_{t\in(0,T)} \int_\Omega |\nabla u_\varepsilon(\cdot,t)|^2 \le c_2 \tag{2.37}$$

for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, which by $u_\varepsilon \le c_1$ can be turned into a uniform bound on $\|u_{\varepsilon t}\|_{L^2(\Omega\times(0,T))}$, from which it follows by means of the fundamental theorem of calculus that after possibly enlarging c_2, we also have

$$\|u_\varepsilon\|_{C^{\frac{1}{2}}([0,T];L^2(\Omega))} \le c_2 \tag{2.38}$$

for such ε.

In order to prove a uniform estimate for u_ε from below, locally in space, we follow a standard comparison procedure: Given a compact set $K \subset \Omega$, we pick any smoothly bounded domain $\Omega' \subset\subset \Omega$ such that $K \subset\subset \Omega'$ and let $\phi \in C^2(\bar\Omega')$ solve $-\Delta\phi = 1$ in Ω' with $\phi|_{\partial\Omega'} = 0$. Then the lower estimate in (2.3) guarantees that writing $c_3(K) := \frac{1}{2\|\phi\|_{L^\infty(\Omega')}} \liminf_{\varepsilon\searrow 0} \inf_K u_{0\varepsilon}$, we can find $\varepsilon_0(K) > 0$ such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ satisfies $\varepsilon < \varepsilon_0(K)$, we have

$$u_{0\varepsilon}(x) \ge \frac{1}{2} \liminf_{\varepsilon\searrow 0} \inf_K u_{0\varepsilon} \ge c_3(K)\phi(x) \quad \text{for all } x \in \Omega'. \tag{2.39}$$

Letting $z(t) := \frac{c_3(K)}{1+c_3(K)t}$, $t \ge 0$, denote the solution of $z' = -z^2$ with $z(0) = c_3(K)$, we thus find that $\underline{u}(x,t) := z(t)\phi(x)$ satisfies $\underline{u} \le u_\varepsilon$ on the parabolic boundary of $\Omega' \times (0,\infty)$. Since

$$\underline{u}_t - \underline{u}\Delta\underline{u} = z'\phi + z^2\phi = 0 \quad \text{in } \Omega' \times (0,\infty)$$

and

$$u_{\varepsilon t} - u_\varepsilon \Delta u_\varepsilon = u_\varepsilon \cdot \rho_\varepsilon\bigg(\int_\Omega |\nabla u_\varepsilon|^2\bigg) \ge 0 \qquad \text{in } \Omega \times (0,\infty),$$

we conclude from the comparison principle (see [51] for an adequate version) that $\underline{u} \le u_\varepsilon$ and thus, in particular, that for each $T' > 0$ there exists a suitably small $c_4(K,T') > 0$ such that

$$u_\varepsilon \ge c_4(K,T') \quad \text{in } K \times (0,T') \tag{2.40}$$

holds for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ satisfying $\varepsilon < \varepsilon_0(K)$. By positivity of each individual u_ε, one can readily verify that upon suitably diminishing $c_4(K,T')$, (2.40) trivially extends so as to actually be valid for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. Now the estimate $u_\varepsilon \le c_1$, (2.37), (2.38), and (2.40) along with standard compactness arguments allow us to extract a subsequence $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}}$ and a function $u : \Omega \times [0,T] \to \mathbb{R}$ such that

$$u_\varepsilon \to u \quad \text{in } C^0([0,T);L^2(\Omega)) \quad \text{and a.e. in } \Omega \times (0,T), \tag{2.41}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^2_{loc}(\bar\Omega \times [0,T)), \quad \text{and} \tag{2.42}$$

$$u_{\varepsilon t} \rightharpoonup u_t \quad \text{in } L^2(\Omega \times (0,T)) \tag{2.43}$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$. From (2.41), the inequality $u_\varepsilon \le c_1$, and (2.40), we know that $u \le c_1$ a.e. in $\Omega \times (0,T)$ and $u \ge c_4(K,T)$ a.e. in $K \times (0,T)$ whenever $K \subset\subset \Omega$. Moreover, since $u_\varepsilon - \varepsilon$ vanishes on $\partial\Omega$, (2.42) implies that $u \in L^2((0,T); W_0^{1,2}(\Omega))$, so that u fulfills all regularity and positivity properties required for a locally positive weak solution in $\Omega \times (0,T)$ in the sense of Definition 1.1.

In order to verify that u is a weak solution of (1.1) it thus remains to check (1.4). To prepare this, we claim that in addition to (2.42), we also have the strong convergence properties

$$\nabla u_\varepsilon \to \nabla u \quad \text{in } L^2_{loc}(\Omega \times [0,T]) \quad \text{and a.e. in } \Omega \times (0,T) \tag{2.44}$$

as well as

$$\int_\Omega |\nabla u_\varepsilon(x,\cdot)|^2 dx \to \int_\Omega |\nabla u(x,\cdot)|^2 dx \quad \text{in } L^1((0,T)) \tag{2.45}$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$. To see (2.44), we let $K \subset\subset \Omega$ be given and fix a nonnegative $\psi \in C_0^\infty(\Omega)$ such that $\psi \equiv 1$ in K. Then

$$\begin{aligned}
\int_0^T \int_K |\nabla u_\varepsilon - \nabla u|^2 &\le \int_0^T \int_\Omega |\nabla u_\varepsilon - \nabla u|^2 \psi \\
&= \int_0^T \int_\Omega \nabla(u_\varepsilon - u) \cdot \nabla u_\varepsilon \cdot \psi - \int_0^T \int_\Omega \nabla u \cdot \nabla(u_\varepsilon - u) \cdot \psi \\
&=: I_1(\varepsilon) - I_2(\varepsilon) \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}},
\end{aligned} \tag{2.46}$$

where $I_2(\varepsilon) \to 0$ as $\varepsilon = \varepsilon_{j_k} \searrow 0$ by (2.42). Using the equation for u_ε, however, after an integration by parts we find that

$$\begin{aligned}
I_1(\varepsilon) &= -\int_0^T \int_\Omega (u_\varepsilon - u)\Delta u_\varepsilon \cdot \psi - \int_0^T \int_\Omega (u_\varepsilon - u)\nabla u_\varepsilon \cdot \nabla\psi \\
&= -\int_0^T \int_\Omega (u_\varepsilon - u) \cdot \frac{u_{\varepsilon t}}{u_\varepsilon} \cdot \psi + \int_0^T \int_\Omega (u_\varepsilon - u) \cdot \rho_\varepsilon\Big(\int_\Omega |\nabla u_\varepsilon|^2\Big) \cdot \psi \\
&\quad - \int_0^T \int_\Omega (u_\varepsilon - u)\nabla u_\varepsilon \cdot \nabla\psi \\
&=: I_{11}(\varepsilon) + I_{12}(\varepsilon) + I_{13}(\varepsilon) \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}.
\end{aligned}$$

Due to (2.41) and (2.42), we have $I_{13}(\varepsilon) \to 0$, and (2.41), together with (2.37) and Hölder's inequality, implies that

$$|I_{12}(\varepsilon)| \le \Big(\int_0^T \int_\Omega (u_\varepsilon - u)^2\Big)^{\frac{1}{2}} \cdot \Big[\int_0^T \Big(\int_\Omega |\nabla u_\varepsilon|^2\Big)^2\Big]^{\frac{1}{2}} \cdot \|\psi\|_{L^2(\Omega)} \to 0$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$, where we again have used the fact that $\rho_\varepsilon(z) \le z$ for any $z \ge 0$ and all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$. We now use Hölder's inequality and the local lower estimate (2.40), which in conjunction with (2.37) yields

$$\begin{aligned}
|I_{11}(\varepsilon)| &\le \Big(\int_0^T \int_\Omega \frac{u_{\varepsilon t}^2}{u_\varepsilon}\Big)^{\frac{1}{2}} \cdot \Big(\int_0^T \int_\Omega \frac{(u_\varepsilon - u)^2}{u_\varepsilon} \cdot \psi^2\Big)^{\frac{1}{2}} \\
&\le c_2^{\frac{1}{2}} \cdot \frac{\|\psi\|_{L^\infty(\Omega)}}{(c_4(\operatorname{supp}\psi, T))^{\frac{1}{2}}} \cdot \Big(\int_0^T \int_\Omega (u_\varepsilon - u)^2\Big)^{\frac{1}{2}} \to 0
\end{aligned}$$

as $\varepsilon = \varepsilon_{j_k} \searrow 0$, by (2.41). Altogether, we obtain that $I_1(\varepsilon) \to 0$ and hence, by (2.46), that $\nabla u_\varepsilon \to \nabla u$ in $L^2(K \times (0,T))$ as $\varepsilon = \varepsilon_{j_k} \searrow 0$ for arbitrary $K \subset\subset \Omega$.

