Blow-up profiles and life beyond blow-up in the fully parabolic Keller-Segel system

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Abstract

The fully parabolic Keller-Segel system is considered in n-dimensional balls with $n \geq 2$. Pointwise time-independent estimates are derived for arbitrary radially symmetric solutions. These are firstly used to assert that any radial classical solution which blows up in finite time possesses a uniquely determined blow-up profile which satisfies an associated pointwise upper inequality. Secondly, in conjunction with additional regularity features implied by a very weak but temporally and spatially global quasi-entropy property, these estimates are seen to ensure global extensibility of any such solution within a suitable framework of renormalized solutions.

Key words: chemotaxis; blow-up profile; renormalized solution
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1 Introduction

Keller-Segel-type systems ([19]) have been fascinating mathematicians for more than 40 years now ([14], [13]). According to their original intention to serve as simple models for chemotaxis-driven processes of cell aggregation in biology, a considerable literature is devoted to the identification of conditions under which solutions may or may not reflect such types of phenomena in the extreme sense of blow-up. Here, a large variety of powerful analytic techniques for verifying appropriate dominance of diffusion, as thoroughly developed in numerous contexts of dissipative evolution systems over the past decades, has made it possible to derive large classes of assumptions, either on the initial data or on crucial system parameter functions, which rule out any explosions (see e.g. [29], [30], [5] and [37] or also [2] for a recent survey). On the other hand, by means of refined arguments, well-adapted to the respective particular structure of chemotactic cross-diffusion as the main destabilizing mechanism therein, the occurrence of blow-up could be detected in a considerable number of cases.

Specifically, in the Neumann initial-boundary value problem for the fully parabolic Keller-Segel system given by

$$\begin{cases} u_t \;=\; \Delta u - \nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ v_t \;=\; \Delta v - v + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$, $n\ge 2$, with outward normal vector field ν, some initial data leading to unbounded solutions, though possibly global in time, have been found to exist when $n=2$ and Ω is simply connected ([15]). Significantly more information is available upon restricting to radially symmetric solutions when Ω is a ball: In this context, namely, the sophisticated construction presented in the seminal work [12] shows the existence of some particular solutions which blow up already in finite time T, and the asymptotics of which for times $t\in(0,T)$ close to the blow-up time T can be described quite precisely. That such finite-time explosions in fact do occur not only for very specifically chosen initial constellations, but actually within large classes of radial data, has more recently been verified separately for the cases $n\ge 3$ ([41]) and $n=2$ ([26]). In comparison to this, similar results addressing parabolic-elliptic simplifications of (1.1), with the second equation therein being replaced by either $0=\Delta v - v + u$ or $0=\Delta v - \frac{1}{|\Omega|}\int_\Omega u_0 + u$, have been known for much a longer time ([18], [27], [4]). Related results on existence of explosions have been obtained for numerous variants of (1.1) accounting for various types of refinements in the relevant system ingredients mainly related to the cell migration and, in particular, the cross-diffusive interaction therein (see [7], [8], [21], [17], [40], [38], [3] and the surveys [14] and [2], for instance).

Beyond this, however, quite little seems known about the behavior of non-global solutions to (1.1) near their blow-up time. After all, the two-dimensional example reported in [12] admits a rather detailed description of a collapse into a Dirac-type singularity in that for the particular solution (u,v), as constructed there on $\Omega\times(0,T)$ for some $T>0$, one can find $\psi\in L^1(\Omega)$ such that in the sense of measures, $u(\cdot,t)\to 8\pi\delta+\psi$ as $t\nearrow T$, where δ denotes the Dirac distribution, and where with some $C>0$,

$$\psi(x) = \frac{C}{|x|^2} e^{-2\sqrt{\ln|x|}}(1+o(1)) \qquad \text{as } |x|\to 0; \tag{1.2}$$

moreover, this behavior appears to be stable at least with respect to certain suitably small perturbations ([33], cf. also [32]). Available information on blow-up asymptotics applicable to more general solutions seems to be restricted to statements on mass concentration at blow-up points in two-dimensional cases ([28]), to results on absence of equi-integrability properties in the general case $n \ge 2$ ([6], cf. also [34]), and to issues related to temporal blow-up rates ([25]). In particular, it even seems unknown whether at all for an arbitrary solution u ceasing to exist at time $T < \infty$ a blow-up profile $u(\cdot, T) := \lim_{t \nearrow T} u(\cdot, t)$ can adequately be defined, and, in the affirmative case, which common properties such profile functions share.

Apart from that, it seems widely unclear how far it is possible to extend exploding solutions to (1.1) beyond their blow-up time. In fact, precedent results in this direction concentrate on two-dimensional parabolic-elliptic simplifications thereof, and making substantial use of the circumstance that then the problem setting essentially reduces to that of a single scalar parabolic equation, extension beyond blow-up, albeit not necessarily in a unique manner, could be shown to be possible for large classes of initial data ([24], [10], cf. also [35], [1]). To the best of our knowledge, however, extensions beyond blow-up have neither been discussed anywhere in higher-dimensional settings, nor in the fully parabolic problem (1.1) for any $n \ge 2$.

The purpose of this paper is to address the latter two contexts of problems for the original system (1.1) in radially symmetric frameworks for arbitrary $n \ge 2$. Our first goal will consist in establishing pointwise upper bounds for solutions thereof, both global and non-global, which will in particular entail some quantitative information on the spatial behavior of exploding solutions near their blow-up time. Thereafter, global extensibility of arbitrary radial solutions will be discussed.

Main results: Pointwise estimates in singular drift-diffusion problems. Generalizing the sub-problem in (1.1) concerned with the evolution of the cell population density u, let us firstly consider solutions of

$$\begin{cases} u_t \le \Delta u + \nabla \cdot (f(x,t)u), & x \in \Omega, \ t \in (0,T), \\ \frac{\partial u}{\partial \nu} \le 0, & x \in \partial\Omega, \ t \in (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

for $T \in (0, \infty]$, where $f : \overline{\Omega} \times (0,T) \to \mathbb{R}^n$ and nonnegative initial data u_0 are given. With regard to effects of a possibly singular behavior of f near an isolated point inside Ω, refined arguments from parabolic regularity theory will reveal the following in Section 2.

Theorem 1.1 *Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary such that $0 \in \Omega$. Then if $\alpha > 0, \beta \ge 1$ and $q > n$ are such that*

$$\alpha > \frac{nq\beta}{q-n}, \tag{1.4}$$

for all $K > 0$ and $L > 0$ one can find $C(K, L, \alpha, \beta, q) > 0$ with the following property: If for some $T \in (0, \infty]$, a nonnegative function $u \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ satisfies (1.3) with some

$$f \in C^{1,0}(\overline{\Omega} \times (0,T); \mathbb{R}^n) \cap \bigcap_{\vartheta > n} L^\infty_{loc}([0,T); L^\vartheta(\Omega; \mathbb{R}^n)) \tag{1.5}$$

fulfilling

$$f(\cdot, t) \cdot \nu = 0 \quad \textit{on } \partial\Omega \qquad \textit{for all } t \in (0,T) \tag{1.6}$$

as well as

$$\int_\Omega |x|^{q\beta}|f(x,t)|^q dx \le K \qquad \textit{for all } t\in(0,T), \tag{1.7}$$

and with some nonnegative $u_0 \in C^0(\overline{\Omega})$ *such that*

$$\int_\Omega u_0 \le m \tag{1.8}$$

and

$$u_0(x) \le L|x|^{-\alpha} \qquad \textit{for all } x\in\Omega, \tag{1.9}$$

then

$$u(x,t) \le C(K,\alpha,\beta,q)|x|^{-\alpha} \qquad \textit{for all } x\in\Omega \textit{ and } t\in(0,T). \tag{1.10}$$

Main results II: Blow-up profiles and global extension of radial solutions to (1.1). In order to appropriately apply the above to exploding radial solutions to (1.1), let us recall the following results on maximal local classical solvability and on the occurrence of finite-time blow-up within a conveniently large set of radial initial data ([16], [41], [26]).

Proposition 1.2 *Let* $n\ge 2$, $R>0$ *and* $\Omega = B_R(0)\subset\mathbb{R}^n$.

i) Let $u_0 \in C^0(\overline{\Omega})$ *and* $v_0\in W^{1,\infty}(\Omega)$ *be nonnegative and radially symmetric. Then there exixt* $T_{max}\in(0,\infty]$ *and a uniquely determined classical solution* (u,v) *of (1.1) such that*

$$\begin{cases} u\in C^0(\overline{\Omega}\times[0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max})) \qquad \textit{and} \\ v\in \bigcap_{p>n} C^0([0,T_{max});W^{1,p}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max})), \end{cases}$$

that $u(\cdot,t)$ *and* $v(\cdot,t)$ *are nonnegative and radially symmetric in* Ω *for all* $t\in(0,T_{max})$, *and that*

$$\textit{if } T_{max}<\infty \qquad \textit{then} \qquad \limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{1.11}$$

ii) Given any radially symmetric positive functions $u_0\in C^0(\overline{\Omega})$ *and* $v_0\in W^{1,\infty}(\Omega)$, *for all* $p\in(0,\frac{2n}{n+2})$ *one can find sequences* $(u_{0k})_{k\in\mathbb{N}}\subset C^0(\overline{\Omega})$ *and* $(v_{0k})_{k\in\mathbb{N}}\subset W^{1,\infty}(\Omega)$ *of positive radial functions* u_{0k} *and* v_{0k} *with the properties that* $u_{0k}\to u_0$ *in* $L^p(\Omega)$ *and* $v_{0k}\to v_0$ *in* $W^{1,2}(\Omega)$ *as* $k\to\infty$, *and that for the corresponding solutions* (u_k,v_k) *of (1.1) emanating from* $(u_k,v_k)|_{t=0}=(u_{0k},v_{0k})$, *maximally extended up to* $T_{max,k}\in(0,\infty]$ *according to i), we actually have* $T_{max,k}<\infty$ *for all* $k\in\mathbb{N}$; *that is, each of these solutions blows up within finite time in the sense of (1.11).*

In this particular framework of radial solutions to (1.1), an estimate asserting (1.7) for $f := -\nabla v$ will be obtained in Section 3, at the core again relying on parabolic smoothing estimates, and again in a setting slightly more general than actually required (cf. (3.1) and Lemma 3.4). This will enable us to draw the following conclusion of Theorem 1.1 for (1.1) in Section 4.

Corollary 1.3 *Let* $\Omega = B_R(0)\subset\mathbb{R}^n$ *for some* $n\ge 2$ *and* $R>0$, *and let* $u_0\in C^0(\overline{\Omega})$ *and* $v_0\in W^{1,\infty}(\Omega)$ *be nonnegative and radially symmetric. Then given any* $\eta>0$, *one can find* $C(\eta)>0$ *such that with* $T_{max}\in(0,\infty]$ *as determined in Proposition 1.2, the solution of (1.1) satisfies*

$$u(x,t)\le C(\eta)\cdot|x|^{-n(n-1)-\eta} \qquad \textit{for all } x\in\Omega \textit{ and } t\in(0,T_{max}) \tag{1.12}$$

as well as

$$|\nabla v(\cdot,t)| \le C(\eta)\cdot|x|^{-(n-1)-\eta} \qquad \textit{for all } x\in\Omega \textit{ and } t\in(0,T_{max}). \tag{1.13}$$

In cases of non-global solutions, the latter can be combined with an additional argument on regularity with respect to the time variable, inter alia ruling out temporal oscillations, to infer in Section 4 that blow-up profiles are indeed always well-defined and can be estimated in the spirit of Corollary 1.3.

Corollary 1.4 *Let $\Omega = B_R(0)\subset\mathbb{R}^n$ with some $n\ge 2$ and $R>0$, and suppose that $u_0\in C^0(\overline{\Omega})$ and $v_0\in W^{1,\infty}(\Omega)$ are nonnegative and radially symmetric and such that the corresponding solution of (1.1) blows up in finite time; that is, that with T_{max} taken from Proposition 1.2 we have $T_{max}<\infty$. Then there exists a nonnegative radially symmetric function $U\in C^2(\overline{\Omega}\setminus\{0\})$ such that*

$$u(\cdot,t)\to U \quad \textit{in } C^2_{loc}(\overline{\Omega}\setminus\{0\}) \qquad \textit{as } t\nearrow T_{max}, \tag{1.14}$$

and that for each $\eta>0$ we can find $C(\eta)>0$ fulfilling

$$U(x)\le C(\eta)\cdot|x|^{-n(n-1)-\eta} \qquad \textit{for all } x\in\overline{\Omega}\setminus\{0\}. \tag{1.15}$$

We underline that in the spatially two-dimensional case, in view of the particular example of a blow-up mechanism expressed in (1.2) the upper estimate (1.15), asserting behavior not substantially stronger than of type $|x|^{-2}$ near the origin, is essentially optimal with respect to the exponent 2 therein. We do not know how far (1.15) continues to be accordingly sharp in higher-dimensional settings; after all, as a caveat we note that in view of well-known numerical evidence one may expect highly oscillatory spatial dynamics near blow-up, thus possibly enforcing corresponding profiles to exceed, at least along suitable sequences of points x, any multiple of $|x|^{-\alpha}$ for each α located within the range $\alpha\in(0,n)$ that is compatible with the evident mass conservation property associated with (1.1). Only in considerably simplifed settings involving radially decreasing solutions to a parabolic-elliptic variant of (1.1), such oscillations have essentially been ruled out by a recent finding, according to which (1.12), (1.14) and (1.15) even when the exponent $n(n-1)+\eta$ therein is replaced with the optimal number 2 for arbitrary $n\ge 3$ ([36]).

Our final result now asserts extensibility of any such non-global solution beyond its blow-up time in a framework which, by resembling standard concepts of renormalized solutions ([9]) in central aspects, is mild enough so as to avoid a breakdown of solvability also in cases when Dirac-type mass aggregation occurs. This will be achieved in Section 5 on the basis of a quasi-entropy property, as formally expressed in inequalities of the form

$$\frac{d}{dt}\int_\Omega (u+1)^{-p}e^{-\kappa v} + \frac{1}{C}\cdot\left\{\int_\Omega (u+1)^{-p-2}e^{-\kappa v}|\nabla u|^2 + \int_\Omega (u+1)^{-p}e^{-\kappa v}|\nabla v|^2\right\} \le C, \quad t>0, \tag{1.16}$$

with arbitrary $p>0$ and appropriately chosen and suitably large $\kappa=\kappa(p)>0$ and $C=C(p)>0$ (see Lemma 5.1 and especially (5.10)). Through estimates for solution gradients implied by accordingly obtained bounds on the dissipation rate therein, this structural feature (1.16), apparently yet undiscovered in the literature, provides some rather weak but after all spatially global regularity information which, when combined with local estimates outside the origin implied by the outcome of the above, turns out to be sufficient to derive the following.

Theorem 1.5 *Let $n \geq 2$ and $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $R > 0$, and suppose that $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are nonnegative and radially symmetric. Then there exists at least one pair $(\widehat{u}, \widehat{v})$ of nonnegative functions*

$$\begin{cases} \widehat{u} \in C^0((\overline{\Omega} \setminus \{0\}) \times [0,\infty)) \cap C^{2,1}((\overline{\Omega} \setminus \{0\}) \times (0,\infty)) \qquad \textit{and} \\ \widehat{v} \in C^0((\overline{\Omega} \setminus \{0\}) \times [0,\infty)) \cap C^{2,1}((\overline{\Omega} \setminus \{0\}) \times (0,\infty)) \end{cases} \tag{1.17}$$

such that $\widehat{u}(\cdot,t)$ and $\widehat{v}(\cdot,t)$ are radially symmetric for all $t > 0$, and that $(\widehat{u}, \widehat{v})$ is a global renormalized solution of (1.1) in the sense of Definition 5.1 below. Moreover, with $T_{max} \in (0,\infty]$, u and v taken from Proposition 1.2 we have $(\widehat{u}, \widehat{v}) \equiv (u,v)$ in $(\overline{\Omega} \setminus \{0\}) \times [0, T_{max})$.