Having thus proved (2.44), with the aid of Lemma 2.5 we obtain (2.45) as a straightforward consequence:

Given $\delta > 0$, we let $K = K(c_1, c_2, T, \frac{\delta}{4})$ and $\eta > 0$ be the set and the constant provided by Lemma 2.5, and we employ the convergence asserted by (2.42) to choose $k_0 \in \mathbb{N}$ such that for all $k, l > k_0$ we have $\int_0^T \int_K ||\nabla u_{\varepsilon_k}|^2 - |\nabla u_{\varepsilon_l}|^2| \le \frac{\delta}{2}$. Then for all $k, l > k_0$,

$$\begin{aligned}\int_0^T \left| \int_\Omega |\nabla u_{\varepsilon_k}|^2 - \int_\Omega |\nabla u_{\varepsilon_l}|^2 \right| &\le \int_0^T \int_K \left| |\nabla u_{\varepsilon_k}|^2 - |\nabla u_{\varepsilon_l}|^2 \right| + \int_0^T \int_{\Omega \setminus K} |\nabla u_{\varepsilon_k}|^2 \\ &\quad + \int_0^T \int_{\Omega \setminus K} |\nabla u_{\varepsilon_l}|^2 \\ &\le \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4}\end{aligned}$$

and thanks to the completeness of $L^2((0,T))$ we obtain (2.45). We can now proceed to verify that (1.4) holds for all $\varphi \in C_0^\infty(\Omega \times (0,T))$. To this end, we multiply (2.6) by $\varphi \in C_0^\infty(\Omega \times (0,T))$ and integrate to obtain

$$\int_0^T \int_\Omega u_{\varepsilon t}\varphi + \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \varphi + \int_0^T \int_\Omega u_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi = \int_0^T \int_\Omega u_\varepsilon \cdot \rho_\varepsilon\bigg(\int_\Omega |\nabla u_\varepsilon|^2 \bigg) \cdot \varphi.$$

Here, as $\varepsilon = \varepsilon_{j_k} \searrow 0$ we have

$$\int_0^T \int_\Omega u_{\varepsilon t}\varphi \to \int_0^T \int_\Omega u_t \varphi$$

by (2.43), whereas (2.44) and (2.41) allow us to conclude that

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \varphi \to \int_0^T \int_\Omega |\nabla u|^2 \varphi$$

and

$$\int_0^T \int_\Omega u_\varepsilon \nabla u_\varepsilon \cdot \nabla\varphi \to \int_0^T \int_\Omega u \nabla u \cdot \nabla\varphi,$$

because φ vanishes near $\partial\Omega$ and near $t = T$. Finally,

$$\int_0^T \int_\Omega u_\varepsilon \cdot \rho_\varepsilon\bigg(\int_\Omega |\nabla u_\varepsilon|^2 \bigg) \cdot \varphi \to \int_0^T \int_\Omega u \bigg(\int_\Omega |\nabla u|^2 \bigg) \cdot \varphi$$

because of (2.41), (2.45), and the fact that $\rho_\varepsilon(z) \to z$ for all $z \ge 0$ as $\varepsilon \searrow 0$. We thereby see that (1.4) holds and thus infer that u in fact is a weak solution of (1.1) in $\Omega \times (0,T)$. The inequality (2.35) results from Lemma 2.6 and the convergence statements. The estimate (2.36) results from Lemma 2.7: By (2.37) and (2.2) we have the necessary bounds on gradient and initial values, independent of $\varepsilon \in (\varepsilon_j)_{j \in \mathbb{N}}$. Furthermore, for any $t \in [0,T]$ we can find a subsequence $(\varepsilon_{j_k})_{k \in \mathbb{N}}$ of $(\varepsilon_j)_{j \in \mathbb{N}}$ such that

$$\frac{u_{\varepsilon_{j_k}}(\cdot, t) - \varepsilon_{j_k}}{\Phi} \rightharpoonup^* \frac{u(\cdot, t)}{\Phi} \quad \text{in } L^\infty(\Omega),$$

and finally the same bound as in Lemma 2.7 holds for $u(t)$ because

$$\begin{aligned}\|u(\cdot,t)\|_{\Phi,\infty} = \left\|\frac{u(\cdot,t)}{\Phi}\right\|_\infty &\le \liminf_{\varepsilon=\varepsilon_{j_k}\searrow 0}\left\|\frac{u_\varepsilon(\cdot,t)-\varepsilon}{\Phi}\right\|_\infty \\ &\le \liminf_{\varepsilon=\varepsilon_{j_k}\searrow 0}\max\left\{\sup_{0<\tau<t}\int_\Omega|\nabla u_\varepsilon(\tau)|^2, \|u_{0\varepsilon}-\varepsilon\|_{\Phi,\infty}\right\} \\ &\le \liminf_{\varepsilon=\varepsilon_{j_k}\searrow 0}\max\left\{\sup_{0<\tau<t}\int_\Omega|\nabla u_\varepsilon(\tau)|^2, \|u_0\|_{\Phi,\infty}+\varepsilon\right\} \\ &\le \max\left\{\operatorname*{ess\,sup}_{0<\tau<t}\int_\Omega|\nabla u(\tau)|^2, \|u_0\|_{\Phi,\infty}\right\},\end{aligned}$$

where for the last inequality we relied on the pointwise a.e. convergence of $\int_\Omega|\nabla u_\varepsilon(\cdot,t)|^2$ in $(0,T)$, due to (2.45) valid along a subsequence. ∎

We are now in the position to prove Theorem 1.3, which asserts the existence of a locally positive weak solution and $T_{max}\in(0,\infty]$ such that the solution blows up at T_{max} or exists globally.

Proof of Theorem 1.3. According to the statement of Lemma 2.8 there exists $T>0$ such that (1.1) possesses a locally positive weak solution u on $\Omega\times(0,T)$ which satisfies (1.7) and (1.8) for a.e. $t\in(0,T)$. Hence, the set

$$S := \Big\{\widetilde{T}>0 \,\Big|\, \text{there exists a locally positive solution } u \text{ to (1.1) on } \Omega\times(0,\widetilde{T}) \text{ satisfying (1.7) and (1.8) for a.e. } t\in(0,\widetilde{T})\Big\}$$

is not empty and

$$T_{max} = \sup S \in (0,\infty]$$

is well defined. Assume that $T_{max}<\infty$ and $\limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}<\infty$.

This implies the existence of a constant $M>0$ such that $u\le M$ and hence, due to (1.7), also that there is $C>0$ with $\int_\Omega|\nabla u|^2\le C$ on $[0,T_{max})$. Lemma 2.8 provides $T>0$ such that for any initial data u_0 satisfying $u_0\le M$, $\int_\Omega|\nabla u_0|^2\le C$, a locally positive weak solution existing on $\Omega\times(0,T)$ can be constructed.