Exploring how far the statement from Theorem 1.5 remains valid in more general geometric settings, constituting a natural but interesting potential next step, has to be left as an open problem here; in fact, through essentially relying on the analysis that underlies the crucial pointwise estimates in (1.12) and (1.12), our presently pursued approach toward global extensibility in (1.1) seems restricted to radial frameworks.

2 Singular drift-diffusion problems. Proof of Theorem 1.1

To begin with, in this section we shall address the framework specified in Theorem 1.1, that is, we shall consider (1.3) under the standing hypothesis that Ω, u and f satisfy the requirements formulated in Theorem 1.1. Our approach toward the estimate (1.10) for u will involve the use of a weight function sharing the behavior of $\overline{\Omega} \ni x \mapsto |x|^\alpha$ near the origin but leaving all quantities relevant to the boundary condition in (1.3) essentially unaffected. To achieve this, using that $0 \in \Omega$ and hence $\overline{B}_{2r_0}(0) \subset \Omega$ for some $r_0 \in (0,1)$, one can readily verify that it is possible to fix $\zeta \in C^0(\overline{\Omega}) \cap C^\infty(\overline{\Omega} \setminus \{0\})$ such that

$$0 < \zeta \leq 1 \text{ in } \overline{\Omega} \setminus \{0\} \quad \text{and} \quad \zeta(x) = |x| \text{ for all } x \in B_{r_0}(0) \quad \text{as well as} \quad \zeta \equiv 1 \text{ in } \overline{\Omega} \setminus B_{2r_0}(0). \tag{2.1}$$

Then for arbitrary $\alpha > 2$,

$$w(x,t) := \zeta^\alpha(x) u(x,t), \qquad x \in \overline{\Omega},\ t \in [0,T), \tag{2.2}$$

defines a nonnegative element of $C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ which is such that

$$\begin{aligned} &\nabla u = \zeta^{-\alpha} \nabla w - \alpha \zeta^{-\alpha-1} \nabla \zeta w \qquad \text{and} \\ &\Delta u = \zeta^{-\alpha} \Delta w - 2\alpha \zeta^{-\alpha-1} \nabla \zeta \cdot \nabla w + \alpha(\alpha+1) \zeta^{-\alpha-2} |\nabla \zeta|^2 w - \alpha \zeta^{-\alpha-1} \Delta \zeta w \end{aligned} \tag{2.3}$$

and hence

$$w_t = \zeta^\alpha u_t \leq \Delta w - 2\alpha \zeta^{-1} \nabla \zeta \cdot \nabla w + \alpha(\alpha+1)\zeta^{-2} |\nabla \zeta|^2 w - \alpha \zeta^{-1} \Delta \zeta w + \zeta^\alpha \nabla \cdot (\zeta^{-\alpha} f w)$$

in $\Omega \times (0,T)$. Since herein

$$\zeta^{-1} \nabla \zeta \cdot \nabla w = \nabla \cdot (\zeta^{-1} \nabla \zeta w) + \zeta^{-2} |\nabla \zeta|^2 w - \zeta^{-1} \Delta \zeta w$$

and

$$\zeta^\alpha \nabla \cdot (\zeta^{-\alpha} f w) = \nabla \cdot (f w) - \alpha \zeta^{-1} (\nabla \zeta \cdot f) w$$

in $\Omega \times (0,T)$, and since moreover the fact that $\zeta \equiv 1$ near $\partial\Omega$ warrants that $\frac{\partial w}{\partial \nu} = \frac{\partial u}{\partial \nu}$ on $\partial\Omega$ by (2.3), we thus infer that w satisfies

$$\begin{cases} w_t \le \Delta w - w + \nabla \cdot (a_1(x)w) + \nabla \cdot (f(x,t)w) + a_2(x)w + (a_3(x)\cdot f(x,t))w, & x \in \Omega,\ t \in (0,T), \\ \frac{\partial w}{\partial \nu} \le 0, & x \in \partial\Omega,\ t \in (0,T), \\ w(x,0) = \zeta^\alpha(x) u_0(x), & x \in \Omega, \end{cases} \tag{2.4}$$

where

$$\begin{aligned} & a_1(x) := -2\alpha\zeta^{-1}(x)\nabla\zeta(x), \\ & a_2(x) := \alpha(\alpha-1)\zeta^{-2}(x)|\nabla\zeta(x)|^2 + \alpha\zeta^{-1}(x)\Delta\zeta(x) + 1 \qquad \text{and} \\ & a_3(x) := -\alpha\zeta^{-1}(x)\nabla\zeta(x) \end{aligned} \tag{2.5}$$

for $x \in \Omega$. Let us first make sure that these functions can be controlled in the following sense.

Lemma 2.1 *Let $\alpha > 2$. Then there exists $C > 0$ such that*

$$\zeta(x) \ge \frac{|x|}{C} \qquad \textit{for all } x \in \Omega, \tag{2.6}$$

and such that the functions from (2.5) satisfy

$$|a_1(x)| \le \frac{C}{|x|} \qquad \textit{for all } x \in \Omega \tag{2.7}$$

and

$$|a_2(x)| \le \frac{C}{|x|^2} \qquad \textit{for all } x \in \Omega \tag{2.8}$$

as well as

$$|a_3(x)| \le \frac{C}{|x|} \qquad \textit{for all } x \in \Omega. \tag{2.9}$$

Proof. Since $\zeta(x) = |x|$ in $B_{r_0}(0)$ and $\zeta > 0$ in $\overline{\Omega} \times B_{r_0}(0)$ according to (2.1), (2.6) is an immediate consequence of the continuity of ζ. As moreover $\nabla\zeta$ and $\Delta\zeta$ are bounded in Ω, all three estimates in (2.7)-(2.9) result from (2.5) and (2.6) upon suitably enlarging C if necessary. $\square$

Now an application of parabolic regularity arguments to (2.4) will yield the claimed estimates for (1.3):

Proof of Theorem 1.1. Observing that (1.4) together with the assumption $\beta \ge 1$ implies that $\alpha > n$, we can choose $p_1 > 1$ such that

$$n < p_1 \le \alpha, \tag{2.10}$$

and making full use of (1.4), namely ensuring that

$$\frac{q\alpha}{\alpha + q\beta} = \frac{q}{1 + \frac{q\beta}{\alpha}} > \frac{q}{1 + \frac{q-n}{n}} = n,$$

we can fix $p_2 > 1$ fulfilling

$$n < p_2 \le \frac{q\alpha}{\alpha + q\beta}. \tag{2.11}$$

Again since $\alpha > n$, we can thereafter pick $p_3 > 1$ such that

$$\frac{n}{2} < p_3 \le \frac{\alpha}{2}, \tag{2.12}$$

and then take some $p_4 > 1$ satisfying

$$\frac{n}{2} < p_4 \le \frac{q\alpha}{\alpha + q\beta + q}, \tag{2.13}$$

noting that the latter is possible due to the fact that (1.4) along with the restrictions $q > n$ and $\beta \ge 1$ warrant that

$$\frac{q\alpha}{\alpha + q\beta + q} = \frac{q}{1 + \frac{q\beta + q}{\alpha}} > \frac{q}{1 + \frac{(q\beta+q)(q-n)}{nq\beta}} = \frac{q}{\frac{q}{n} + \frac{q-n}{n\beta}} > \frac{q}{\frac{q}{n} + \frac{q}{n}} = \frac{n}{2}.$$

We next make use of known smoothing properties of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge0}$ over Ω ([11, Lemma 3.3], [39, Lemma 1.3]) to find positive constants c_1, c_2, c_3 and c_4 such that for $i \in \{1,2\}$ and any $\tau > 0$ we have

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le c_i\cdot(1+\tau^{-\theta_i})\|\varphi\|_{L^{p_i}(\Omega)} \qquad \text{for all } \varphi \in C^1(\overline{\Omega};\mathbb{R}^n) \text{ such that } \varphi\cdot\nu = 0 \text{ on } \partial\Omega, \tag{2.14}$$

and that for $i \in \{3,4\}$ and $\tau > 0$,

$$\|e^{\tau\Delta}\varphi\|_{L^\infty(\Omega)} \le c_i \cdot (1 + \tau^{-\theta_i})\|\varphi\|_{L^{p_i}(\Omega)} \qquad \text{for all } \varphi \in C^0(\overline{\Omega}), \tag{2.15}$$

where

$$\theta_i := \frac{1}{2} + \frac{n}{2p_i} \quad \text{for } i \in \{1,2\} \qquad \text{and} \qquad \theta_i := \frac{n}{2p_i} \quad \text{for } i \in \{3,4\}$$

satisfy $\theta_i \in (0,1)$ for all $i \in \{1,2,3,4\}$ due to the left inequalities in (2.10)-(2.13).
As a final preliminary, we recall Lemma 2.1 to fix positive numbers c_5, c_6, c_7 and c_8 fulfilling

$$\zeta(x) \ge c_5|x| \qquad \text{for all } x \in \Omega \tag{2.16}$$

and

$$|a_1(x)| \le \frac{c_6}{|x|} \qquad \text{for all } x \in \Omega \tag{2.17}$$

as well as

$$|a_2(x)| \le \frac{c_7}{|x|^2} \qquad \text{for all } x \in \Omega \tag{2.18}$$

and

$$|a_3(x)| \le \frac{c_8}{|x|} \qquad \text{for all } x \in \Omega. \tag{2.19}$$

We now suppose that $0 \le u_0 \in C^0(\overline{\Omega})$ satisfies (1.8) and (1.9), and that for some $T \in (0,\infty]$, $0 \le u \in C^0(\overline{\Omega}\times[0,T)) \cap C^{2,1}(\overline{\Omega}\times(0,T))$ and $f : \overline{\Omega}\times(0,T) \to \mathbb{R}^n$ are such that (1.5), (1.6), (1.7) and (1.3) are valid, and note that then due to (1.3) and (1.6),

$$\frac{d}{dt}\int_\Omega u = \int_{\partial\Omega}\frac{\partial u}{\partial\nu} + \int_{\partial\Omega} u(f\cdot\nu) \le 0 \qquad \text{for all } t \in (0,T)$$

and hence

$$\int_\Omega u(\cdot,t) \le \int_\Omega u_0 \le m \qquad \text{for all } t\in(0,T). \tag{2.20}$$

Now relying on the comparison principle and a variation-of-constants representation, from (2.4) we obtain the pointwise inequality

$$\begin{aligned} w(\cdot,t) &\le e^{t(\Delta-1)}[\zeta^\alpha u_0] \\ &\quad + \int_0^t e^{(t-s)(\Delta-1)}\Big\{\nabla\cdot(a_1 w(\cdot,s)) + \nabla\cdot(f(\cdot,s)w(\cdot,s)) + a_2 w(\cdot,s) + (a_3\cdot f(\cdot,s))w(\cdot,s)\Big\}ds \end{aligned}$$

in Ω for all $t\in(0,T)$, which by nonnegativity of w and by (2.14) and (2.15) implies that for all $t\in(0,T)$,

$$\begin{aligned} \|w(\cdot,t)\|_{L^\infty(\Omega)} &\le e^{-t}\|e^{t\Delta}[\zeta^\alpha u_0]\|_{L^\infty(\Omega)} \\ &\quad + c_1\int_0^t \Big(1+(t-s)^{-\theta_1}\Big)e^{-(t-s)}\|a_1 w(\cdot,s)\|_{L^{p_1}(\Omega)}ds \\ &\quad + c_2\int_0^t \Big(1+(t-s)^{-\theta_2}\Big)e^{-(t-s)}\|f(\cdot,s) w(\cdot,s)\|_{L^{p_2}(\Omega)}ds \\ &\quad + c_3\int_0^t \Big(1+(t-s)^{-\theta_3}\Big)e^{-(t-s)}\|a_2 w(\cdot,s)\|_{L^{p_3}(\Omega)}ds \\ &\quad + c_4\int_0^t \Big(1+(t-s)^{-\theta_4}\Big)e^{-(t-s)}\|(a_3\cdot f(\cdot,s)) w(\cdot,s)\|_{L^{p_4}(\Omega)}ds. \end{aligned} \tag{2.21}$$

In order to develop this into an estimate for the finite numbers

$$M(T') := \sup_{t\in(0,T')}\|w(\cdot,t)\|_{L^\infty(\Omega)}, \qquad T'\in(0,T),$$

we first observe that again thanks to the comparison principle, with $L>0$ taken from (1.9) we have

$$e^{-t}\|e^{t\Delta}[\zeta^\alpha u_0]\|_{L^\infty(\Omega)} \le r_0^{-\alpha}L \qquad \text{for all } t>0, \tag{2.22}$$

because (2.1) along with our restriction that $r_0<1$ ensures that

$$\zeta(x) \le \frac{|x|}{r_0} \qquad \text{for all } x\in\Omega\setminus\{0\}, \tag{2.23}$$

and that hence $0\le\zeta^\alpha u_0 \le r_0^{-\alpha}L$ in Ω. Next, combining (2.17) with (2.20) and (2.23) we see that

$$\begin{aligned} \|a_1 w(\cdot,s)\|_{L^{p_1}(\Omega)}^{p_1} &\le c_6^{p_1}\int_\Omega |x|^{-p_1}w^{p_1}(x,s)dx \\ &\le c_6^{p_1}M^{p_1-1}(T')\int_\Omega |x|^{-p_1}w(x,s)dx \\ &= c_6^{p_1}M^{p_1-1}(T')\int_\Omega |x|^{-p_1}\zeta^\alpha(x)u(x,s)dx \\ &\le c_6^{p_1}r_0^{-\alpha}M^{p_1-1}(T')\int_\Omega |x|^{\alpha-p_1}u(x,s)dx \\ &\le c_9 M^{p_1-1}(T') \qquad \text{for all } s\in(0,T') \end{aligned} \tag{2.24}$$

with $c_9 := c_6^{p_1} r_0^{-\alpha} m \sup_{x\in\Omega} |x|^{\alpha-p_1}$ being finite due to the right inequality in (2.10).
Likewise, (2.18) along with (2.20) and (2.23) entails that

$$\begin{aligned}
\|a_2 w(\cdot,s)\|_{L^{p_3}(\Omega)}^{p_3} &\le c_7^{p_3} \int_\Omega |x|^{-2p_3} w^{p_3}(x,s)dx \\
&\le c_7^{p_3} r_0^{-\alpha} M^{p_3-1}(T') \int_\Omega |x|^{\alpha-2p_3} u(x,s)dx \\
&\le c_{10} M^{p_3-1}(T') \qquad \text{for all } s\in(0,T'),
\end{aligned} \tag{2.25}$$

where $c_{10} := c_7^{p_3} r_0^{-\alpha} m \sup_{x\in\Omega} |x|^{\alpha-2p_3} < \infty$ by (2.12).
In the second integrand on the right of (2.21), we employ the Hölder inequality to make use of (1.7) according to

$$\begin{aligned}
\|f(\cdot,s)w(\cdot,s)\|_{L^{p_2}(\Omega)}^{p_2} &= \int_\Omega |f(x,s)|^{p_2} w^{p_2}(x,s)dx \\
&= \int_\Omega \Big||x|^\beta f(x,s)\Big|^{p_2} \cdot |x|^{-p_2\beta} w^{p_2}(x,s)dx \\
&\le \Big\{\int_\Omega |x|^{q\beta}|f(x,s)|^q\Big\}^{\frac{p_2}{q}} \cdot \Big\{\int_\Omega |x|^{-\frac{p_2 q\beta}{q-p_2}} w^{\frac{p_2 q}{q-p_2}}(x,s)dx\Big\}^{\frac{q-p_2}{q}} \\
&\le K^{\frac{p_2}{q}} \cdot \Big\{\int_\Omega |x|^{-\frac{p_2 q\beta}{q-p_2}} w^{\frac{p_2 q}{q-p_2}}(x,s)dx\Big\}^{\frac{q-p_2}{q}} \qquad \text{for all } s\in(0,T).
\end{aligned}$$

Here we once more interpolate in the above flavor to obtain

$$\int_\Omega |x|^{-\frac{p_2 q\beta}{q-p_2}} w^{\frac{p_2 q}{q-p_2}}(x,s)dx \le r_0^{-\alpha} M^{\frac{p_2 q}{q-p_2}-1}(T') \int_\Omega |x|^{\alpha-\frac{p_2 q\beta}{q-p_2}} u(x,s)dx \qquad \text{for all } s\in(0,T'),$$

whence again by (2.20) we see that

$$\|f(\cdot,s)w(\cdot,s)\|_{L^{p_2}(\Omega)}^{p_2} \le c_{11} M^{p_2-\frac{q-p_2}{q}}(T') \qquad \text{for all } s\in(0,T'), \tag{2.26}$$

with finiteness of $c_{11} := r_0^{-\frac{(q-p_2)\alpha}{q}} K^{\frac{p_2}{q}} \Big\{\sup_{x\in\Omega} |x|^{\alpha-\frac{p_2 q\beta}{q-p_2}}\Big\}^{\frac{q-p_2}{q}}$ resulting from the circumstance that

$$\alpha - \frac{p_2 q\beta}{q-p_2} = \alpha - \frac{q\beta}{\frac{q}{p_2}-1} \ge \alpha - \frac{q\beta}{\frac{\alpha+q\beta}{\alpha}-1} = 0$$

by (2.11).
In quite a similar manner, the Hölder inequality in conjunction with (2.19), (1.7) and (2.20) ensures that

$$\begin{aligned}
\|(a_3\cdot f(\cdot,s))w(\cdot,s)\|_{L^{p_4}(\Omega)}^{p_4} &\le c_8^{p_4} \int_\Omega |x|^{-p_4}|f(x,s)|^{p_4} w^{p_4}(x,s)dx \\
&= c_8^{p_4} \int_\Omega \Big||x|^\beta f(x,s)\Big|^{p_4} \cdot |x|^{-p_4(\beta+1)} w^{p_4}(x,s)dx
\end{aligned}$$

$$
\begin{aligned}
&\le c_8^{p_4}\cdot\Big\{\int_\Omega |x|^{q\beta}|f(x,s)|^q\Big\}^{\frac{p_4}{q}}\cdot\Big\{\int_\Omega |x|^{-\frac{p_4q(\beta+1)}{q-p_4}} w^{\frac{p_4q}{q-p_4}}(x,s)dx\Big\}^{\frac{q-p_4}{q}}\\
&\le c_8^{p_4} r_0^{-\frac{(q-p_4)\alpha}{q}} K^{\frac{p_4}{q}} M^{p_4-\frac{q-p_4}{q}}(T')\cdot\Big\{\int_\Omega |x|^{\alpha-\frac{p_4q(\beta+1)}{q-p_4}} u(x,s)dx\Big\}^{\frac{q-p_4}{q}}\\
&\le c_{12}M^{p_4-\frac{q-p_4}{q}}(T') \qquad \text{for all } s\in(0,T') \qquad (2.27)
\end{aligned}
$$

if we let $c_{12} := c_8^{p_4} r_0^{-\frac{(q-p_4)\alpha}{q}} K^{\frac{p_4}{q}}\Big\{\sup_{x\in\Omega}|x|^{\alpha-\frac{p_4q(\beta+1)}{q-p_4}}\Big\}^{\frac{q-p_4}{q}}$, noting that $c_{12}<\infty$ thanks to the inequality

$$
\alpha-\frac{p_4q(\beta+1)}{q-p_4} = \alpha-\frac{q(\beta+1)}{\frac{q}{p_4}-1} \ge \alpha-\frac{q(\beta+1)}{\frac{\alpha+q\beta+q}{\alpha}-1} = 0
$$

guaranteed by (2.13).

In summary, from (2.22)-(2.27) we infer that (2.21) entails the estimate

$$
\begin{aligned}
\|w(\cdot,t)\|_{L^\infty(\Omega)} \le\ & r_0^{-\alpha}L\\
&+c_1c_9^{\frac{1}{p_1}}M^{1-\frac{1}{p_1}}(T')\int_0^t\Big(1+(t-s)^{-\theta_1}\Big)e^{-(t-s)}ds\\
&+c_2c_{11}^{\frac{1}{p_2}}M^{1-\frac{q-p_2}{p_2q}}(T')\int_0^t\Big(1+(t-s)^{-\theta_2}\Big)e^{-(t-s)}ds\\
&+c_3c_{10}^{\frac{1}{p_3}}M^{1-\frac{1}{p_3}}(T')\int_0^t\Big(1+(t-s)^{-\theta_3}\Big)e^{-(t-s)}ds\\
&+c_4c_{12}^{\frac{1}{p_4}}M^{1-\frac{q-p_4}{p_4q}}(T')\int_0^t\Big(1+(t-s)^{-\theta_4}\Big)e^{-(t-s)}ds \qquad \text{for all } t\in(0,T'),
\end{aligned}
$$

where for each $i\in\{1,2,3,4\}$,

$$
\int_0^t\Big(1+(t-s)^{-\theta_i}\Big)e^{-(t-s)}ds \le c_{13} := \max_{j\in\{1,2,3,4\}}\int_0^\infty(1+\sigma^{-\theta_j})e^{-\sigma}d\sigma \qquad \text{for all } t>0,
$$

with c_{13} being finite due to the above observation that $\theta_i<1$ for all $i\in\{1,2,3,4\}$. Abbreviating $c_{14} := \max\Big\{r_0^{-\alpha}L, c_1c_9^{\frac{1}{p_1}}c_{13}, c_2c_{11}^{\frac{1}{p_2}}c_{13}, c_3c_{10}^{\frac{1}{p_3}}c_{13}, c_4c_{12}^{\frac{1}{p_4}}c_{13}\Big\}$ and $\lambda := \max\{1-\frac{1}{p_1}, 1-\frac{q-p_2}{p_2q}, 1-\frac{1}{p_3}, 1-\frac{q-p_4}{p_4q}\}$, by Young's inequality we thereby readily obtain that

$$
M(T') \le c_{14}\cdot\Big\{4+M^\lambda(T')\Big\} \qquad \text{for all } T'\in(0,T)
$$

and that hence

$$
M(T') \le c_{15} := \max\Big\{4^{\frac{1}{\lambda}}\,,\ (2c_{14})^{\frac{1}{1-\lambda}}\Big\} \qquad \text{for all } T'\in(0,T)
$$

due to the evident fact that $\lambda\in(0,1)$. On taking $T'\nearrow T$ and making use of (2.16), we thus arrive at the estimate

$$
|x|^\alpha u(x,t) = \Big(\frac{|x|}{\zeta(x)}\Big)^\alpha w(x,t) \le c_5^{-\alpha}c_{15} \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T)
$$

and conclude as intended. $\square$

3 Estimating ∇v for radial solutions

In this section, again slightly generalizing part of (1.1), we will consider solutions $v \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ of

$$\begin{cases} v_t = \Delta v - v + g(x,t), & x \in \Omega,\ t \in (0,T), \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0,T), \\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

for $T \in (0,\infty]$, where a crucial additional assumption will be that Ω be a ball and that $v(\cdot,t)$ be radially symmetric with respect to the origin for all $t \in [0,T)$. Aiming at an application of Theorem 1.1 to (1.1), we shall particularly investigate quantitative effects on regularity of v, and especially on integrability features of ∇v, implied by the hypotheses that with some positive constant m, here playing a role slightly more general than exclusively denoting population sizes,

$$\begin{cases} g \in C^0(\overline{\Omega} \times [0,T)) \quad \text{is radial and nonnegative with} \quad \|g(\cdot,t)\|_{L^1(\Omega)} \le m \text{ for all } t \in (0,T), \qquad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \quad \text{is radial and nonnegative and such that} \quad \|v_0\|_{W^{1,\infty}(\Omega)} \le m, \end{cases} \tag{3.2}$$

Here and throughout the sequel, without further comment we shall switch to the radial notation in e.g. writing $v(r,t)$ instead of $v(x,t)$ whenever convenient.

Let us first recall two essentially well-known basic implications of the L^1 information concerning g, as contained in (3.2). The first, inter alia, provides an integral estimate for ∇v in some L^p space for suitably small p.

Lemma 3.1 *Let $p \in [1, \frac{n}{n-1})$. Then for all $m > 0$ there exists $C(p,m) > 0$ such that if for some $T \in (0,\infty]$, (3.1) is satisfied with some g and v_0 fulfilling (3.2), then*

$$\|v(\cdot,t)\|_{W^{1,p}(\Omega)} \le C(p,m) \qquad \textit{for all } t \in (0,T). \tag{3.3}$$

Proof. This can be seen by means of a standard argument based on well-known regularization features of the Neumann heat semigroup (cf. e.g. [16, Lemma 4.1] for details in a closely related setting). $\Box$

As seen in [41], the latter implies a pointwise bound for v itself:

Lemma 3.2 *For all $\beta > n-2$ and $m > 0$ there exists $C(\beta,m) > 0$ such that if $T \in (0,\infty]$ and (3.1) holds with some g and v_0 such that (3.2) is valid, we have*

$$v(x,t) \le C(\beta,m) \cdot |x|^{-\beta} \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T). \tag{3.4}$$

Proof. The claimed estimate can be derived in a straightforward manner from Lemma 3.1 by making use of the radial symmetry of v; details of an elementary argument can be found in [41, Lemma 3.2, Corollary 3.3], for instance. $\Box$

In order to prepare our derivation of some refined estimates for ∇v, similar to our procedure in Section 2 let us fix a nondecreasing function $\rho \in C^\infty([0,R])$ such that $\rho(r) = r$ for all $r \in [0, \frac{R}{2}]$ and $\rho_r(R) = 0$, whence for any $\beta > 1$, the function z defined by

$$z(r,t) := \rho^\beta(r) v(r,t), \qquad r \in [0,R],\ t \in [0,T), \tag{3.5}$$

belongs to $C^0([0,R]\times[0,T))\cap C^1([0,R]\times(0,T))\cap C^{2,1}((0,R)\times(0,T))$. Moreover, computing

$$z_r=\rho^\beta v_r+\beta\rho^{\beta-1}\rho_r v \qquad \text{and} \qquad z_{rr}=\rho^\beta v_{rr}+2\beta\rho^{\beta-1}\rho_r v_r+\beta(\beta-1)\rho^{\beta-2}\rho_r^2 v+\beta\rho^{\beta-1}\rho_{rr}v,$$

from (3.1) we see that

$$\begin{aligned}
z_t &= \rho^\beta v_{rr}+\frac{n-1}{r}\rho^\beta v_r-\rho^\beta v+\rho^\beta g \\
&= z_{rr}-2\beta\rho^{\beta-1}\rho_r v_r-\beta(\beta-1)\rho^{\beta-2}\rho_r^2 v-\beta\rho^{\beta-1}\rho_{rr}v+\frac{n-1}{r}\rho^\beta v_r-z+\rho^\beta g \quad \text{in } (0,R)\times(0,T).
\end{aligned}$$

As our restriction $\beta>1$ warrants that $z_r(0,t)=z_r(R,t)=0$ for all $t\in(0,T)$ due to the identities $v_r(0,t)=v_r(R,t)=0$ for all $t\in(0,T)$ and $\rho(0)=\rho_r(R)=0$, we accordingly obtain that for any such β, z is a solution of

$$\begin{cases}
z_t=z_{rr}-z+b_1(r)v_r+b_2(r)v+\rho^\beta g, & r\in(0,R),\ t\in(0,T),\\
z_r=0, & r\in\{0,R\},\ t\in(0,T),\\
z(r,0)=\rho^\beta(r)v_0(r) & r\in(0,R),
\end{cases} \tag{3.6}$$

with

$$\begin{cases}
b_1(r):=-2\beta\rho^{\beta-1}(r)\rho_r(r)+\frac{n-1}{r}\rho^\beta(r) \qquad \text{and}\\
b_2(r):=-\beta(\beta-1)\rho^{\beta-2}(r)\rho_r^2(r)-\beta\rho^{\beta-1}(r)\rho_{rr}(r)
\end{cases} \tag{3.7}$$

for $r\in(0,R)$.

As done in Lemma 2.1, let us first state some useful estimates for these coefficient functions.

Lemma 3.3 *Let $\beta>1$. Then there exists $C>0$ such that the functions b_1 and b_2 defined in (3.7) satisfy*

$$|b_1(r)|\le Cr^{\beta-1} \qquad \textit{for all } r\in(0,R) \tag{3.8}$$

and

$$|b_2(r)|\le Cr^{\beta-2} \qquad \textit{for all } r\in(0,R), \tag{3.9}$$

and that moreover

$$\frac{1}{C}\cdot r\le\rho(r)\le C\cdot r \qquad \textit{for all } r\in(0,R). \tag{3.10}$$

PROOF. As ρ is smooth and nondecreasing on $[0,R]$ with $\frac{\rho(r)}{r}=1$ for all $r\in(0,\frac{R}{2})$, we can find positive constants c_1,c_2,c_3 and c_4 such that

$$c_1\le\frac{\rho(r)}{r}\le c_2, \quad |\rho_r(r)|\le c_3 \quad \text{and} \quad \rho(r)|\rho_{rr}(r)|\le c_4 \qquad \text{for all } r\in(0,R).$$

Therefore, (3.10) becomes obvious, whereas (3.7) implies that

$$\begin{aligned}
|r^{1-\beta}b_1(r)| &\le 2\beta\cdot\Big(\frac{\rho(r)}{r}\Big)^{\beta-1}\cdot|\rho_r(r)|+(n-1)\cdot\Big(\frac{\rho(r)}{r}\Big)^\beta \\
&\le 2\beta\cdot c_2^{\beta-1}c_3+(n-1)c_2^\beta \qquad \text{for all } r\in(0,R)
\end{aligned}$$

and that

$$
\begin{aligned}
|r^{2-\beta}b_2(r)| &\le \beta(\beta-1)\cdot\Big(\frac{\rho(r)}{r}\Big)^{\beta-2}\cdot\rho_r^2(r)+\beta\cdot\Big(\frac{\rho(r)}{r}\Big)^{\beta-2}\cdot\rho(r)|\rho_{rr}(r)| \\
&\le \beta(\beta-1)\cdot\max\{c_1^{\beta-2},c_2^{\beta-2}\}\cdot c_3^2+\beta\cdot\max\{c_1^{\beta-2},c_2^{\beta-2}\}\cdot c_4 \qquad \text{for all } r\in(0,R)
\end{aligned}
$$

in both cases $\beta<2$ and $\beta\ge 2$. $\square$

Now the main results of this section indeed provide two types of estimates for ∇v in (3.1). Having in mind our application thereof in Lemma 4.2 below, in deriving the first of these we will merely rely on (3.2), and hence exclusively make use of the L^1 bound on g therein, in order to derive some weighted integral bound for ∇v in L^q spaces with arbitrarily high $q<\infty$. Part ii) of the following lemma will thereafter be used to derive even some weighted L^∞ bound for the signal gradient by making use of certain improved knowledge on u gained through combining i) with Theorem 1.1 for suitably large q.