Choose $t_0\in(T_{max}-\frac{T}{2},T_{max})$ such that $u(x,t_0)\le M$ and $\int_\Omega|\nabla u(x,t_0)|^2\le C$ and such that u satisfies (1.7) and (1.8) at $t=t_0$.

Let v denote the corresponding solution with initial value $u(\cdot,t_0)$ and define

$$\widehat{u}(x,t) = \begin{cases} u(x,t), & x\in\Omega, t<t_0,\\ v(x,t-t_0), & x\in\Omega, t\in(t_0,t_0+T).\end{cases}$$

Then $\widehat{u}$ is a solution of (1.1), and (1.7) and (1.8) obviously hold for a.e. $t\in(0,t_0)$, whereas for $t\in(t_0,t_0+T)$ we have

$$\begin{aligned}&\int_\Omega|\nabla\widehat{u}(\cdot,t)|^2 \le \int_\Omega|\nabla u(t_0)|^2 \\ &\quad\times\exp\left[\frac{1}{2C_{\Omega'}}\left(\sup_{\tau\in(t_0,t)}\int_\Omega\widehat{u}(\cdot,\tau)\right)\left(\int_{\Omega'}\phi\ln\widehat{u}(\cdot,t)-\int_{\Omega'}\phi\ln u(\cdot,t_0)+\int_{t_0}^t\int_{\Omega'}\widehat{u}\right)\right] \\ &\le \int_\Omega|\nabla u_0|^2\exp\left[\frac{1}{2C_{\Omega'}}\left(\sup_{\tau\in(0,t_0)}\int_\Omega u(\cdot,\tau)\right)\left(\int_{\Omega'}\phi\ln u(\cdot,t_0)-\int_{\Omega'}\phi\ln u_0+\int_0^{t_0}\int_{\Omega'}u\right)\right]\end{aligned}$$

$$\times \exp\left[\frac{1}{2C_{\Omega'}}\left(\sup_{\tau\in(t_0,t)}\int_\Omega \widehat{u}(\cdot,\tau)\right)\left(\int_{\Omega'}\phi\ln\widehat{u}(\cdot,t)-\int_{\Omega'}\phi\ln u(\cdot,t_0)+\int_{t_0}^t\int_{\Omega'}\widehat{u}\right)\right]$$

$$\le \int_\Omega |\nabla u_0|^2 \exp\left[\frac{1}{2C_{\Omega'}}\left(\sup_{\tau\in(0,t)}\int_\Omega \widehat{u}(\cdot,\tau)\right)\right.$$

$$\left.\times\left(\int_{\Omega'}\phi\ln u(\cdot,t_0)-\int_{\Omega'}\phi\ln u_0+\int_0^{t_0}\int_{\Omega'}u+\int_{\Omega'}\phi\ln\widehat{u}(\cdot,t)-\int_{\Omega'}\phi\ln u(\cdot,t_0)+\int_{t_0}^t\int_{\Omega'}\widehat{u}\right)\right]$$

$$= \int_\Omega |\nabla u_0|^2 \exp\left[\frac{1}{2C_{\Omega'}}\left(\sup_{\tau\in(0,t)}\int_\Omega \widehat{u}(\cdot,\tau)\right)\left(\int_{\Omega'}\phi\ln\widehat{u}(\cdot,t)-\int_{\Omega'}\phi\ln u_0+\int_0^t\int_{\Omega'}u\right)\right].$$

Also, for a.e. $t\in(0,t_0+T)$,

$$\|\widehat{u}(\cdot,t)\|_{\Phi,\infty}\le \max\left\{\|u(\cdot,t_0)\|_{\Phi,\infty},\ \sup_{\tau\in(t_0,t)}\int_\Omega|\nabla\widehat{u}(\cdot,\tau)|^2\right\}$$

$$\le \max\left\{\max\left\{\|u_0\|_{\Phi,\infty},\ \sup_{\tau\in(0,t_0)}\int_\Omega|\nabla u(\cdot,\tau)|^2\right\},\ \sup_{\tau\in(t_0,t)}\int_\Omega|\nabla\widehat{u}(\cdot,\tau)|^2\right\}$$

$$\le \max\left\{\|u_0\|_{\Phi,\infty},\ \sup_{\tau\in(0,t)}\int_\Omega|\nabla\widehat{u}(\cdot,\tau)|^2\right\}.$$

Thus $\widehat{u}$ is defined on $(0,T_{max}+\frac{T}{2})$, contradicting the definition of T_{max}. ∎

As a direct consequence of (1.8) we obtain that finite-time gradient blow-up cannot occur. More precisely, we have the following.

Corollary 2.9. *Let u and T_{max} be as given by Theorem* 1.3. *If $\limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty$, then also*

$$\limsup_{t\nearrow T_{max}}\int_\Omega|\nabla u(x,t)|^2dx=\infty.$$

Combining now Corollary 2.9 with the estimate (1.7), we can conclude that if finite-time L^∞-blow-up occurs, then also L^1-blow-up takes place at the same finite time.

Corollary 2.10. *Let u and T_{max} be as given by Theorem* 1.3. *If $\limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty$, then also*

$$\limsup_{t\nearrow T_{max}}\int_\Omega u(x,t)dx=\infty.$$

3. Total mass. Proof of Theorem 1.5. Let u be a solution of (1.1) on $[0,T]$. Consider its mass

$$y(t)=\int_\Omega u(x,t)\,dx,\qquad t\in[0,T), \tag{3.1}$$

and note that (3.1) defines a continuous function on $[0,T]$. Indeed, we have the following.

LEMMA 3.1. *For any weak solution u of* (1.1) *on* $[0,T]$, (3.1) *defines an absolutely continuous function* $y\colon [0,T]\to\mathbb{R}$ *that satisfies*

$$y'(t)=(y(t)-1)\int_\Omega |\nabla u(x,t)|^2\,dx \tag{3.2}$$

for almost every $t\in(0,T)$.

Proof. We will show that whenever $0<s<t<T$,

$$y(t)-y(s)=\int_s^t \left((y(\tau)-1)\int_\Omega |\nabla u(x,\tau)|^2\,dx\right)d\tau, \tag{3.3}$$

where absolute continuity follows from the representation as an integral and the assertion about the derivative is a direct consequence of division by $t-s$ and passing to the limit $s\to t$.

Let $0<s<t<T$ and $0<\delta<\min\{s,T-t\}$. Define the function $\chi\colon\mathbb{R}\to\mathbb{R}$ by setting

$$\chi(\tau)=\begin{cases}0, & \tau<s-\delta,\\ 1+\frac{\tau-s}{\delta}, & s-\delta\le\tau<s,\\ 1, & s\le\tau<t,\\ 1-\frac{\tau-t}{\delta}, & t\le\tau<t+\delta,\\ 0, & \tau\ge t+\delta.\end{cases}$$

Then, according to standard approximation arguments, $\varphi(x,t):=\chi(t)$ defines an admissible test function for (1.3) and we obtain

$$-\frac{1}{\delta}\int_{s-\delta}^s\int_\Omega u+\frac{1}{\delta}\int_t^{t+\delta}\int_\Omega u+\int_{s-\delta}^{t+\delta}\int_\Omega |\nabla u|^2\varphi=\int_{s-\delta}^{t+\delta}\left(\int_\Omega u\varphi\right)\cdot\left(\int_\Omega |\nabla u|^2\right).$$

Since $u\in C_{loc}([0,T),L^2(\Omega))$, we have

$$\frac{1}{\delta}\int_t^{t+\delta}\int_\Omega u\to y(t)\quad\text{and}\quad\frac{1}{\delta}\int_{s-\delta}^s\int_\Omega u\to y(s)$$

as $\delta\searrow 0$.

Also by Lebesgue's dominated convergence theorem,

$$\int_{s-\delta}^{t+\delta}\int_\Omega |\nabla u|^2\varphi\to\int_s^t\int_\Omega |\nabla u|^2$$

and

$$\int_{s-\delta}^{t+\delta}\left(\int_\Omega u\varphi\right)\cdot\left(\int_\Omega |\nabla u|^2\right)\to\int_s^t\left(\int_\Omega u\right)\cdot\left(\int_\Omega |\nabla u|^2\right)$$

as $\delta\searrow 0$. Hence, (3.3) holds. ∎

This lemma is the main ingredient in the following proof of Theorem 1.5.

Proof of Theorem 1.5. (i) In the case of subcritical initial mass, Lemma 3.1 shows that y as defined in (3.1) is decreasing, which by Corollary 2.10 entails global existence, and from the nonnegativity of y we derive that $y(t)\to c$ as $t\to\infty$ for some $c\ge 0$. Note that Poincaré's and Hölder's inequalities imply that for some $C_P>0$ we have

$$\int_\Omega |\nabla u|^2\,dx\ge\frac{1}{C_P}\int_\Omega u^2\,dx\ge\frac{1}{C_P|\Omega|}\left(\int_\Omega u\,dx\right)^2=\frac{1}{C_P|\Omega|}y^2\qquad\text{on }(0,\infty),$$

and hence Lemma 3.1, due to the negativity of $y(t)-1$, entails that

$$y'(t) \le (y(t)-1)\frac{1}{C_P|\Omega|}\,y^2(t) \le -\frac{1-y(0)}{C_P|\Omega|}\,y^2(t) \le -\frac{1-y(0)}{C_P|\Omega|}\,c^2$$

for almost every $t>0$. This would lead to a contradiction to the nonnegativity of $y(t)$ if c were positive, whence actually $c=0$.

(ii) If $\int_\Omega u_0 = 1$, then Lemma 3.1 implies that

$$y(t)-1 = \int_0^t \left[(y(s)-1)\int_\Omega |\nabla u(x,s)|^2\,dx\right] ds,$$

and by virtue of Gronwall's lemma we conclude that $y(t)-1\equiv 0$ throughout the time interval on which the solution exists, which combined with Corollary 2.10 also implies global existence.

(iii) In the case when the total mass is supercritical initially, Lemma 3.1 entails that y is nondecreasing, and again Poincaré's and Hölder's inequalities imply that

$$y'(t) \ge \frac{y(0)-1}{C_P|\Omega|}y^2(t) \qquad \text{for a.e. } t\in[0,T_{max})$$

with some $C_P>0$. Now let z denote the solution to

$$z'(t) = \frac{y(0)-1}{C_P|\Omega|}z(t)^2, \quad z(0)=z_0,$$

for some $1<z_0<y(0)$, defined up to its maximal existence time $T_0>0$. Then $T := T_{max} < T_0$, because $y\ge z$, and the assertion follows by Theorem 1.3 in combination with Corollary 2.10. ∎

4. Global blow-up. Proof of Theorem 1.7. We proceed to prove that blow-up of our solutions always occurs globally, as stated in Theorem 1.7.

Proof of Theorem 1.7. Assume to the contrary that the closed set $\mathscr{B}$ is strictly contained in $\overline{\Omega}$. Then there exists a smoothly bounded subdomain $\Omega'\subset\Omega\setminus\mathscr{B}$ such that u is bounded in $\Omega'\times(0,T_{max})$. Let ϕ be a solution to $-\Delta\phi=1$ in Ω', $\phi=0$ on $\partial\Omega'$.

Consider $T'<T_{max}$. Due to the local positivity of u, we have $\frac{\phi}{u}\in L^\infty(\Omega\times(0,T'))$ and $\nabla\frac{\phi}{u} = \frac{\nabla\phi}{u} - \frac{\phi}{u^2}\nabla u \in L^2(\Omega'\times(0,T'))$ and hence $\frac{\phi}{u}\in L^2((0,T'),W_0^{1,2}(\Omega'))\cap L^\infty(\Omega\times(0,T'))\subset L^2((0,T'),W_0^{1,2}(\Omega))\cap L^\infty(\Omega\times(0,T'))$. Therefore, we can readily verify by approximation arguments that it is possible to use $\varphi=\frac{\phi}{u}$ as a test function in (1.4), which then leads to

$$\int_0^t\int_{\Omega'}\frac{u_t}{u}\phi\,dx\,ds + \int_0^t\int_{\Omega'}\nabla u\cdot\nabla\phi\,dx\,ds = \int_0^t\left(\int_{\Omega'}\phi\,dx\right)\cdot\left(\int_\Omega|\nabla u|^2\,dx\right)ds$$

for any $t\in(0,T_{max})$. Hence, with $C_{\Omega'} := \int_{\Omega'}\phi$ and because of $\partial_\nu\phi\big|_{\partial\Omega'}\le 0$,

$$\int_{\Omega'}\phi\ln u(t)\,dx - \int_{\Omega'}\phi\ln u_0\,dx - \int_0^t\int_{\Omega'}u\cdot\Delta\phi\,dx\,ds \ge C_{\Omega'}\int_0^t\int_\Omega|\nabla u|^2\,dx\,ds,$$

that is

$$\int_0^t\int_{\Omega'}u\,dx\,ds + \int_{\Omega'}\phi\ln u(t)\,dx - \int_{\Omega'}\phi\ln u_0\,dx \ge C_{\Omega'}h(t), \tag{4.1}$$

where $h(t) := \int_0^t \int_\Omega |\nabla u(x,s)|^2\,dx\,ds$ and where—due to the choice of Ω'—the left-hand side is bounded from above.

On the other hand, from Lemma 3.1 we know that

$$\frac{y'(t)}{y(t)-1} = \int_\Omega |\nabla u|^2\,dx$$

for $y(t) = \int_\Omega u(x,t)\,dx$. Therefore,

$$h(t) = \int_0^t \int_\Omega |\nabla u|^2\,dx\,ds = \int_0^t \frac{y'(\tau)}{y(\tau)-1}\,ds = \ln(y(t)-1) - \ln(y(0)-1) = \ln \frac{\int_\Omega u(x,t)\,dx - 1}{\int_\Omega u_0\,dx - 1}$$

and, by Theorem 1.5 (iii), $\limsup_{t \nearrow T_{max}} h(t) = \infty$, contradicting the boundedness of the left-hand side of (4.1). ∎

We have seen that the question of global existence versus blow-up of solutions to (1.1) is intimately connected with the size of the initial data. If $\int_\Omega u_0 > 1$, the solution blows up globally; if $\int_\Omega u_0 < 1$, we have proven convergence toward 0. The missing case of solutions emanating from initial data with unit mass must exhibit a behavior different from either, as Theorem 1.5 (ii) shows. For a study of these solutions, which are actually very important for the described replicator dynamics model, we refer the reader to the article [27].

Appendix A: Modeling background. *Evolutionary game dynamics* is a major part of modern game theory. It was appropriately fostered by evolutionary biologists such as W. D. Hamilton and J. Maynard Smith (see [10] for a collection of survey papers and [42] for a popularized account) and it actually brought a conceptual revolution to game theory analogous to that of population dynamics in biology. The resulting population-based approach has also found many applications in nonbiological fields such as economics and learning theory and introduces a significant enrichment of *classical* game theory which focuses on the concept of a rational individual.

The main focus of evolutionary game dynamics is to explain how a population of players updates their strategies in the course of a game according to the strategies' success. This contrasts with classical noncooperative game theory, which analyzes how rational players will behave through static solution concepts such as the Nash equilibrium (NE) (i.e., a strategy choice for each player whereby no individual has a unilateral incentive to change his or her behavior).