Our argument in the following is again based on smoothing estimates for heat semigroups, but unlike in Theorem 1.1 this time in spatially one-dimensional intervals.

Lemma 3.4 *Let $\beta>n-1$ and $m>0$.*

i) For all $q\in(1,\infty)$ there exists $C(\beta,q,m)>0$ with the property that whenever $T\in(0,\infty]$ and (3.1) holds with some radial $v\in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$ and some g and v_0 such that (3.2) is valid, we have

$$
\int_\Omega |x|^{q\beta}|\nabla v(x,t)|^q dx \le C(\beta,q,m) \qquad \text{for all } t\in(0,T). \tag{3.11}
$$

ii) For any choice of $\gamma>\beta$ and $K>0$ one can find $(\beta,\gamma,m,K)>0$ such that if (3.1) holds with some $T\in(0,\infty]$, some radial $v\in C^0(\overline{\Omega}\times[0,T))\cap C^{2,1}(\overline{\Omega}\times(0,T))$ and some g and v_0 satisfying (3.2) as well as

$$
g(x,t)\le K|x|^{-\gamma} \qquad \text{for all } x\in\Omega \text{ and each } t\in(0,T), \tag{3.12}
$$

then

$$
|\nabla v(x,t)|\le C(\beta,\gamma,m,K)|x|^{-\beta} \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T). \tag{3.13}
$$

Proof. In order to prepare a simultaneous analysis of both the case i) in which $q<\infty$ is fixed, and of the situation addressed in ii) in which we write $q:=\infty$, let us first observe that since $\beta>n-1$, we can pick $p_0=p_0(\beta)\in(1,q)$ and $\kappa=\kappa(\beta)>0$ in such a way that

$$
\frac{n}{\beta}<p_0<\frac{n}{n-1} \qquad \text{and} \qquad n-2<\kappa<\beta-1. \tag{3.14}
$$

Since thus $p_0\beta>n$ and hence

$$
\frac{p_0(\beta-1)-(n-1)}{p_0-1}>-1,
$$

and since (3.14) moreover ensures that $\beta-\kappa-2>-1$, these choices enable us to pick numbers $p_1\in(1,p_0)$ and $p_2\in(1,q)$ sufficiently close to 1 such that still

$$
\frac{[p_0(\beta-1)-(n-1)]\cdot p_1}{p_0-p_1}>-1 \tag{3.15}
$$

and

$$p_2(\beta - \kappa - 2) > -1. \tag{3.16}$$

Next, in the situation of i) we let

$$p_3 \equiv p_3(q) := 1 \qquad \text{if } q \in (1,\infty), \tag{3.17}$$

while in the case considered in ii), with $\gamma > \beta$ taken from (3.12) we set

$$p_3 \equiv p_3(\infty) := \frac{\gamma + 1 - n}{\gamma - \beta}, \tag{3.18}$$

observing that the hypotheses $\gamma > \beta > n-1$ assert that then $p_3 > 1$.
Now in both cases $q \in (1,\infty)$ and $q = \infty$, by well-known smoothing properties of the one-dimensional heat semigroup $(e^{-\tau A})_{\tau \ge 0}$ generated by $A := -(\cdot)_{rr}$ under homogeneous Neumann boundary conditions on $(0,R)$ ([39, Lemma 1.3]), we can find positive constants c_1 and $c_{2,i}$, $i \in \{1,2,3\}$, such that for all $\tau > 0$,

$$\|\partial_r e^{-\tau A}\varphi\|_{L^q((0,R))} \le c_1 \|\varphi\|_{W^{1,\infty}((0,R))} \qquad \text{for all } \varphi \in W^{1,\infty}((0,R)) \tag{3.19}$$

and

$$\|\partial_r e^{-\tau A}\varphi\|_{L^q((0,R))} \le c_{2,i}(1 + \tau^{-\theta_i}) \|\varphi\|_{L^{p_i}((0,R))} \qquad \text{for all } \varphi \in C^0([0,R]), \tag{3.20}$$

where $\theta_i := \frac{1}{2} + \frac{1}{2}(\frac{1}{p_i} - \frac{1}{q})$ satisfies $\theta_i \in (\frac{1}{2}, 1)$ for all $i \in \{1,2,3\}$, since $p_i < q$, since clearly $\frac{1}{p_i} - \frac{1}{q} < \frac{1}{p_i} \le 1$ if $q < \infty$, and since (3.18) ensures that when $q = \infty$, we have $\frac{1}{p_i} - \frac{1}{q} = \frac{1}{p_i} < 1$.

We now suppose that $T \in (0,\infty]$ and that $v \in C^0(\overline{\Omega} \times [0,T)) \cap C^{2,1}(\overline{\Omega} \times (0,T))$ is radial and solves (3.1) with certain functions g and v_0 which satisfy (3.2) for some $m > 0$, and which in the case $q = \infty$ moreover comply with (3.12) with some $K > 0$.
Then due to the inequalities $p_0 < \frac{n}{n-1}$ and $\gamma > n-2$ asserted by (3.14), Lemma 3.1 and Lemma 3.2 become applicable so as to yield $c_3 = c_3(m) > 0$ and $c_4 = c_4(m) > 0$ fulfilling

$$\int_0^R r^{n-1} |v_r(r,t)|^{p_0} dr \le c_3 \qquad \text{for all } t \in (0,T) \tag{3.21}$$

and

$$v(r,t) \le c_4 r^{-\kappa} \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T). \tag{3.22}$$

In order to make appropriate use thereof, let us finally invoke Lemma 3.3 to fix positive constants c_5, c_6, c_7 and c_8 such that

$$c_5 r \le \rho(r) \le c_6 r \qquad \text{for all } r \in (0,R) \tag{3.23}$$

and

$$|b_1(r)| \le c_7 r^{\beta-1} \qquad \text{for all } r \in (0,R) \tag{3.24}$$

as well as

$$|b_2(r)| \le c_8 r^{\beta-2} \qquad \text{for all } r \in (0,R). \tag{3.25}$$

Then since by means of a Duhamel formula associated with (3.13) we can represent the function z defined in (3.5) according to

$$z(\cdot,t)=e^{-t(A+1)}[\rho^\beta v_0]+\int_0^t e^{-(t-s)(A+1)}\Big\{b_1 v_r(\cdot,s)+b_2 v(\cdot,s)+\rho^\beta g(\cdot,s)\Big\}ds, \qquad t\in(0,T),$$

from (3.19) and (3.20) we obtain that

$$\begin{aligned}\|z_r(\cdot,t)\|_{L^q((0,R))} \ &\le\ c_1 e^{-t}\|\rho^\beta v_0\|_{W^{1,\infty}((0,R))}\\ &\quad +c_{2,1}\int_0^t \Big(1+(t-s)^{-\theta_1}\Big)e^{-(t-s)}\|b_1 v_r(\cdot,s)\|_{L^{p_1}((0,R))}ds\\ &\quad +c_{2,2}\int_0^t \Big(1+(t-s)^{-\theta_2}\Big)e^{-(t-s)}\|b_1 v(\cdot,s)\|_{L^{p_2}((0,R))}ds\\ &\quad +c_{2,3}\int_0^t \Big(1+(t-s)^{-\theta_3}\Big)e^{-(t-s)}\|\rho^\beta g(\cdot,s)\|_{L^{p_3}((0,R))}ds \quad \text{for all } t\in(0,T). \qquad (3.26)\end{aligned}$$

Here due to (3.24), the Hölder inequality and (3.21),

$$\begin{aligned}\|b_1 v_r(\cdot,s)\|^{p_1}_{L^{p_1}((0,R))} \ &\le\ c_7^{p_1}\int_0^R r^{p_1(\beta-1)}|v_r(r,s)|^{p_1}dr\\ &=\ c_7^{p_1}\int_0^R \Big\{r^{n-1}|v_r(r,s)|^{p_0}\Big\}^{\frac{p_1}{p_0}}\cdot r^{\frac{[p_0(\beta-1)-(n-1)]p_1}{p_0}}dr\\ &\le\ c_7^{p_1}\cdot\Big\{\int_0^R r^{n-1}|v_r(r,s)|^{p_0}dr\Big\}^{\frac{p_1}{p_0}}\cdot\Big\{\int_0^R r^{\frac{[p_0(\beta-1)-(n-1)]p_1}{p_0-p_1}}dr\Big\}^{\frac{p_1-p_0}{p_0}}\\ &\le\ c_9:=c_3^{\frac{p_1}{p_0}}c_7^{p_1}\cdot\Big\{\int_0^R r^{\frac{[p_0(\beta-1)-(n-1)]p_1}{p_0-p_1}}dr\Big\}^{\frac{p_1-p_0}{p_0}} \qquad (3.27)\end{aligned}$$

for all $s\in(0,T)$, with c_9 being finite thanks to (3.15).
Next, combining (3.25) with (3.22) shows that

$$\begin{aligned}\|b_2 v(\cdot,s)\|^{p_2}_{L^{p_2}((0,R))} \ &\le\ c_8^{p_2}\int_0^R r^{p_2(\beta-2)}v^{p_2}(r,s)dr\\ &\le\ c_{10}:=c_4^{p_2}c_8^{p_2}\int_0^R r^{p_2(\beta-\kappa-2)}dr \qquad \text{for all } s\in(0,T), \qquad (3.28)\end{aligned}$$

where (3.16) warrants finiteness of c_{10}.
Finally, in the context of i) when $p_3=1$ by (3.17), we rely on (3.2) as well as on the right inequality in (3.23) and the fact that $\beta>n-1$ to see that

$$\begin{aligned}\|\rho^\beta g(\cdot,s)\|^{p_3}_{L^{p_3}((0,R))} \ &=\ \int_0^R \rho^\beta(r)g(r,s)dr\\ &\le\ c_6^\beta\int_0^R r^\beta g(r,s)dr\\ &\le\ c_6^\beta R^{\beta-(n-1)}\int_0^R r^{n-1}g(r,s)dr\\ &\le\ c_{11}:=c_6^\beta R^{\beta-(n-1)}\cdot\frac{m}{n|B_1(0)|} \qquad \text{for all } s\in(0,T); \qquad (3.29)\end{aligned}$$

in the setting of ii), with p_3 as given by (3.18) we additionally make use of (3.12), which in conjunction with (3.2) enables us to estimate

$$\begin{aligned}
\|\rho^\beta g(\cdot,s)\|_{L^{p_3}((0,R))}^{p_3} &\le c_6^{p_3\beta}\int_0^R r^{p_3\beta}g^{p_3}(r,s)dr \\
&\le c_6^{p_3\beta}K^{p_3-1}\int_0^R r^{p_3\beta-(p_3-1)\beta}g(r,s)dr \\
&= c_3^{p_3\beta}K^{p_3-1}\int_0^R r^{n-1}g(r,s)dr \\
&\le c_{12} := c_3^{p_3\beta}K^{p_3-1}\cdot\frac{m}{n|B_1(0)|} \qquad \text{for all } s\in(0,T). \qquad (3.30)
\end{aligned}$$

Collecting (3.27)-(3.30) and recalling (3.2), we thus infer that with $c_{13} := \|\rho\|_{L^\infty((0,R))}^\beta\cdot m$ and $c_{14} := (\max\{c_{11},c_{12}\})^{\frac{1}{p_3}}$, in both cases addressed in i) and ii) we have

$$\begin{aligned}
\|z_r(\cdot,t)\|_{L^q((0,R))} &\le c_1c_{13}+c_{2,1}c_9^{\frac{1}{p_1}}\int_0^t\Big(1+(t-s)^{-\theta_1}\Big)e^{-(t-s)}ds \\
&\quad +c_{2,2}c_{10}^{\frac{1}{p_2}}\int_0^t\Big(1+(t-s)^{-\theta_2}\Big)e^{-(t-s)}ds+c_{2,3}c_{14}\int_0^t\Big(1+(t-s)^{-\theta_3}\Big)e^{-(t-s)}ds
\end{aligned}$$

for all $t\in(0,T)$. As for all $i\in\{1,2,3\}$ we know that $c_{15,i} := \int_0^\infty(1+\sigma^{-\theta_i})e^{-\sigma}d\sigma$ is finite due to the inclusion $\theta_i\in(0,1)$, this entails that

$$\|z_r(\cdot,t)\|_{L^q((0,R))}\le c_{16} := c_1c_{13}+c_{2,1}c_9^{\frac{1}{p_1}}c_{15,1}+c_{2,2}c_{10}^{\frac{1}{p_2}}c_{15,2}+c_{2,3}c_{14}c_{15,3} \qquad \text{for all } t\in(0,T). \quad (3.31)$$

In order to see that this implies the claimed ineuqualities (3.11) and (3.13), we recall our definition of z to obtain using the left inequality in (3.23) and again (3.22) that

$$\begin{aligned}
|v_r(r,t)| &= \Big|\rho^{-\beta}(r)z(r,t)-\beta\frac{\rho_r(r)}{\rho(r)}v(r,t)\Big| \\
&\le c_5^{-\beta}r^{-\beta}|z_r(r,t)|+\frac{\beta c_4}{c_5}\|\rho_r\|_{L^\infty((0,R))}r^{-\kappa-1} \\
&\le c_5^{-\beta}r^{-\beta}|z_r(r,t)|+c_{17}r^{-\beta} \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T) \qquad (3.32)
\end{aligned}$$

with $c_{17} := \frac{\beta c_4}{c_5}\|\rho_r\|_{L^\infty((0,R))}R^{\beta-\kappa-1}$, because $\kappa<\beta-1$ by (3.14). Therefore, (3.31) immediately yields the statement from ii), while in the situation from i) when q is finite, we combine (3.32) with (3.31) and Young's inequality to estimate

$$\begin{aligned}
\int_0^R r^{n-1+q\beta}|v_r(r,t)|^q dr &\le 2^{q-1}\cdot\Big\{\int_0^R r^{n-1+q\beta}\cdot\Big(c_5^{-\beta}r^{-\beta}|z_r(r,t)|\Big)^q dr+\int_0^R r^{n-1+q\beta}\cdot(c_{17}r^{-\beta})^q dr\Big\} \\
&= 2^{q-1}c_5^{-q\beta}\int_0^R r^{n-1}|z_r(r,t)|^q dr+2^{q-1}c_{17}^q\int_0^R r^{n-1}dr \\
&\le 2^{q-1}c_5^{-q\beta}c_{16}^qR^{n-1}+\frac{2^{q-1}c_{17}^qR^n}{n} \qquad \text{for all } t\in(0,T),
\end{aligned}$$

and conclude as intended also in this case. □

4 Pointwise estimates and regularity in Keller-Segel-type systems

Now in order to make the above applicable to the original Keller-Segel system (1.1), simultaneously with regard to both the derivation of the pointwise properties of classical solutions from Corollary 1.3 and Corollary 1.4, as well as to the construction of global renormalized solutions in Theorem 1.5, we shall introduce a convenient regularization of (1.1) which actually coincides with the latter as long as the first solution component remains below a large number actually diverging through the intended limit process. More precisely, let us fix a nonincreasing $\chi \in C^\infty(\mathbb{R})$ such that $\chi \equiv 1$ on $(-\infty, 0]$ and $\chi \equiv 0$ throughout $[1, \infty)$, and let

$$G_\varepsilon(\xi) := \int_0^\xi \chi\Big(\sigma - \frac{1}{\varepsilon}\Big) d\sigma, \qquad \xi \geq 0, \tag{4.1}$$

for $\varepsilon \in (0,1)$. Then for any such ε, G_ε is a nondecreasing function belonging to $C^\infty([0,\infty))$ which satisfies

$$G_\varepsilon(\xi) = \xi \quad \text{for all } \xi \in \Big[0, \frac{1}{\varepsilon}\Big] \qquad \text{as well as} \qquad G_\varepsilon(\xi) \leq 1 + \frac{1}{\varepsilon} \quad \text{for all } \xi \geq 0. \tag{4.2}$$

Keeping this family $(G_\varepsilon)_{\varepsilon \in (0,1)}$ fixed henceforth, for $\varepsilon \in (0,1)$ we shall subsequently consider the approximate variants of (1.1) given by

$$\begin{cases} u_{\varepsilon t} &= \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon \nabla v_\varepsilon), & x \in \Omega, \ t > 0, \\ v_{\varepsilon t} &= \Delta v_\varepsilon - v_\varepsilon + G_\varepsilon(u_\varepsilon), & x \in \Omega, \ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} &= \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u_\varepsilon(x,0) &= u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{4.3}$$

which due to our choice of G_ε can readily be seen to have the following basic properties.