As Hofbauer and Sigmund [19] pointed out, strategies with high payoff will spread within the population through learning, imitation, or inheriting processes, or even by infection. The payoffs depend on the actions of the coplayers, i.e., the frequencies in which the various strategies appear, and since these frequencies change according to the payoffs, a feedback loop appears. The dynamics of this feedback loop will determine the long time progress of the game, and its investigation is exactly the course of evolutionary game theory.

According to the extensive survey paper [19] there is a variety of different dynamics in evolutionary game theory such as replicator dynamics, imitation dynamics, best response dynamics, Brown–von Neumann–Nash dynamics, etc. However, the dynamics most widely used and studied in the literature on evolutionary game theory are the replicator dynamics which were introduced in [49] and christened in [40]. Such dynamics illustrate the idea that in a dynamic process of evolution, a strategy should increase in frequency if it is a successful strategy in the sense that individuals employing this strategy obtain a higher than average payoff.

Let us consider a game with m discrete pure strategies, forming the strategy space $S = \{1, 2, \ldots, m\}$ and corresponding frequencies $p_i(t), i = 1, 2, \ldots, m$, for any $t \geq 0$. (Alternatively, S could be considered as the set of different states (genetic programs) of a biological population.) The frequency (probability) vector $p(t) = (p_1(t), p_2(t), \ldots, p_m(t))^T$ belongs to the invariant simplex

$$S(m) = \left\{ y = (y_1, y_2, \ldots, y_m)^T \in \mathbb{R}^m : y_i \geq 0, i = 1, 2, \ldots, m \quad \text{and} \quad \sum_{i=1}^{m} y_i = 1 \right\}.$$

The game is actually determined by the payoff matrix $A = (a_{ij})$, which is a real $m \times m$ symmetric matrix. Payoff means expected gain, and if an individual plays strategy i against another individual following strategy j, then the payoff to i is defined to be a_{ij}, while the payoff to j is a_{ji}. For symmetric games, matrix A is considered to be symmetric. (In the case of a biological population, payoff represents fitness, or reproductive success.)

Then the expected payoff for an individual playing strategy i can be expressed as

$$(A \cdot p(t))_i = \sum_{j=1}^{m} a_{ij} p_j(t),$$

whereas the average payoff over the whole population is given by

$$(p(t)^T \cdot A \cdot p(t)) = \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} p_i(t) p_j(t).$$

Consider that our game is symmetric with infinitely many players (or that the biological population is infinitely large and its generations blend continuously into each other); then we obtain that $p_i(t)$ evolve as differentiable functions. Note that the rate of increase of the per capita rate of growth $\dot{p}_i/p_i$ of strategy (type) i is a measure of its evolutionary success; here $\dot{p}_i$ stands for the time derivative of p_i. A reasonable assumption, which is also in agreement with the basic tenet of Darwinism, is that the per capita rate of growth (i.e., the logarithmic derivative) $\dot{p}_i/p_i$ is given by the difference between the payoff for strategy (type) i and the average payoff. This yields the *replicator dynamical system*,

$$\text{(A.1)} \quad \frac{dp_i}{dt} = \left(\sum_{j=1}^{m} a_{ij} p_j(t) - \sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij} p_i(t) p_j(t) \right) p_i(t), \quad i = 1, 2, \ldots, m, \quad t > 0.$$

The dynamical system (A.1) actually describes the mechanism by which individuals tend to switch to strategies that are doing well, or that individuals bear offspring who tend to use the same strategies as their parents, and the fitter the individual, the more numerous the offspring.

Most of the work on replicator dynamics has focused on games that have a finite strategy space, thus leading to a dynamical system for the frequencies of the population which is finite dimensional. However, interesting applications arise in both biology and economics where the strategy space is not finite or, even, not discrete; see [7, 33, 34, 35]. In the case when the strategy space S is discrete but consists of an infinite number of strategies, e.g., $S = \mathbb{Z}$, then the replicator dynamics describing the evolution of the infinite dimensional vector $p(t) = (\dots, p_1(t), p_2(t), \dots)$ are described by

$$\frac{dp_i}{dt} = \left(\sum_{j\in\mathbb{Z}} a_{ij}p_j(t) - \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} a_{ij}p_i(t)p_j(t) \right) p_i(t), \quad t > 0,$$

which is an infinite dynamical system with $p_i(t) \geq 0$ for $i \in \mathbb{Z}$ and $||p(t)||_{\ell^1(\mathbb{Z})} = 1$ for any $t > 0$.

In the current paper we are concentrating on games whose pure strategies belong to a continuum. For instance, this could be the aspiration level of a player or the size of an investment in economics, or it might arise in situations where the pure strategies correspond to geographical points as in economic geography [24]. On the other hand, in biology such strategies correspond to some continuously varying trait such as the sex ratio in a litter or the virulence of an infection [19]. There are different ways of modeling the evolutionary dynamics in this case; however, in the current work we adapt the approach introduced in [7]. In that case the strategy set Ω is an arbitrary, not necessarily bounded, Borel set of $\mathbb{R}^N$, $N \geq 2$, and hence strategies can be identified by $x \in \Omega$. For the case of symmetric two-player games, the payoff can be given by a Borel measurable function $f : \Omega\times\Omega \to \mathbb{R}$, where $f(x, y)$ is the payoff for player 1 when she follows strategy x and player 2 plays strategy y. A population is now characterized by its state, a probability measure $\mathscr{P}$ in the measure space $(\Omega, \mathscr{A})$, where $\mathscr{A}$ is the Borel algebra of subsets of Ω. The average (mean) payoff of a subpopulation in state $\mathscr{P}$ against the overall population in state $\mathscr{Q}$ is given by the form

$$E(\mathscr{P}, \mathscr{Q}) := \int_\Omega \int_\Omega f(x, y)\mathscr{Q}(dy)\mathscr{P}(dx).$$

Then, the success (or lack of success) of a strategy x followed by population $\mathscr{Q}$ is provided by the difference

$$\sigma(x, \mathscr{Q}) := \int_\Omega f(x, y)\mathscr{Q}(dy) - \int_\Omega \int_\Omega f(x, y)\mathscr{Q}(dy)\mathscr{Q}(dx) = E(\delta_x, \mathscr{Q}) - E(\mathscr{Q}, \mathscr{Q}),$$

where δ_x is the unit mass concentrated on the strategy x.

The evolution in time of the population state $\mathscr{Q}(t)$ is given by the replicator dynamics equation

$$\text{(A.2)} \qquad \frac{d\mathscr{Q}}{dt}(A) = \int_A \sigma(x, \mathscr{Q}(t))\mathscr{Q}(t)(dx),\ t > 0, \quad \mathscr{Q}(0) = \mathscr{P},$$

for any $A \in \mathscr{A}$, where the time derivative should be understood with respect to the variational norm of a subspace of the linear span $\mathscr{M}$ of $\mathscr{A}$. The well-posedness of (A.2) and related stability issues were investigated in [34, 35] under the assumption that the payoff function $f(x, y)$ is bounded.

The abstract form of (A.2) does not actually allow us to obtain insight into the form of its solutions and thus a better understanding of the evolutionary dynamics of the corresponding game. In order to give a better overview of the evolutionary game, following the approach in [21, 22] we restrict our attention to measures $\mathscr{Q}(t)$ which, for each $t > 0$, are absolutely continuous with respect to the Lebesgue measure, with probability density $u(x, t)$. Then the replicator dynamics equation (A.2) can be reduced to the integro-differential equation

(A.3)
$$\frac{\partial u}{\partial t} = \left(\int_\Omega f(x, y)u(y, t)\, dy - \int_\Omega \int_\Omega f(z, y)u(y, t)u(z, t)dy\, dz \right) u(x, t),\ t > 0,\ \ x \in \Omega,$$

also taking into account that the probability density u satisfies

$$\int_\Omega \int_\Omega u(y,t)\, u(z,t)\, dy\, dz = 1, \tag{A.4}$$

and hence we can skip the denominator from the average payoff term in (A.3).