Lemma 4.1 *Let $\varepsilon \in (0,1)$. Then (4.3) possesses a global classical solution $(u_\varepsilon, v_\varepsilon)$ with*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \text{and} \\ v_\varepsilon \in \bigcap_{p>n} C^0([0,\infty); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \end{cases} \tag{4.4}$$

such that both u_ε and v_ε are nonnegative and radially symmetric. Moreover, if $T_{max} \in (0,\infty]$ and (u,v) are as in Proposition 1.2, then

$$u_\varepsilon \equiv u \quad \text{and} \quad v_\varepsilon \equiv v \qquad \text{in } \overline{\Omega} \times [0, T_\varepsilon], \tag{4.5}$$

where $T_\varepsilon \in (0,\infty]$ is given by

$$T_\varepsilon := \sup \Big\{ T \in (0, T_{max}) \ \Big| \ u \leq \frac{1}{\varepsilon} \text{ in } \Omega \times (0,T) \Big\}. \tag{4.6}$$

Proof. A straightforward contraction mapping argument ([16]) yields $T_{max,\varepsilon} \in (0,\infty]$ and a classical solution $(u_\varepsilon, v_\varepsilon)$ of (4.3) in $\Omega \times (0, T_{max,\varepsilon})$, uniquely determined by the inclusions

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega} \times [0, T_{max,\varepsilon})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon})), \\ v_\varepsilon \in \bigcap_{p>n} C^0([0, T_{max,\varepsilon}); W^{1,p}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon})), \end{cases}$$

such that either $T_{max,\varepsilon} = \infty$ or $\limsup_{t\nearrow T_{max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. According to the latter uniqueness property, a standard reasoning implies radial symmetry of $u_\varepsilon(\cdot,t)$ and $v_\varepsilon(\cdot,t)$ for all $t \in (0, T_{max,\varepsilon})$, and since $0 \le G_\varepsilon \le 1 + \frac{1}{\varepsilon}$ on $[0,\infty)$ by (4.2), it follows e.g. from [23] that actually $T_{max,\varepsilon} = \infty$. The identities in (4.5) again result from a uniqueness argument. $\square$

Now on the basis of two applications of Lemma 3.4, Theorem 1.1 applies so as to yield the following.

Lemma 4.2 *Let $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $n \ge 2$ and $R > 0$, and suppose that $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are nonnegative and radially symmetric. Then for all $\eta > 0$ one can find $C(\eta) > 0$ such that whenever $\varepsilon \in (0,1)$, the solution of (4.3) satisfies*

$$u_\varepsilon(x,t) \le C(\eta) \cdot |x|^{-n(n-1)-\eta} \qquad \textit{for all } x \in \Omega \textit{ and } t > 0 \tag{4.7}$$

as well as

$$|\nabla v_\varepsilon(\cdot,t)| \le C(\eta) \cdot |x|^{-(n-1)-\eta} \qquad \textit{for all } x \in \Omega \textit{ and } t > 0. \tag{4.8}$$

Proof. Given $\eta > 0$, we let $\alpha := n(n-1) + \eta > 2$ and $\beta := n - 1 + \eta > 1$, and observe that since $\alpha > n(n-1)$ it is possible to pick $\beta_0 > n-1$ such that $\alpha > n\beta_0$. We can therefore find $q_0 \in (n,\infty)$ fulfilling $q_0 > \frac{n\alpha}{\alpha - n\beta_0}$, which is equivalent to the inequality

$$\alpha > \frac{nq_0\beta}{q_0 - n}. \tag{4.9}$$

Now if $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are radial and nonnegative, then from (4.3) it is evident that for all $\varepsilon \in (0,1)$ we have $\frac{d}{dt}\int_\Omega u_\varepsilon = 0$ for all $t > 0$, so that

$$\int_\Omega G_\varepsilon(u_\varepsilon) \le \int_\Omega u_\varepsilon \le \int_\Omega u_0 \qquad \text{for all } t > 0. \tag{4.10}$$

As $\beta_0 > n-1$, we may therefore employ Lemma 3.4 i) to obtain $c_1 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\int_\Omega |x|^{-q_0\beta_0} |\nabla v_\varepsilon(x,t)|^{q_0} dx \le c_1 \qquad \text{for all } t > 0,$$

which in view of (4.9) and the inequalities $\beta_0 \ge 1$ and $q_0 > n$ thereafter enables us to invoke Theorem 1.1, thereby concluding that with some $c_2 > 0$ we have

$$u_\varepsilon(x,t) \le c_2|x|^{-\alpha} \qquad \text{for all } x \in \Omega \text{ and } t > 0 \tag{4.11}$$

and any $\varepsilon \in (0,1)$. Combined with (4.10) and the observation that $\beta = n - 1 + \eta < n(n-1) + \eta = \alpha$, this allows for a second application of Lemma 3.4, which through its part ii) namely guarantees that thanks to (4.10) and (4.11) we can find $c_3 > 0$ such that for all $\varepsilon \in (0,1)$,

$$|\nabla v_\varepsilon(x,t)| \le c_3|x|^{-\beta} \qquad \text{for all } x \in \Omega \text{ and any } t > 0. \tag{4.12}$$

Again by definition of α and β, the claimed estimates are precisely asserted by (4.11) and (4.12). $\square$

Again through parabolic regularity theory, this time addressing genuinely localizations by means of suitable cut-off procedures, the latter implies estimates involving higher order norms.

Lemma 4.3 *For all $\delta \in (0,R)$ and $\tau > 0$ one can find $C(\delta,\tau) > 0$ and $\theta = \theta(\delta,\tau) \in (0,1)$ such that for all $\varepsilon \in (0,1)$,*

$$\|u_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}((\overline{\Omega}\setminus B_\delta(0))\times[t,t+1])} \le C \qquad \text{for all } t > \tau \tag{4.13}$$

and

$$\|v_\varepsilon\|_{C^{2+\theta,1+\frac{\theta}{2}}((\overline{\Omega}\setminus B_\delta(0))\times[t,t+1])} \le C \qquad \text{for all } t > \tau. \tag{4.14}$$

Proof. We first note that whenever $\zeta \in C^\infty(\overline{\Omega} \times [0,\infty))$ is such that $\frac{\partial\zeta}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$, then for each $\varepsilon \in (0,1)$,

$$w_\varepsilon := \zeta(x,t)u_\varepsilon(x,t) \quad \text{and} \quad z_\varepsilon(x,t) := \zeta(x,t)v_\varepsilon(x,t), \qquad x \in \overline{\Omega},\ t \ge 0, \tag{4.15}$$

satisfy $\frac{\partial w_\varepsilon}{\partial\nu} = \frac{\partial z_\varepsilon}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$ as well as

$$w_{\varepsilon t} = \Delta w_\varepsilon + \nabla \cdot a_\varepsilon(x,t) + b_\varepsilon(x,t) \qquad \text{in } \Omega \times (0,\infty) \tag{4.16}$$

and

$$z_{\varepsilon t} = \Delta z_\varepsilon + h_\varepsilon(x,t) \qquad \text{in } \Omega \times (0,\infty), \tag{4.17}$$

where

$$a_\varepsilon(x,t) := -2u_\varepsilon\nabla\zeta - \zeta u_\varepsilon \nabla v_\varepsilon \tag{4.18}$$

and

$$b_\varepsilon(x,t) := u_\varepsilon\Delta\zeta + u_\varepsilon\nabla v_\varepsilon \cdot \nabla\zeta + \zeta_t u_\varepsilon \tag{4.19}$$

as well as

$$h_\varepsilon(x,t) := -2\nabla\zeta \cdot \nabla v_\varepsilon - \zeta v_\varepsilon + \zeta G_\varepsilon(u_\varepsilon) + \zeta_t v_\varepsilon \tag{4.20}$$

for $(x,t) \in \Omega \times (0,\infty)$.
We furthermore observe that if we fix $\xi \in C^\infty(\mathbb{R})$ such that $0 \le \xi \le 1$ on $\mathbb{R}$, $\xi \equiv 0$ on $(-\infty,\frac{1}{2}]$ and $\xi \equiv 1$ on $[1,\infty)$, then for any $\delta \in (0,R)$ and $\tau > 0$,

$$\zeta(x,t) \equiv \zeta^{(\delta,\tau)}(x,t) := \xi\Big(\frac{|x|}{\delta}\Big) \cdot \xi\Big(\frac{t}{\tau}\Big), \qquad x \in \overline{\Omega},\ t \ge 0, \tag{4.21}$$

defines a function $\zeta \in C^\infty(\overline{\Omega} \times [0,\infty))$ wich is such that $\zeta \equiv 0$ in $\overline{B}_{\frac{\delta}{2}}(0) \times [0,\infty)$ and on $\overline{\Omega} \times [0,\frac{\tau}{2}]$, and that $\frac{\partial\zeta}{\partial\nu} = 0$ throughout $\partial\Omega \times (0,\infty)$, and it can readily be verified that

$$|\nabla\zeta| \le \frac{c_1}{\delta}, \quad |\Delta\zeta| \le \frac{2(n-1)c_1 + c_2}{\delta^2} \quad \text{and} \quad |\zeta_t| \le \frac{c_1}{\tau} \qquad \text{in } \Omega \times (0,\infty), \tag{4.22}$$

where $c_1 := \|\xi'\|_{L^\infty(\mathbb{R})}$ and $c_2 := \|\xi''\|_{L^\infty(\mathbb{R})}$.
As a final preparation, according to Lemma 4.2 we fix positive constants α, β, c_3 and c_4 such that for all $\varepsilon \in (0,1)$ we have

$$u_\varepsilon(x,t) \le c_3|x|^{-\alpha} \qquad \text{for any } x \in \Omega \text{ and } t > 0 \tag{4.23}$$

as well as

$$v_\varepsilon(x,t) \le c_4|x|^{-\beta} \qquad \text{for all } x \in \Omega \text{ and each } t > 0. \tag{4.24}$$

To derive (4.13) and (4.14), we now proceed in five steps.

Step 1. Let us first make sure that for all $\delta \in (0,R)$ and each $\tau > 0$ there exist $\theta_1 = \theta_1(\delta,\tau) \in (0,1)$ and $c_5(\delta,\tau) > 0$ such that for any choice of $\varepsilon \in (0,1)$,

$$\|u_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}((\overline{\Omega}\setminus B_\delta(0))\times[t,t+1])} \le c_5(\delta,\tau) \qquad \text{for all } t \ge \tau. \tag{4.25}$$

To see this, given any such δ and τ we let ζ be as defined in (4.21). Then in (4.15), (4.18) and (4.19), according to (4.22), (4.23) and (4.24) we can estimate

$$|w_\varepsilon(x,t)| \le c_3\Big(\frac{\delta}{2}\Big)^{-\alpha}$$

and

$$|a_\varepsilon(x,t)| \le 2\cdot c_3\Big(\frac{\delta}{2}\Big)^{-\alpha}\cdot\frac{c_1}{\delta} + c_3\Big(\frac{\delta}{2}\Big)^{-\alpha}\cdot c_4\Big(\frac{\delta}{2}\Big)^{-\beta}$$

as well as

$$|b_\varepsilon(x,t)| \le c_3\Big(\frac{\delta}{2}\Big)^{-\alpha}\cdot\frac{2(n-1)c_1+c_2}{\delta^2} + c_3\Big(\frac{\delta}{2}\Big)^{-\alpha}\cdot c_4\Big(\frac{\delta}{2}\Big)^{-\beta}\cdot\frac{c_1}{\delta} + c_3\Big(\frac{\delta}{2}\Big)^{-\alpha}\cdot\frac{c_1}{\tau} \tag{4.26}$$

for all $x \in \Omega$ and $t > 0$. As moreover $w_\varepsilon \equiv 0$ in $\overline{\Omega}\times[0,\frac{\tau}{2}]$, a standard result on Hölder regularity in scalar parabolic equations ([31]) becomes applicable to (4.16) so as to provide $\theta_1 = \theta_1(\delta,\tau) \in (0,1)$ and $c_5(\delta,\tau) > 0$ fulfilling

$$\|w_\varepsilon\|_{C^{\theta_1,\frac{\theta_1}{2}}(\overline{\Omega}\times[t,t+1])} \le c_5(\delta,\tau) \qquad \text{for all } t \ge 0$$

and any $\varepsilon \in (0,1)$. Since $w_\varepsilon \equiv u_\varepsilon$ in $(\overline{\Omega}\setminus B_\delta(0))\times[\tau,\infty)$ by (4.15) and (4.21), this directly entails (4.25).

Step 2. We next verify that for any $\delta \in (0,R)$ and $\tau > 0$ there exist $\theta_2 = \theta_2(\delta,\tau) \in (0,1)$ and $c_6(\delta,\tau) > 0$ such that for all $\varepsilon \in (0,1)$,

$$\|v_\varepsilon\|_{C^{1+\theta_2,\theta_2}((\overline{\Omega}\setminus B_\delta(0))\times[t,t+1])} \le c_6(\delta,\tau) \qquad \text{for all } t \ge \tau. \tag{4.27}$$

In fact, again by means of (4.22), (4.23) and (4.24), since $0 \le G_\varepsilon(u_\varepsilon) \le u_\varepsilon$ we can estimate the functions z_ε and h_ε from (4.15) and (4.20) according to

$$|z_\varepsilon(x,t)| \le c_4\Big(\frac{\delta}{2}\Big)^{-\beta}$$

and

$$|h_\varepsilon(x,t)| \le 2\cdot\frac{c_1}{\delta}\cdot c_4\Big(\frac{\delta}{2}\Big)^{-\beta} + c_4\Big(\frac{\delta}{2}\Big)^{-\beta} + c_3\Big(\frac{\delta}{2}\Big)^{-\alpha} + \frac{c_1}{\tau}\cdot c_4\Big(\frac{\delta}{2}\Big)^{-\beta}$$

for all $x \in \Omega$ and $t > 0$. Using that thanks to (4.15) we have $z_\varepsilon \equiv 0$ in $\overline{\Omega}\times[0,\frac{\tau}{2}]$, we may therefore employ gradient estimates for scalar parabolic equations ([22]) to find $\theta_2 = \theta_2(\delta,\tau) \in (0,1)$ and $c_6(\delta,\tau) > 0$ satisfying

$$\|z_\varepsilon\|_{C^{1+\theta_2,\theta_2}(\overline{\Omega}\times[t,t+1])} \le c_6(\delta,\tau) \qquad \text{for all } t \ge 0,$$

which implies (4.27) due to the fact that $z_\varepsilon \equiv v_\varepsilon$ in $(\overline{\Omega} \setminus B_\delta(0)) \times [\tau, \infty)$.

Step 3. We proceed to complete our reasoning with regard to v_ε by showing that for all $\delta \in (0, R)$ and $\tau > 0$ one can find $\theta_3 = \theta_3(\delta, \tau) \in (0, 1)$ and $c_7(\delta, \tau) > 0$ such that

$$\|v_\varepsilon\|_{C^{2+\theta_3, 1+\frac{\theta_3}{2}}((\overline{\Omega}\setminus B_\delta(0))\times[t,t+1])} \le c_7(\delta, \tau) \qquad \text{for all } t \ge \tau \tag{4.28}$$

whenever $\varepsilon \in (0, 1)$.

Indeed, using that $0 \le G'_\varepsilon \le 1$ we infer from Step 1 and Step 2 that actually

$$\|h_\varepsilon\|_{C^{\theta_3, \frac{\theta_3}{2}}(\overline{\Omega}\times[t,t+1])} \le c_8(\delta, \tau) \qquad \text{for all } t \ge 0 \text{ and } \varepsilon \in (0, 1)$$

with some $\theta_3 = \theta_3(\delta, \tau) \in (0, 1)$ and $c_8(\delta, \tau) > 0$. Therefore, standard parabolic Schauder theory ([20]) yields $c_9(\delta, \tau) > 0$ such that

$$\|z_\varepsilon\|_{C^{2+\theta_3, 1+\frac{\theta_3}{2}}(\overline{\Omega}\times[t,t+1])} \le c_9(\delta, \tau) \qquad \text{for all } t \ge 0 \text{ and } \varepsilon \in (0, 1),$$

from which (4.28) immediately results.

Step 4. In our next step, given $\delta \in (0, R)$ and $\tau > 0$ we derive the existence of $\theta_4 = \theta_4(\delta, \tau) \in (0, 1)$ and $c_{10}(\delta, \tau) > 0$ fulfilling

$$\|u_\varepsilon\|_{C^{1+\theta_4, \theta_4}((\overline{\Omega}\setminus B_\delta(0))\times[t,t+1])} \le c_{10}(\delta, \tau) \qquad \text{for all } t \ge \tau$$

and any $\varepsilon \in (0, 1)$.

This again follows from parabolic gradient regularity theory ([22]) by once more relying on (4.26), and by observing that thanks to Step 1 and Step 2, there exist $\theta_5 = \theta_5(\delta, \tau) \in (0, 1)$ and $c_{11}(\delta, \tau) > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\|a_\varepsilon\|_{C^{\theta_5, \frac{\theta_5}{2}}(\overline{\Omega}\times[t,t+1])} \le c_{11}(\delta, \tau) \qquad \text{for all } t \ge 0.$$

Step 5. We can finally assert that to each $\delta \in (0, R)$ and $\tau > 0$ there correspond some $\theta_6 = \theta_6(\delta, \tau) \in (0, 1)$ and $c_{12}(\delta, \tau) > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\|u_\varepsilon\|_{C^{2+\theta_6, 1+\frac{\theta_6}{2}}((\overline{\Omega}\setminus B_\delta(0))\times[t,t+1])} \le c_{12}(\delta, \tau) \qquad \text{for all } t \ge \tau. \tag{4.29}$$

Indeed, the information gathered in Step 4 and Step 3 now warrants that with some $\theta_7 = \theta_7(\delta, \tau) \in (0, 1)$ and $c_{13}(\delta, \tau) > 0$, for all $\varepsilon \in (0, 1)$ we have

$$\|\nabla \cdot a_\varepsilon\|_{C^{\theta_7, \frac{\theta_7}{2}}(\overline{\Omega}\times[t,t+1])} \le c_{13}(\delta, \tau) \qquad \text{for all } t \ge 0$$

and

$$\|b_\varepsilon\|_{C^{\theta_7, \frac{\theta_7}{2}}(\overline{\Omega}\times[t,t+1])} \le c_{13}(\delta, \tau) \qquad \text{for all } t \ge 0.$$

Accordingly, (4.29) once again becomes a consequence of classical parabolic Schauder theory ([20]). $\square$

In consequence, in regions not containing the origin this allows for passing to the limit in convenient topologies, and to thereby construct a candidate for our globally extended solution.

Lemma 4.4 *There exists $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and such that as $\varepsilon = \varepsilon_j \searrow 0$,*

$$\begin{aligned} &u_\varepsilon \to \widehat{u} \quad \text{in } C^{2,1}_{loc}((\overline{\Omega}\setminus\{0\})\times(0,\infty)) \qquad \text{as well as} \\ &v_\varepsilon \to \widehat{v} \quad \text{in } C^{2,1}_{loc}((\overline{\Omega}\setminus\{0\})\times(0,\infty)) \end{aligned} \tag{4.30}$$

with some nonnegative radial functions $\widehat{u} \in C^{2,1}((\overline{\Omega}\setminus\{0\})\times(0,\infty))$ and $\widehat{v} \in C^{2,1}((\overline{\Omega}\setminus\{0\})\times(0,\infty))$ satisfying $\frac{\partial\widehat{u}}{\partial\nu} = \frac{\partial\widehat{v}}{\partial\nu} = 0$ on $\partial\Omega\times(0,\infty)$. Moreover, with $T_{max} \in (0,\infty]$ and (u,v) taken from Proposition 1.2 we have

$$\widehat{u} \equiv u \quad \text{and} \quad \widehat{v} \equiv v \qquad \text{in } \Big(\overline{\Omega}\setminus\{0\}\Big)\times(0,T_{max}); \tag{4.31}$$

in particular,

$$(\widehat{u},\widehat{v}) \in \Big(C^0(\overline{\Omega}\times[0,T_{max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max}))\Big)^2. \tag{4.32}$$

Proof. In view of Lemma 4.3, the Arzelà-Ascoli theorem evidently implies the existence of a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and nonnegative radial functions $\widehat{u} \in C^{2,1}((\overline{\Omega}\setminus\{0\})\times(0,\infty))$ and $\widehat{v} \in C^{2,1}((\overline{\Omega}\setminus\{0\})\times(0,\infty))$ satisfying $\varepsilon_j \searrow 0$ as $j\to\infty$ as well as (4.30) and, as a consequence thereof and of (4.3), also $\frac{\partial\widehat{u}}{\partial\nu} = \frac{\partial\widehat{v}}{\partial\nu} = 0$ on $\partial\Omega\times(0,\infty)$.
Since furthermore by continuity of u in $\overline{\Omega}\times[0,T_{max})$ the numbers T_ε in (4.6) have the property that $T_\varepsilon \nearrow T_{max}$ as $\varepsilon\searrow 0$, from (4.6) we trivially infer that

$$u_\varepsilon \to u \quad \text{and} \quad v_\varepsilon \to v \quad \text{in } C^0_{loc}(\overline{\Omega}\times[0,T_{max})) \qquad \text{as } \varepsilon\searrow 0,$$

so that (4.31) and (4.32) become evident by-products of (4.30). □

This limit couple trivially inherits some properties of its approximations.

Lemma 4.5 *Let $\delta \in (0,R)$. Then there exist $C(\delta) > 0$ and $\theta = \theta(\delta) \in (0,1)$ such that the limit functions $\widehat{u}$ and $\widehat{v}$ obtained in Lemma 4.4 satisfy*

$$\|\widehat{u}\|_{C^{2+\theta,1+\frac{\theta}{2}}((\overline{\Omega}\setminus B_\delta(0))\times[t,t+1])} \le C \qquad \text{for all } t>1 \tag{4.33}$$

and

$$\|\widehat{v}\|_{C^{2+\theta,1+\frac{\theta}{2}}((\overline{\Omega}\setminus B_\delta(0))\times[t,t+1])} \le C \qquad \text{for all } t>1. \tag{4.34}$$

In particular, if $T_{max} \in (0,\infty]$, u and v are as in Proposition 1.2, then in the case $T_{max} < \infty$ we have

$$\begin{aligned} &u(\cdot,t) \to \widehat{u}(\cdot,T_{max}) \quad \text{in } C^2_{loc}(\overline{\Omega}\setminus\{0\}) \qquad \text{and} \\ &v(\cdot,t) \to \widehat{v}(\cdot,T_{max}) \quad \text{in } C^2_{loc}(\overline{\Omega}\setminus\{0\}) \end{aligned} \tag{4.35}$$

as $t \nearrow T_{max}$.

Proof. The estimates in (4.33) and (4.34) are direct consequences of Lemma 4.3 when combined with Lemma 4.4. Thereupon, the convergence properties in (4.35) become obvious. □

With the above preparations at hand, we can immediately derive both of our main results concerning pointwise bounds for solutions as well as the existence of blow-up profiles and estimates therefor.

Proof of Corollary 1.3. In view of (4.30) and the identities in (4.31), this immediately results from Lemma 4.2. □

Proof of Corollary 1.4. Taking $\widehat{u}$ as constructed in Lemma 4.4, from Lemma 4.5 we directly obtain the claim if we let $U(x) := \widehat{u}(x,T_{max})$ for $x \in \overline{\Omega}\setminus\{0\}$. □

5 A quasi-entropy structure

In finally assigning the above limit $(\widehat{u}, \widehat{v})$ the role of a solution to (1.1) in an appropriate, in particular spatially global sense, we shall resort to the following adaptation of the celebrated notion of renormalized solutions ([9]) to the present context.

Definition 5.1 *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $u_0 \in L^1(\Omega)$ and $v_0 \in L^1(\Omega)$ be nonnegative. Then a pair (u, v) of nonnegative functions*

$$\begin{cases} u \in L^1_{loc}(\overline{\Omega} \times [0, \infty)), \\ v \in L^1_{loc}(\overline{\Omega} \times [0, \infty)) \end{cases} \tag{5.1}$$

such that for all $\psi \in C_0^\infty([0, \infty)^2)$ we have

$$\begin{cases} \psi(u, v)\nabla u \in L^2_{loc}(\overline{\Omega} \times [0, \infty)) \qquad \text{and} \\ \psi(u, v)\nabla v \in L^2_{loc}(\overline{\Omega} \times [0, \infty)), \end{cases} \tag{5.2}$$

will be called a global renormalized solution *of (1.1) if for any choice of $\phi \in C_0^\infty([0, \infty)^2)$, the identity*

$$\begin{aligned} -\int_0^\infty \int_\Omega \phi(u, v)\varphi_t - \int_\Omega \phi(u_0, v_0)\varphi(\cdot, 0) \\ = & -\int_0^\infty \int_\Omega \phi_{uu}(u, v)|\nabla u|^2\varphi - \int_0^\infty \int_\Omega \phi_{uv}(u, v)(\nabla u \cdot \nabla v)\varphi - \int_0^\infty \int_\Omega \phi_u(u, v)\nabla u \cdot \nabla\varphi \\ & + \int_0^\infty \int_\Omega u\phi_{uu}(u, v)(\nabla u \cdot \nabla v)\varphi + \int_0^\infty \int_\Omega u\phi_{uv}(u, v)|\nabla v|^2\varphi + \int_0^\infty \int_\Omega u\phi_u(u, v)\nabla v \cdot \nabla\varphi \\ & - \int_0^\infty \int_\Omega \phi_{vv}(u, v)|\nabla v|^2\varphi - \int_0^\infty \int_\Omega \phi_{uv}(u, v)(\nabla u \cdot \nabla v)\varphi - \int_0^\infty \int_\Omega \phi_v(u, v)\nabla v \cdot \nabla\varphi \\ & - \int_0^\infty \int_\Omega v\phi_v(u, v)\varphi + \int_0^\infty \int_\Omega u\phi_v(u, v)\varphi \end{aligned} \tag{5.3}$$

is valid for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0, \infty))$.

Remark. It can readily be verified by formally computing $\frac{\partial\phi(u,v)}{\partial t}$ on the basis of (1.1) that the above concept in fact generalizes that of classical solvability in the sense that whenever (u, v) is a pair of sufficiently smooth functions fulfilling (5.3), then indeed both initial-boundary value problems contained in (1.1) are satisfied in the classical pointwise sense.

Now the following observation, based on a rigorous analysis of an approximate counterpart of (1.16), forms a key ingredient for our derivation of Theorem 1.5.

Lemma 5.1 *Let $p > 0$ and*

$$\kappa > (p + 1) \cdot \left(p + \sqrt{p^2 + \frac{p}{4}}\right). \tag{5.4}$$

Then there exists $C(p, \kappa) > 0$ such that for all $\varepsilon \in (0, 1)$,

$$\int_t^{t+1} \int_\Omega (u_\varepsilon + 1)^{-p-2} e^{-\kappa v_\varepsilon} |\nabla u_\varepsilon|^2 \leq C(p, \kappa) \qquad \text{for all } t \geq 0 \tag{5.5}$$

and

$$\int_t^{t+1}\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2 \le C(p,\kappa) \qquad \textit{for all } t\ge 0. \tag{5.6}$$

PROOF. Using (4.3) and integrating by parts, for $\varepsilon\in(0,1)$ we compute

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon} &= -p\int_\Omega (u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}\cdot\Big\{\Delta u_\varepsilon-\nabla\cdot(u_\varepsilon\nabla v_\varepsilon)\Big\}\\
&\quad -\kappa\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}\cdot\Big\{\Delta v_\varepsilon-v_\varepsilon+G_\varepsilon(u_\varepsilon)\Big\}\\
&= p\int_\Omega \nabla\Big\{(u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}\Big\}\cdot\Big\{\nabla u_\varepsilon-u_\varepsilon\nabla v_\varepsilon\Big\}\\
&\quad +\kappa\int_\Omega \nabla\Big\{(u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}\Big\}\cdot\nabla v_\varepsilon\\
&\quad +\kappa\int_\Omega (u_\varepsilon+1)^{-p}v_\varepsilon e^{-\kappa v_\varepsilon}-\kappa\int_\Omega (u_\varepsilon+1)^{-p}G_\varepsilon(u_\varepsilon)e^{-\kappa v_\varepsilon}\\
&= -p(p+1)\int_\Omega (u_\varepsilon+1)^{-p-2}e^{-\kappa v_\varepsilon}|\nabla u_\varepsilon|^2\\
&\quad -p\kappa\int_\Omega (u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\\
&\quad +p(p+1)\int_\Omega u_\varepsilon(u_\varepsilon+1)^{-p-2}e^{-\kappa v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\\
&\quad +p\kappa\int_\Omega u_\varepsilon(u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2\\
&\quad -p\kappa\int_\Omega (u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\\
&\quad -\kappa^2\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2\\
&\quad +\kappa\int_\Omega (u_\varepsilon+1)^{-p}v_\varepsilon e^{-\kappa v_\varepsilon}-\kappa\int_\Omega (u_\varepsilon+1)^{-p}G_\varepsilon(u_\varepsilon)e^{-\kappa v_\varepsilon} \qquad \text{for all } t>0.
\end{aligned}$$