There are applications in both biology and computer science in which the payoff kernel has the form $f(x,y) = G(x-y)$, with G being a steep function of Gaussian type; see [17, 18, 20, 32]. This case, in general, models games where the payoff is measured as the distance from some reference strategy and, finally, under some proper scaling leads to

$$\int_\Omega f(x,y) u(y,t)\, dy \approx \Delta u(x,t) \tag{A.5}$$

(see also [23]), which by virtue of (A.2) yields

$$\frac{\partial u}{\partial t} \approx \left(\Delta u - \int_\Omega u\, \Delta u\, dx \right) u. \tag{A.6}$$

Another alternative way to get payoffs of this type is to consider a game with a discrete strategy space and take the appropriate scaling limit. In that case a Taylor expansion and a proper scaling give a similar approximation to (A.5); see also [21, 22].

Therefore, in the case when Ω is a bounded and smooth domain of $\mathbb{R}^N$, it is easily seen that via integration by parts the nonlocal integro-differential dynamics equation (A.3) is approximated by the degenerate nonlocal parabolic equation

$$\frac{\partial u}{\partial t} = u\Big(\Delta u + \int_\Omega |\nabla u|^2\, dx \Big), \qquad x \in \Omega,\ t > 0. \tag{A.7}$$

The nonlocal equation (A.7) is associated with initial data

$$u(x,0) = u_0(x),\ \ x \in \Omega, \tag{A.8}$$

which in the relevant case satisfy

$$\int_\Omega u_0(x)\, dx = 1, \tag{A.9}$$

and with homogeneous Dirichlet boundary conditions

$$u(x,t) = 0, \quad x \in \partial\Omega,\ \ t > 0, \tag{A.10}$$

when the agents avoid playing the strategies located on the boundary of the strategy space, since they are assumed to be too risky, or when the individuals of the biological population do not interact when they are close to the spatial boundary, where the "food" is assumed to be sparse. We remark that when individuals on the boundary of the strategy space do not distinguish between nearby strategies but rather populate them equally, the nonlocal equation (A.7) should rather be complemented by homogeneous Neumann boundary conditions not explicitly considered here; see [22].

Our analysis will inter alia reveal that initial unit mass is preserved and guarantees that

$$\int_\Omega u(x,t)\, dx = 1; \tag{A.11}$$

see also Theorem 1.5 (ii), which in this case provides an a posteriori justification for (A.4).

Appendix B: A convenient approximation of the initial data. In this paper, we have kept the proof of Lemma 2.1 very short. Here we give a more detailed version, which still suppresses some of the more involved technical calculations.

Proof. Choose $\gamma > 0$ and a domain $U_\vartheta \subset \Omega$ such that $dist(U_\vartheta, \partial\Omega) > \gamma$. Let $\vartheta \in C_0^\infty(U_\vartheta)$ with $\vartheta \geq 0$ and $\int_\Omega \vartheta = 1$. Let $\varepsilon > 0$, and let $\varphi \in C_0^\infty(\Omega)$ be such that $\|\varphi - u_0\|_{W^{1,2}(\Omega)} < \varepsilon$ and $\|\varphi\|_{\Phi,\infty} \leq C + \zeta(\varepsilon)$, where $\zeta\colon [0,\infty) \to [0,\infty)$ is a function satisfying $\lim_{\varepsilon\to 0}\zeta(\varepsilon) = 0$. In order to see that this is possible, recall how smooth approximations φ of $W^{1,2}(\Omega)$-functions u_0 are usually constructed (see [56, section 3]): With the aid of a partition of unity $\{\alpha_i\}$, the function is written as a sum, where the single summands are supported in small patches only and those close to the boundary are shifted toward the interior by application of shift operators s_i; finally, the function is smoothed by convolution with a standard mollifier j_ε.

We observe that the same procedure applied to Φ does not violate too much the inequality $\|u_0\|_{\Phi,\infty} \leq C$, i.e., $u_0 \leq C\Phi$; that is,

$$C \sum j_\varepsilon \star (\alpha_i s_i(\Phi)) \leq C \sum \alpha_i \Phi + \zeta\Phi = C\Phi + \zeta\Phi$$

holds for some ζ with $\lim_{\varepsilon\searrow 0}\zeta(\varepsilon) = 0$. (The calculations showing this use the facts that mollification of smooth functions converge in C^1 and that Φ grows toward the interior, and use the mean value theorem.) Hence, the fact that mollification preserves pointwise estimates that hold everywhere shows that also φ satisfies $\varphi(x) \leq C\Phi(x)$.

Let K be a compact subset of Ω such that $|\Omega \setminus K| < \varepsilon$ and $dist(\partial\Omega, K) < \varepsilon$. Let $\rho \in C_0^\infty(\Omega)$ such that $\rho = 1$ on $\hat{K} \cup \operatorname{supp}\,\varphi$, $|\nabla\rho(x)| < \frac{2}{dist(\hat{K},\partial\Omega)}$, and $0 \leq \rho \leq 1$. Denoting

$$\begin{aligned}
A = A(\varepsilon) &= \int_\Omega \Phi^2|\nabla\rho|^2 + \int_\Omega (1-\rho)^2|\nabla\Phi|^2 + \int_\Omega |\nabla\vartheta|^2 \left(\int_\Omega (1-\rho)\Phi\right)^2,\\
B = B(\varepsilon) &= -1 - 2\int_\Omega (1-\rho)\Phi \int_\Omega \nabla\varphi\nabla\vartheta\\
&\quad - 2\int_\Omega (1-\rho)\Phi \int_\Omega (u_0-\varphi)\int_\Omega |\nabla\vartheta|^2 + 2\varepsilon|\Omega| \int_\Omega (1-\rho)\Phi \int_\Omega |\nabla\vartheta|^2,\\
\Gamma = \Gamma(\varepsilon) &= \int_\Omega |\nabla\varphi|^2 + 2\int_\Omega (u_0-\varphi)\int_\Omega \nabla\varphi\nabla\vartheta\\
&\quad - 2\varepsilon|\Omega| \int_\Omega \nabla\varphi\nabla\vartheta - 2\varepsilon|\Omega|\int_\Omega (u_0-\varphi)\int_\Omega |\nabla\vartheta|^2\\
&\quad + \left(\int_\Omega (u_0-\varphi)\right)^2 \int_\Omega |\nabla\vartheta|^2 + \varepsilon^2|\Omega|^2 \int_\Omega |\nabla\vartheta|^2,
\end{aligned}$$

we let $C = C(\varepsilon) = -\frac{2\Gamma}{B - \sqrt{B^2 - 4A\Gamma}}$. Then C solves

$$AC^2 + BC + \Gamma = 0. \tag{B.1}$$

As Φ and $\nabla\Phi$ are bounded, as $1-\rho$ is supported on a small set with measure smaller than ε, and as we have $\Phi|\nabla\rho| \leq 2D_2$, where $\Phi(x) \leq D_2 dist(x, \partial\Omega)$, most integrals from the definition of A, B, Γ can be estimated, yielding $A \to 0$, $B \to -1$, $\Gamma \to \int_\Omega |\nabla u_0|^2$ as $\varepsilon \to 0$. Therefore,

$$C = -\frac{2\Gamma}{B - \sqrt{B^2 - 4A\Gamma}} \to -\frac{2\int_\Omega |\nabla u_0|^2}{-1 - \sqrt{1-0}} = \int_\Omega |\nabla u_0|^2 > 0,$$

as $\varepsilon \to 0$, and in particular, $\limsup(C - L) \leq 0$. Furthermore, for sufficiently small ε, we have $C > 0$. We also observe that

$$\alpha = \int_\Omega (u_0 - \varphi) - \varepsilon|\Omega| - C \int_\Omega (1-\rho)\Phi \to 0$$