Here we recall that $G_\varepsilon\ge 0$ and use that $\xi e^{-\kappa\xi}\le\frac{1}{\kappa e}$ for all $\xi\ge 0$ to see upon evident rearrangements that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon} &\le -p(p+1)\int_\Omega (u_\varepsilon+1)^{-p-2}e^{-\kappa v_\varepsilon}|\nabla u_\varepsilon|^2\\
&\quad -\kappa^2\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2\\
&\quad +p\kappa\int_\Omega u_\varepsilon(u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2\\
&\quad -2p\kappa\int_\Omega (u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\\
&\quad +p(p+1)\int_\Omega u_\varepsilon(u_\varepsilon+1)^{-p-2}e^{-\kappa v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon
\end{aligned}$$

$$+\frac{|\Omega|}{e} \qquad \text{for all } t>0, \tag{5.7}$$

because $(u_\varepsilon+1)^{-p}\le 1$. Now since trivially $\xi(\xi+1)^{-p-1}\le(\xi+1)^{-p}$ for all $\xi\ge 0$, we obtain

$$\begin{aligned} -\kappa^2\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2 + p\kappa\int_\Omega u_\varepsilon(u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2 & \\ \le -\kappa(\kappa-p)\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2 \qquad \text{for all } t>0, & \end{aligned} \tag{5.8}$$

and in order to make appropriate use of this in absorbing the integrals in (5.7) containing $\nabla u_\varepsilon\cdot\nabla v_\varepsilon$, we observe that our assumption (5.4), that is, condition $\kappa>p(p+1)+\sqrt{p^2(p+1)^2+\frac{p(p+1)^2}{4}}$, warrants that $\kappa^2-2p(p+1)\kappa-\frac{p(p+1)^2}{4}>0$ and hence

$$p\cdot\Big(4\kappa^2+4p\kappa+4\kappa+(p+1)^2\Big)<4(p+1)\kappa(\kappa-p).$$

As this is equivalent to the inequality

$$\frac{[p(2\kappa+p+1)]^2}{4p(p+1)}<\kappa(\kappa-p),$$

we can therefore find some $\eta=\eta(p,\kappa)\in(0,1)$ suitably close to 1 such that still

$$\frac{[p(2\kappa+p+1)]^2}{4p(p+1)\eta}<\kappa(\kappa-p). \tag{5.9}$$

Now with this value of η being fixed, in (5.7) we use Young's inequality to see that since $\xi(\xi+1)^{-p-2}\le(\xi+1)^{-p-1}$ for all $\xi\ge 0$,

$$\begin{aligned} -2p\kappa\int_\Omega (u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon + p(p+1)\int_\Omega u_\varepsilon(u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}\nabla u_\varepsilon\cdot\nabla v_\varepsilon & \\ \le p(2\kappa+p+1)\int_\Omega (u_\varepsilon+1)^{-p-1}e^{-\kappa v_\varepsilon}|\nabla u_\varepsilon|\cdot|\nabla v_\varepsilon| & \\ \le p(p+1)\eta\int_\Omega (u_\varepsilon+1)^{-p-2}e^{-\kappa v_\varepsilon}|\nabla u_\varepsilon|^2 & \\ +\frac{[p(2\kappa+p+1)]^2}{4p(p+1)\eta}\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2 \qquad \text{for all } t>0. & \end{aligned}$$

Together with (5.8), this shows that (5.7) implies that

$$\begin{aligned} \frac{d}{dt}\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon} \le & -c_1(p,\kappa)\int_\Omega (u_\varepsilon+1)^{-p-2}e^{-\kappa v_\varepsilon}|\nabla u_\varepsilon|^2 \\ & -c_2(p,\kappa)\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2 \\ & +\frac{|\Omega|}{e} \qquad \text{for all } t>0 \end{aligned} \tag{5.10}$$

with

$$c_1(p,\kappa) := p(p+1)(1-\eta(p,\kappa)) \qquad \text{and} \qquad c_2(p,\kappa) := \kappa(\kappa-p) - \frac{[p(2\kappa+p+1)]^2}{4p(p+1)\eta(p,\kappa)}$$

both being positive according to our restriction (5.9) and the fact that $\eta(p,\kappa) \in (0,1)$. On integration in time, however, (5.10) entails that

$$\begin{aligned}
c_1(p,\kappa)\int_t^{t+1}\int_\Omega (u_\varepsilon+1)^{-p-2}e^{-\kappa v_\varepsilon}|\nabla u_\varepsilon|^2 + c_2(p,\kappa)\int_t^{t+1}\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2 \\
\le \int_\Omega \Big(u_\varepsilon(\cdot,t)+1\Big)^{-p} e^{-\kappa v_\varepsilon(\cdot,t)} - \int_\Omega \Big(u_\varepsilon(\cdot,t+1)+1\Big)^{-p} e^{-\kappa v_\varepsilon(\cdot,t+1)} + \frac{|\Omega|}{e} \\
\le |\Omega| + \frac{|\Omega|}{e} \qquad \text{for all } t \ge 0,
\end{aligned}$$

because clearly $0 \le (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon} \le 1$ on $\Omega\times(0,\infty)$. Letting $C(p,\kappa) := \max\{\frac{1}{c_1(p,\kappa)}, \frac{1}{c_2(p,\kappa)}\}$, we thereby readily arrive at (5.5) and (5.6). $\square$

A straightforward application of Fatou's lemma to suitable versions of (5.5) and (5.6) immediately verifies the regularity requirements in (5.2).

Corollary 5.2 *Let $\widehat{u}$ and $\widehat{v}$ be as in Lemma 4.4. Then for all nonnegative $\psi \in C_0^\infty([0,\infty)^2)$ and each $T>0$,*

$$\int_0^T\int_\Omega \psi(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2 < \infty \tag{5.11}$$

and

$$\int_0^T\int_\Omega \psi(\widehat{u},\widehat{v})|\nabla\widehat{v}|^2 < \infty. \tag{5.12}$$

In particular, both inclusions in (5.2) hold.

Proof. We fix an arbitrary $p>0$ and employ Lemma 5.1 to find $\kappa>0, c_1>0$ and $c_2>0$ such that for all $\varepsilon\in(0,1)$,

$$\int_0^T\int_\Omega (u_\varepsilon+1)^{-p-2}e^{-\kappa v_\varepsilon}|\nabla u_\varepsilon|^2 \le c_1\cdot(T+1) \qquad \text{for all } T>0 \tag{5.13}$$

and

$$\int_0^T\int_\Omega (u_\varepsilon+1)^{-p}e^{-\kappa v_\varepsilon}|\nabla v_\varepsilon|^2 \le c_2\cdot(T+1) \qquad \text{for all } T>0. \tag{5.14}$$

Then given a nonnegative $\psi\in C_0^\infty([0,\infty)^2)$, by boundedness of $\operatorname{supp}\psi$ we see that $[0,\infty)^2 \ni (\xi,\eta) \mapsto (\xi+1)^{p+2}e^{\kappa\eta}\psi(\xi,\eta)$ is bounded and that hence we can pick $c_3>0$ fulfilling

$$(u_\varepsilon+1)^{p+2}e^{\kappa v_\varepsilon}\psi(u_\varepsilon,v_\varepsilon) \le c_3 \qquad \text{in } \Omega\times(0,\infty) \text{ for all } \varepsilon\in(0,1).$$

Therefore, according to (5.13) we can estimate

$$\begin{aligned}
\int_0^T\int_\Omega \psi(u_\varepsilon,v_\varepsilon)|\nabla u_\varepsilon|^2 &= \int_0^T\int_\Omega \Big\{(u_\varepsilon+1)^{p+2}e^{\kappa v_\varepsilon}\psi(u_\varepsilon,v_\varepsilon)\Big\}\cdot(u_\varepsilon+1)^{-p-2}e^{-\kappa v_\varepsilon}|\nabla u_\varepsilon|^2 \\
&\le c_3c_1\cdot(T+1) \qquad \text{for all } T>0 \text{ and each } \varepsilon\in(0,1),
\end{aligned}$$

so that since from Lemma 4.4 we know that with $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ as provided there we have

$$\psi(u_\varepsilon(x,t),v_\varepsilon(x,t))|\nabla u_\varepsilon(x,t)|^2 \to \psi(\widehat{u}(x,t),\widehat{v}(x,t))|\nabla\widehat{u}(x,t)|^2 \qquad \text{for all } x\in\Omega\setminus\{0\} \text{ and any } t\in(0,T)$$

as $\varepsilon=\varepsilon_j \searrow 0$, (5.11) results upon an application of Fatou's lemma. The property (5.12) can be deduced from (5.14) in quite a similar manner. $\Box$

Now by combining the convergence information from Lemma 4.4 with the global regularity properties obtained in Corollary 5.2 we can indeed make sure that $(\widehat{u},\widehat{v})$ solves (1.1) in the sense of Definition 5.1.

Lemma 5.3 *The pair $(\widehat{u},\widehat{v})$ obtained in Lemma 4.4 is a global renormalized solution of (1.1) in the sense of Definition 5.1.*

PROOF. We once more fix $\xi\in C^\infty(\mathbb{R})$ such that $\chi_{[1,\infty)} \le \xi \le \chi_{[\frac{1}{2},\infty)}$, and let $\zeta_\delta(x) := \xi(\frac{|x|}{\delta})$ for $x\in\overline{\Omega}$ and $\delta\in(0,R)$. Then given $\phi\in C_0^\infty([0,\infty)^2)$ and $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, for $\delta\in(0,R)$ we let $\varphi_\delta(x,t) := \zeta_\delta(x)\varphi(x,t)$, $x\in\overline{\Omega}, t\ge 0$, and use (4.3) to see that

$$\begin{aligned}
I_1(\delta,\varepsilon)+I_2(\delta,\varepsilon) &:= \int_0^\infty\int_\Omega \phi(u_\varepsilon,v_\varepsilon)\varphi_{\delta t} - \int_\Omega \phi(u_0,v_0)\varphi_\delta(\cdot,0) \\
&= -\int_0^\infty\int_\Omega \phi_{uu}(u_\varepsilon,v_\varepsilon)|\nabla u_\varepsilon|^2\varphi_\delta - \int_0^\infty\int_\Omega \phi_{uv}(u_\varepsilon,v_\varepsilon)(\nabla u_\varepsilon\cdot\nabla v_\varepsilon)\varphi_\delta \\
&\quad -\int_0^\infty\int_\Omega \phi_u(u_\varepsilon,v_\varepsilon)\nabla u_\varepsilon\cdot\nabla\varphi_\delta \\
&\quad +\int_0^\infty\int_\Omega u_\varepsilon\phi_{uu}(u_\varepsilon v_\varepsilon)(\nabla u_\varepsilon\cdot\nabla v_\varepsilon)\varphi_\delta + \int_0^\infty\int_\Omega u_\varepsilon\phi_{uv}(u_\varepsilon,v_\varepsilon)|\nabla v_\varepsilon|^2\varphi_\delta \\
&\quad +\int_0^\infty\int_\Omega u_\varepsilon\phi_u(u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon\cdot\nabla\varphi_\delta \\
&\quad -\int_0^\infty\int_\Omega \phi_{vv}(u_\varepsilon,v_\varepsilon)|\nabla v_\varepsilon|^2\varphi_\delta - \int_0^\infty\int_\Omega \phi_{uv}(u_\varepsilon,v_\varepsilon)(\nabla u_\varepsilon\cdot\nabla v_\varepsilon)\varphi_\delta \\
&\quad -\int_0^\infty\int_\Omega \phi_v(u_\varepsilon,v_\varepsilon)\nabla v_\varepsilon\cdot\nabla\varphi_\delta \\
&\quad -\int_0^\infty\int_\Omega v_\varepsilon\phi_v(u_\varepsilon,v_\varepsilon)\varphi_\delta + \int_0^\infty\int_\Omega G_\varepsilon(u_\varepsilon)\phi_v(u_\varepsilon,v_\varepsilon)\varphi_\delta \\
&=: \sum_{i=3}^{13} I_i(\delta,\varepsilon)
\end{aligned} \tag{5.15}$$

for all $\varepsilon\in(0,1)$ and $\delta\in(0,R)$. Here in order to let $\varepsilon=\varepsilon_j\searrow 0$ along the sequence $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ provided by Lemma 4.4, according to Lemma 4.1 and Lemma 4.4 we can pick $\tau>0$ such that $u_\varepsilon\equiv\widehat{u}$ and $v_\varepsilon\equiv\widehat{v}$ in $\Omega\times(0,\tau)$ for all $\varepsilon\in(0,1)$, and thereafter choose $T>\tau$ such that $\varphi\equiv 0$ in $\Omega\times(T,\infty)$. We may then invoke Lemma 4.4 to infer that for fixed $\delta\in(0,R)$,

$$u_\varepsilon\to\widehat{u} \quad\text{and}\quad v_\varepsilon\to\widehat{v} \quad \text{in } C^1\Big((\overline{\Omega}\setminus B_{\frac{\delta}{2}}(0))\times[\tau,T]\Big) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0, \tag{5.16}$$

whence in particular

$$\begin{aligned}
I_3(\delta,\varepsilon) &= -\int_0^\tau \int_\Omega \phi_{uu}(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2\varphi_\delta - \int_\tau^T \int_{\Omega\setminus B_{\frac{\delta}{2}}(0)} \phi_{uu}(u_\varepsilon,v_\varepsilon)|\nabla u_\varepsilon|^2\varphi_\delta \\
&\to -\int_0^\tau \int_\Omega \phi_{uu}(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2\varphi_\delta - \int_\tau^T \int_{\Omega\setminus B_{\frac{\delta}{2}}(0)} \phi_{uu}(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2\varphi_\delta \\
&= -\int_0^\infty \int_\Omega \phi_{uu}(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2\varphi_\delta
\end{aligned}$$

as $\varepsilon=\varepsilon_j \searrow 0$.
In the expressions $I_1(\delta,\varepsilon)$ and $I_i(\delta,\varepsilon)$ for $i\in\{4,...,13\}$, we argue similarly, in addition observing that due to (5.16) and our construction of G_ε we have $G_\varepsilon(u_\varepsilon)\equiv u_\varepsilon$ in $(\overline{\Omega}\setminus B_{\frac{\delta}{2}}(0))\times(0,T)$ for all sufficiently small $\varepsilon\in(0,1)$ and hence clearly also

$$G_\varepsilon(u_\varepsilon)\to\widehat{u} \quad \text{in } C^0\Big((\overline{\Omega}\setminus B_{\frac{\delta}{2}}(0))\times[0,T]\Big) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0.$$

In consequence, on taking $\varepsilon=\varepsilon_j\searrow 0$ separately in each of the summands making up (5.15) we readily arrive at the identity

$$\begin{aligned}
J_1(\delta)+J_2(\delta) &:= \int_0^\infty \int_\Omega \phi(\widehat{u},\widehat{v})\varphi_{\delta t} - \int_\Omega \phi(u_0,v_0)\varphi_\delta(\cdot,0) \\
&= -\int_0^\infty \int_\Omega \phi_{uu}(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2\varphi_\delta - \int_0^\infty \int_\Omega \phi_{uv}(\widehat{u},\widehat{v})(\nabla\widehat{u}\cdot\nabla\widehat{v})\varphi_\delta \\
&\quad -\int_0^\infty \int_\Omega \phi_u(\widehat{u},\widehat{v})\nabla\widehat{u}\cdot\nabla\varphi_\delta \\
&\quad +\int_0^\infty \int_\Omega \widehat{u}\phi_{uu}(\widehat{u},\widehat{v})(\nabla\widehat{u}\cdot\nabla\widehat{v})\varphi_\delta + \int_0^\infty \int_\Omega \widehat{u}\phi_{uv}(\widehat{u},\widehat{v})|\nabla\widehat{v}|^2\varphi_\delta \\
&\quad +\int_0^\infty \int_\Omega \widehat{u}\phi_u(\widehat{u},\widehat{v})\nabla\widehat{v}\cdot\nabla\varphi_\delta \\
&\quad -\int_0^\infty \int_\Omega \phi_{vv}(\widehat{u},\widehat{v})|\nabla\widehat{v}|^2\varphi_\delta - \int_0^\infty \int_\Omega \phi_{uv}(\widehat{u},\widehat{v})(\nabla\widehat{u}\cdot\nabla\widehat{v})\varphi_\delta \\
&\quad -\int_0^\infty \int_\Omega \phi_v(\widehat{u},\widehat{v})\nabla\widehat{v}\cdot\nabla\varphi_\delta \\
&\quad -\int_0^\infty \int_\Omega \widehat{v}\phi_v(\widehat{u},\widehat{v})\varphi_\delta + \int_0^\infty \int_\Omega \widehat{u}\phi_v(\widehat{u},\widehat{v})\varphi_\delta \\
&=: \sum_{i=3}^{13} J_i(\delta) \qquad \text{for all } \delta\in(0,R).
\end{aligned} \tag{5.17}$$