as $\varepsilon \to 0$. If ε is small enough, therefore, $|\alpha| < \frac{\frac{1}{2} essinf_{\left\{x; dist(x,\partial\Omega) > \frac{\gamma}{2}\right\}} u_0}{\sup \vartheta}$, and hence $|\alpha\vartheta| \leq \frac{1}{2} \inf_{\{x \in \Omega, dist(x,\partial\Omega) > \frac{\gamma}{2}\}} \varphi$ on Ω (as supp $\vartheta \subset$ supp φ). Therefore,

$$\varphi(x) + \alpha\vartheta(x) \geq \frac{1}{2} \inf_{\left\{x \in \Omega; dist(x,\partial\Omega) > \frac{d}{2}\right\}} \varphi =: C_K \tag{B.2}$$

for $x \in K$ and

$$\varphi + \alpha\vartheta \geq 0 \tag{B.3}$$

on Ω, because $\varphi \geq 0$ and $\alpha\vartheta \neq 0$ only on U_ϑ, where (B.2) guarantees (B.3) already. We also have

$$\begin{aligned} \varphi + \alpha\vartheta \leq& \left(L + \frac{\varepsilon}{2}\right)\Phi + \alpha\vartheta \leq \left(L + \frac{\varepsilon}{2}\right)\Phi + \vartheta \int_\Omega |u_0 - \varphi| \\ \leq& (L + \zeta(\varepsilon))\Phi \end{aligned}$$

with some ζ fulfilling $\lim_{\varepsilon \searrow 0} \zeta(\varepsilon) = 0$. Finally, define

$$u_{0\varepsilon} = \varepsilon + C(1-\rho)\Phi + \rho(\varphi + \alpha\vartheta). \tag{B.4}$$

Estimate (B.3) and the positivity of C and of Φ in Ω, together with (B.4), entail $u_{0\varepsilon} \geq \varepsilon$. Accordingly (2.1) holds, for we clearly obtain $u_{0\varepsilon} = \varepsilon$, and $\Delta u_{0\varepsilon} = -C = -\int_\Omega |\nabla u_{0\varepsilon}|^2$ on $\partial\Omega$, because

$$\int_\Omega |\nabla u_{0\varepsilon}|^2 = \int_\Omega |\nabla(\varepsilon + C(1-\rho)\Phi + \rho(\varphi + \alpha\vartheta))|^2 = AC^2 + (B+1)C + \Gamma = C$$

by (B.1). Furthermore,

$$\int_\Omega u_{0\varepsilon} = \int_\Omega \varepsilon + \int_\Omega C(1-\rho)\Phi + \int_\Omega \rho\varphi + \alpha \int_\Omega \rho\vartheta = \int_\Omega u_0$$

which is (2.5). The smoothness assertion follows from the smoothness of φ (as mollification) and that of Φ and that of $\rho, \vartheta \in C_0^\infty(\Omega)$. By definition of $u_{0\varepsilon}$,

$$\|u_{0\varepsilon} - \varepsilon\|_{\Phi,\infty} = \|C\Phi(1-\rho) + \rho(\varphi + \alpha\vartheta)\|_{\Phi,\infty}.$$

In every point $x \in \Omega$, $u_{0\varepsilon} - \varepsilon$ is a convex combination of $C\Phi$ and $\varphi + \alpha\vartheta$, which both satisfy the estimate "$\leq (L + \zeta(\varepsilon))\Phi$." Therefore (2.2) holds. Furthermore,

$$\begin{aligned} \|u_{0\varepsilon} - u_0\|_{W^{1,2}(\Omega)} &= \|\varepsilon + C\Phi(1-\rho) + \rho(\varphi + \alpha\vartheta) - u_0\|_{W^{1,2}(\Omega)} \\ &\leq \varepsilon\sqrt{|\Omega|} + C\,\|\nabla\Phi(1-\rho)\|_{L^2(\Omega)} \\ &\quad + C\,\|\Phi\nabla\rho\|_{L^2(\Omega)} + C\,\|\Phi(1-\rho)\|_{L^2(\Omega)} \\ &\leq \varepsilon\sqrt{|\Omega|} + C \sup|\nabla\Phi|\sqrt{\varepsilon} + 2CD_2\sqrt{\varepsilon} \\ &\quad + C\sup\Phi\sqrt{\varepsilon} + \varepsilon + \varepsilon + \alpha\,\|\vartheta\|_{W^{1,2}(\Omega)} \to 0 \end{aligned}$$

as $\varepsilon \searrow 0$, where we have used, once again, the facts that $\|\Phi\nabla\rho\|_{L^2(\Omega)} \leq 2D_2\sqrt{\varepsilon}$, $\|u_0\|_{W^{1,2}(\Omega\setminus K)} < \varepsilon$, and $\|u_0 - \varphi\|_{W^{1,2}(\Omega)} < \varepsilon$. In total, we obtain (2.4). Finally, given $K \subset\subset \Omega$, the estimate in (2.3) holds for $0 < \varepsilon < dist(K, \partial\Omega)$ and with the choice of C_K as in (B.2). ∎

Acknowledgments. N. I. Kavallaris thanks Professors V. G. Papanicolaou and A. N. Yannacopoulos for introducing him to the topic of infinite dimensional replicator dynamics. He is also grateful to the Institut für Mathematik, Universität Paderborn, for its hospitality and stimulating atmosphere during the preparation of part of this paper. The authors thank the anonymous referees, whose comments improved the manuscript.

REFERENCES

[1] S. B. Angenent and M. Fila, *Interior gradient blow-up in a semilinear parabolic equation*, Diferential Integral Equations, 9 (1996), pp. 865–877.

[2] D. G. Aronson, *The porous medium equation*, in Nonlinear Diffusion Problems, Lecture Notes in Math. 1224, Springer, Berlin, 1986, pp. 1–46.

[3] J. M. Arrieta Algarra, A. Rodríguez Bernal, and Ph. Souplet, *Boundedness of global solutions for nonlinear parabolic equations involving gradient blow-up phenomena*, Ann. Scu. Norm. Sup. Pisa Cl. Sci., 3 (2004), pp. 1–15.

[4] M. Bertsch, R. Dal Passo, and M. Ughi, *Discontinuous "viscosity" solutions of a degenerate parabolic equation*, Trans. Amer. Math. Soc., 320 (1990), pp. 779–798.

[5] M. Bertsch and L. A. Peletier, *A positivity property of solutions of nonlinear diffusion equations*, J. Differential Equations, 53 (1984), pp. 30–47.

[6] M. Bertsch and M. Ughi, *Positivity properties of viscosity solutions of a degenerate parabolic equation*, Nonlinear Anal. Theory Methods Appl., 14 (1990), pp. 571–592.

[7] I. Bomze, *Dynamical aspects of evolutionary stability*, Monatsh. Math., 110 (1990), pp. 189–206.

[8] T. Dlotko, *Examples of parabolic problems with blowing-up derivatives*, J. Math. Anal. Appl., 154 (1991), pp. 226–237.

[9] W. Deng, Z. Duan, and C. Xie, *The blow-up rate for a degenerate parabolic equation with a non-local source*, J. Math. Anal. Appl., 264 (2001), pp. 577–597.

[10] L. A. Dugatin and H. K. Reeve, eds., *Game Theory and Animal Behaviour*, Oxford Univeristy Press, Oxford, UK, 1998.

[11] L. Du and Z. Xiang, *A further blow-up analysis for a localized porous medium equation*, Appl. Math. Comput., 179 (2006), pp. 200–208.

[12] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[13] A. Friedman and B. McLeod, *Blow-up of positive solutions of semilinear heat equations*, Indiana Univ. Math. J., 34 (1985), pp. 425–447.

[14] A. Friedman and B. McLeod, *Blow-up of solutions of nonlinear degenerate parabolic equations*, Arch. Rational Mech. Anal., 96 (1987), pp. 55–80.

[15] V. A. Galaktionov and J. L. Vazquez, *Regional blow up in a semilinear heat equation with convergence to a Hamilton–Jacobi equation*, SIAM J. Math. Anal., 24 (1993), pp. 1254–1276, https://doi.org/10.1137/0524071.

[16] Y. Giga and R. V. Kohn, *Nondegeneracy of blowup for semilinear heat equations*, Comm. Pure Appl. Math., 42 (1989), pp. 845–884.

[17] P. Haccou and Y. Iwasa, *Optical mixed strategies in stochastic environments*, Theor. Population Biol., 47 (1995), pp. 212–243.

[18] P. Haccou and Y. Iwasa, *Robustness of optimal mixed strategies*, J. Math. Biol., 36 (1998), pp. 485–496.

[19] J. Hofbauer and K. Sigmund, *Evolutionary game dynamics*, Bull. Amer. Math. Soc., 40 (2003), pp. 479–519.

[20] A. Krause and C. S. Ong, *Contextual Gaussian process bandit optimization*, in Proceedings of the 24th International Conference on Neural Information Processing Systems (NIPS'11), Granada, Spain, pp. 2447–2455.

[21] D. Kravvaritis, V. G. Papanicolaou, and A. N. Yannacopoulos, *Similarity solutions for a replicator dynamics equation*, Ind. Univ. Math. J., 57 (2008), pp. 1929–1946.

[22] D. Kravvaritis, V. G. Papanicolaou, A. Xepapadeas, and A. N. Yannacopoulos, *On a class of operator equations arising in infinite dimensional replicator dynamics*, Nonlinear Anal. Real World Appl., 11 (2010), pp. 2537–2556.

[23] D. Kravvaritis and V. G. Papanicolaou, *Singular equilibrium solutions for a replicator dynamics model*, Electron. J. Differential Equations, 87 (2011), pp. 1–8.

[24] P. Krugman, *The Self-Organizing Economy*, Mitsui Lectures in Economics, Wiley-Blackwell, Cambridge, MA, 1996.

[25] A. A. Lacey, *Global blow-up of a nonlinear heat equation*, Proc. Roy. Soc. Edinburgh Sect. A, 104 (1986), pp. 161–167.

[26] O. A. Ladyzenskaja, V. A. Solonnikov, and N. N. Ural'ceva, *Linear and Quasi-linear Equations of Parabolic Type*, AMS, Providence, RI, 1968.
[27] J. Lankeit, *Equilibration of unit mass solutions to a degenerate parabolic equation with a nonlocal gradient nonlinearity*, Nonlinear Anal., 135 (2016), pp. 236–248.
[28] F. Liang and Y. Li, *Blow-up for a nonlocal parabolic equation*, Nonlinear Anal., 71 (2009), pp. 3551–3562.
[29] Y. Li and Ph. Souplet, *Single-point gradient blow-up on the boundary for diffusive Hamilton-Jacobi equations in planar domains*, Comm. Math. Phys., 293 (2010), pp. 499–517.
[30] Q. Liu, Y. Li, and H. Gao, *Uniform blow-up rate for a nonlocal degenerate parabolic equations*, Nonlinear Anal., 66 (2007), pp. 881–889.
[31] S. Luckhaus and R. Dal Passo, *A degenerate diffusion equation not in divergence form*, J. Differential Equations, 69 (1987), pp. 1–14.
[32] J. Marecki, *Playing in the Dark: Game Theory for Security Applications*, Technical report, IBM, 2011.
[33] J. Maynard Smith, *Evolution and The Theory of Games*, Cambridge University Press, Cambridge, MA, 1982.
[34] J. Oechssler and F. Riedel, *Evolutionary dynamics on infinite strategy spaces*, Economic Theory, 17 (2001), pp. 141–162.
[35] J. Oechssler and F. Riedel, *On the dynamic foundation of evolutionary stability in continuous models*, J. Economic Theory, 107 (2002), pp. 223–252.
[36] V. G. Papanicolaou and G. Smyrlis, *Similarity solutions for a multi-dimensional replicator dynamics equation*, Nonlinear Anal. Theory Methods Appl., 71 (2009), pp. 3185–3196.
[37] V. G. Papanicolaou and K. Vasilakopoulou, *Similarity solutions of a replicator dynamics equation associated to a continuum of pure strategies*, Electron. J. Differential Equations, 231 (2015), pp. 1–16.
[38] V. G. Papanicolaou and K. Vasilakopoulou, *Similarity solutions of a multidimensional replicator dynamics integrodifferential equation*, J. Dyn. Games, 3 (2016), pp. 51–74.
[39] P. Quittner and Ph. Souplet, *Superlinear Parabolic Problems. Blow-Up, Global Existence and Steady States*, Birkhäuser Advanced Texts, Birkhäuser, Basel, 2007.
[40] P. Schuster and K. Sigmund, *Replicator dynamics*, J. Theoret. Biol., 100 (1983), pp. 533–538.
[41] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mihailov, *Blow-Up in Quasilinear Parabolic Equations*, De Gruyter Exp. Math. 19, Walter De Gruyter & Co., Berlin, 1995.
[42] K. Sigmund, *Games of Life: Explorations in Ecology, Evolution, and Behaviour*, Oxford University Press, New York, 1993.
[43] Ph. Souplet, *Blow-up in nonlocal reaction-diffusion equations*, SIAM J. Math. Anal., 29 (1998), pp. 1301–1334, https://doi.org/10.1137/S0036141097318900.
[44] Ph. Souplet, *Uniform blow-up profiles and boundary behaviour for diffusion equations with nonlocal nonlinear source*, J. Differential Equations, 153 (1999), pp. 374–406.
[45] Ph. Souplet, *Gradient blow-up for multidimensional nonlinear parabolic equations with general boundary conditions*, Differential Integral Equations, 15 (2002), pp. 237–256.
[46] Ph. Souplet, *Uniform blow-up profile and boundary behaviour for a non-local reaction-diffusion equation with critical damping*, Math. Methods Appl. Sci., 27 (2004), pp. 1819–1829.
[47] C. Stinner and M. Winkler, *Boundedness vs. blow-up in a degenerate diffusion equation with gradient nonlinearity*, Indiana Univ. Math. J., 56 (2007), pp. 2233–2264.
[48] C. Stinner and M. Winkler, *Finite time vs. infinite time gradient blow-up in a degenerate diffusion equation*, Indiana Univ. Math. J., 57 (2008), pp. 2321–2354.
[49] P. D. Taylor and L. Jonker, *Evolutionary stable strategies and game dynamics*, Math. Biosci., 40 (1978), pp. 145–156.
[50] M. Wang and Y. Wang, *Properties of positive solutions for non-local reaction-diffusion problems*, Math. Methods Appl. Sci., 19 (1996), pp. 1141–1156.
[51] M. Wiegner, *A degenerate diffusion equation with a nonlinear source term*, Nonlinear Anal. Theory Methods Appl., 28 (1997), pp. 1977–1995.
[52] M. Winkler, *A critical exponent in a degenerate parabolic equation*, Math. Methods Appl. Sci., 25 (2002), pp. 911–925.
[53] M. Winkler, *Blow-up of solutions to a degenerate parabolic equation not in divergence form*, J. Differential Equations, 192 (2003), pp. 445–474.
[54] M. Winkler, *Boundary behaviour in strongly degenerate parabolic equations*, Acta Math. Univ. Comenian., 72 (2003), pp. 129–139.
[55] M. Winkler, *Propagation vs. constancy of support in the degenerate parabolic equation* $u_t = f(u)\Delta u$, Rend. Istit. Mat. Univ. Trieste, 36 (2004), pp. 1–15.
[56] J. Wloka, *Partial Differential Equations*, Cambridge University Press, Cambridge, UK, 1987.