Here since $\phi_{uu}(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2$ belongs to $L^1(\Omega\times(0,T)$ by Corollary 5.2, and since clearly $\varphi_\delta\to\varphi$ a.e. in $\Omega\times(0,T)$ as $\delta\searrow 0$ with $|\varphi_\delta|\le\|\varphi\|_{L^\infty(\Omega\times((0,T))}$ in $\Omega\times(0,T)$, invoking the dominated convergence theorem we find that

$$J_3(\delta) = -\int_0^T \int_\Omega \phi_{uu}(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2\varphi_\delta \to -\int_0^T \int_\Omega \phi_{uu}(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2\varphi = -\int_0^\infty \int_\Omega \phi_{uu}(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2\varphi \tag{5.18}$$

as $\delta \searrow 0$. In quite a similar manner, by means of the dominated convergence theorem and Corollary 5.2 we see that passing to the limit $\delta \searrow 0$ leads to the respectively expected results in each of the summands in (5.17) containing either φ_δ itself or $\varphi_{\delta t} \equiv \zeta_\delta \varphi_t$ but not $\nabla \varphi_\delta$; that is, as $\delta \searrow 0$ we have

$$J_1(\delta) + J_2(\delta) \to -\int_0^\infty \int_\Omega \phi(\widehat{u}, \widehat{v})\varphi_t - \int_\Omega \phi(u_0, v_0)\varphi(\cdot, 0) \tag{5.19}$$

and

$$\begin{aligned}
&J_4(\delta) + J_6(\delta) + J_7(\delta) + J_9(\delta) + J_{10}(\delta) + J_{12}(\delta) + J_{13}(\delta) \\
&\qquad \to \ -\int_0^\infty \int_\Omega \phi_{uv}(\widehat{u}, \widehat{v})(\nabla \widehat{u} \cdot \nabla \widehat{v})\varphi \\
&\qquad\qquad + \int_0^\infty \int_\Omega \widehat{u}\phi_{uu}(\widehat{u}, \widehat{v})(\nabla \widehat{u} \cdot \nabla \widehat{v})\varphi + \int_0^\infty \int_\Omega \widehat{u}\phi_{uv}(\widehat{u}, \widehat{v})|\nabla \widehat{v}|^2\varphi \\
&\qquad\qquad - \int_0^\infty \int_\Omega \phi_{vv}(\widehat{u}, \widehat{v})|\nabla \widehat{v}|^2\varphi - \int_0^\infty \int_\Omega \phi_{uv}(\widehat{u}, \widehat{v})(\nabla \widehat{u} \cdot \nabla \widehat{v})\varphi \\
&\qquad\qquad - \int_0^\infty \int_\Omega \widehat{v}\phi_v(\widehat{u}, \widehat{v})\varphi + \int_0^\infty \int_\Omega \widehat{u}\phi_v(\widehat{u}, \widehat{v})\varphi,
\end{aligned} \tag{5.20}$$

and that moreover in

$$\begin{aligned}
J_5(\delta) &= -\int_0^\infty \int_\Omega \phi_u(\widehat{u}, \widehat{v})(\nabla \widehat{u} \cdot \nabla \varphi)\zeta_\delta - \int_0^\infty \int_\Omega \phi_u(\widehat{u}, \widehat{v})(\nabla \widehat{u} \cdot \nabla \zeta_\delta)\varphi \\
&=: J_{5,1}(\delta) + J_{5,2}(\delta), \qquad \delta \in (0, R),
\end{aligned} \tag{5.21}$$

and

$$\begin{aligned}
J_8(\delta) &= -\int_0^\infty \int_\Omega \widehat{u}\phi_u(\widehat{u}, \widehat{v})(\nabla \widehat{v} \cdot \nabla \varphi)\zeta_\delta + \int_0^\infty \int_\Omega \widehat{u}\phi_u(\widehat{u}, \widehat{v})(\nabla \widehat{v} \cdot \nabla \zeta_\delta)\varphi \\
&=: J_{8,1}(\delta) + J_{8,2}(\delta), \qquad \delta \in (0, R),
\end{aligned} \tag{5.22}$$

as well as in

$$\begin{aligned}
J_{11}(\delta) &= -\int_0^\infty \int_\Omega \phi_v(\widehat{u}, \widehat{v})(\nabla \widehat{v} \cdot \nabla \varphi)\zeta_\delta - \int_0^\infty \int_\Omega \phi_v(\widehat{u}, \widehat{v})(\nabla \widehat{v} \cdot \nabla \zeta_\delta)\varphi \\
&=: J_{11,1}(\delta) + J_{11,2}(\delta), \qquad \delta \in (0, R),
\end{aligned} \tag{5.23}$$

we have

$$\begin{aligned}
J_{5,1}(\delta) + J_{8,1}(\delta) + J_{11,1}(\delta) \ &\to \ -\int_0^\infty \int_\Omega \phi_u(\widehat{u}, \widehat{v})\nabla \widehat{u} \cdot \nabla \varphi + \int_0^\infty \int_\Omega \widehat{u}\phi_u(\widehat{u}, \widehat{v})\nabla \widehat{v} \cdot \nabla \varphi \\
&\qquad - \int_0^\infty \int_\Omega \phi_v(\widehat{u}, \widehat{v})\nabla \widehat{v} \cdot \nabla \varphi
\end{aligned} \tag{5.24}$$

as $\delta \searrow 0$, so that it remains to consider $J_{i,2}(\delta)$ for $i \in \{5, 8, 11\}$. for this purpose, we recall the definition of ζ_δ to estimate

$$|\nabla \zeta_\delta(x)| = \frac{1}{\delta}\Big|\xi'\Big(\frac{|x|}{\delta}\Big)\Big| \le \frac{c_1}{\delta} \qquad \text{for all } x \in \Omega \text{ and } \delta \in (0, R)$$

with $c_1 := \|\xi'\|_{L^\infty(\mathbb{R})}$, so that since $\operatorname{supp} \nabla\zeta_\delta \subset \overline{B}_\delta(0)$ and $|\overline{B}_\delta(0)| = c_2\delta^n$ for all $\delta \in (0,R)$ with $c_2 := |B_1(0)|$, thanks to our overall assumption that $n \ge 2$ we find that

$$\int_\Omega |\nabla\zeta_\delta|^2 \le \frac{c_1^2}{\delta^2}|\overline{B}_\delta(0)| \le c_1^2 c_2 \delta^{n-2} \le c_3 := c_1^2 c_2 R^{n-2} \qquad \text{for all } \delta \in (0,R).$$

Using the Cauchy-Schwarz inequality, from this we obtain that

$$\begin{aligned} |J_{5,1}(\delta)| &\le \|\varphi\|_{L^\infty(\Omega\times(0,T))} \cdot \left\{ \int_0^T \int_{\overline{B}_\delta(0)} \phi_u^2(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2 \right\}^{\frac{1}{2}} \cdot \left\{ \int_0^T \int_\Omega |\nabla\zeta_\delta|^2 \right\}^{\frac{1}{2}} \\ &\le \sqrt{c_3 T}\|\varphi\|_{L^\infty(\Omega\times(0,T))} \cdot \left\{ \int_0^T \int_{\overline{B}_\delta(0)} \phi_u^2(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2 \right\}^{\frac{1}{2}} \\ &\to 0 \qquad \text{as } \delta \searrow 0, \end{aligned} \tag{5.25}$$

because again from Corollary 5.2 we know that $\phi_u^2(\widehat{u},\widehat{v})|\nabla\widehat{u}|^2 \in L^1(\Omega\times(0,T))$. Similarly,

$$|J_{8,1}(\delta)| + |J_{11,1}(\delta)| \to 0 \qquad \text{as } \delta \searrow 0,$$

which, when together with (5.25) and (5.24) inserted into (5.21)-(5.23) and combined with (5.17)-(5.20), yields the claimed identity (5.3). $\square$

Our final result on global extensibility thereby becomes evident.

Proof of Theorem 1.5. We take $\widehat{u}$ and $\widehat{v}$ from Lemma 4.4 and then obtain (1.17) as a consequence of Lemma 4.5. The claimed solution properties of $(\widehat{u},\widehat{v})$ are then precisely asserted by Lemma 4.2, whereas the desired characterization of $(\widehat{u},\widehat{v})$ before blow-up has already been contained in Lemma 4.4. $\square$

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References

[1] Bedrossian, J., Masmoudi, N.: *Existence, uniqueness and Lipschitz dependence for Patlak-Keller-Segel and Navier-Stokes in $\mathbb{R}^2$ with measure-valued initial data.* Arch. Rat. Mech. Anal. **214**, 717-801 (2014)

[2] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: *Toward a Mathematical Theory of Keller-Segel Models of Pattern Formation in Biological Tissues.* Math. Mod. Meth. Appl. Sci. **25**, 1663-1763 (2015)

[3] Bellomo, N., Winkler, M.: *Finite-time blow-up in a degenerate chemotaxis system with flux limitation.* Trans. Amer. Math. Soc. B **4**, 31-67 (2017)

[4] Biler, P.: *Global solutions to some parabolic-elliptic systems of chemotaxis.* Adv. Math. Sci. Appl. **9** (1), 347-359 (1999)

[5] Cao, X.: *Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces.* Discr. Cont. Dyn. Syst. A **35**, 1891-1904 (2015)

[6] Cao, X.: *A refined extensibility criterion for the Keller-Segel system.* Preprint

[7] Cieślak, T., Stinner, C.: *Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions.* J. Differ. Eq. **252** (10), 5832-5851 (2012)

[8] Cieślak, T., Stinner, C.: *New critical exponents in a fully parabolic quasilinear Keller-Segel and applications to volume filling models.* J. Differ. Eq. **258** (6), 2080-2113 (2015)

[9] Di Perna, R.-J., Lions, P.-L.: *On the Cauchy problem for Boltzmann equations: Global existence and weak stability.* Ann. Math. **130**, 321-366 (1989)

[10] Dolbeault, J., Schmeiser, C.: *The two-dimensional Keller-Segel model after blow-up.* Discr. Cont. Dyn. Syst **25**, 109-121 (2009)

[11] Fujie, K., Ito, A., Winkler, M., Yokota, T.: *Stabilization in a chemotaxis model for tumor invasion.* Discrete Contin. Dyn. Syst. A **36**, 151-169 (2016)

[12] Herrero, M. A., Velázquez, J. J. L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scu. Norm. Sup. Pisa Cl. Sci. **24**, 633-683 (1997)

[13] Hillen, T., Painter, K.: *A user's guide to PDE models for chemotaxis.* J. Math. Biol. **58**, 183-217 (2009)

[14] Horstmann, D.: *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences I.* Jahresberichte DMV **105** (3), 103-165 (2003)

[15] Horstmann, D., Wang, G.: Blow-up in a chemotaxis model without symmetry assumptions. Eur. J. Appl. Math. **12**, 159-177 (2001)

[16] Horstmann, D., Winkler, M.: *Boundedness vs. blow-up in a chemotaxis system.* J. Differential Equations **215** (1), 52-107 (2005)

[17] Ishida, S., Yokota, T.: *Blow-up in finite or infinite time for quasilinear degenerate Keller-Segel systems of parabolic-parabolic type.* Discr. Cont. Dyn. Syst. B **18**, 2569-2596 (2013)

[18] Jäger, W., Luckhaus, S.: *On explosions of solutions to a system of partial differential equations modelling chemotaxis.* Trans. Am. Math. Soc. **329**, 819-824 (1992)

[19] Keller, E. F., Segel, L. A.: *Initiation of slime mold aggregation viewed as an instability.* J. Theor. Biol. **26**, 399-415 (1970)

[20] Ladyzenskaja, O. A., Solonnikov, V. A., Ural'ceva, N. N.: *Linear and Quasi-Linear Equations of Parabolic Type.* Amer. Math. Soc. Transl., Vol. **23**, Providence, RI, 1968

[21] Li, Y., Li, Y.: *Finite-time blow-up in higher dimensional fully-parabolic chemotaxis system for two species.* Nonlin. Anal. **109**, 72-84 (2014)

[22] Lieberman, G.: *Second order parabolic differential equations.* World Scientific Publishing, River Edge, NJ, 1996

[23] Liu, D., Tao, Y.: *Boundedness in a chemotaxis system with nonlinear signal production.* Appl. Math. J. Chinese Univ. Ser. B **31**, 379-388 (2016)

[24] Luckhaus, S., Sugiyama, Y., Velázquez, J.J.L.: *Measure valued solutions of the 2D Keller-Segel system.* Arch. Ration. Mech. Anal. **206**, 31-80 (2012)

[25] Mizoguchi, N., Souplet, Ph.: *Nondegeneracy of blow-up points for the parabolic Keller-Segel system.* Ann. Inst. H. Poincaré Anal. Non Linéaire **31**, 851-875 (2014)

[26] Mizoguchi, N., Winkler, M.: *Blow-up in the two-dimensional parabolic Keller-Segel system.* Preprint

[27] Nagai, T.: *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains.* J. Inequal. Appl. **6**, 37-55 (2001)

[28] Nagai, T., Senba, T., Suzuki, T.: *Chemotactic collapse in a parabolic system of mathematical biology.* Hiroshima Math. J. **30**, 463-497 (2000)

[29] Nagai, T., Senba, T., Yoshida, K.: *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis.* Funkc. Ekvacioj, Ser. Int. **40**, 411-433 (1997)

[30] Osaki, K., Yagi, A.: *Finite dimensional attractor for one-dimensional Keller-Segel equations,* Funkcialaj Ekvacioj **44**, 441 - 469 (2001)

[31] Porzio, M.M., Vespri, V.: *Holder Estimates for Local Solutions of Some Doubly Nonlinear Degenerate Parabolic Equations,* J. Differential Eq. **103**, 146-178 (1993)

[32] Raphaël, P., Schweyer, R.: *On the stability of critical chemotactic aggregation.* Math. Ann. **359**, 267-377 (2014)

[33] Schweyer, R.: *Stable blow-up dynamic for the parabolic-parabolic Patlak-Keller-Segel model.* `arXiv:1403.4975`

[34] Senba, T., Suzuki, T.: *Local and norm behavior of blowup solutions to a parabolic system of chemotaxis.* J. Korean Math. Soc. **37**, 929-941 (2000)

[35] Senba, T., Suzuki, T.: *Weak solutions to a parabolic-elliptic system of chemotaxis.* J. Funct. Anal. **191**, 17-51 (2002)

[36] Souplet, Ph., Winkler, M.: *Blow-up profiles for the parabolic-elliptic Keller-Segel system in dimensions $n \geq 3$.* Comm. Math. Phys., to appear

[37] Tao, Y., Winkler, M.: *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity.* J. Differential Eq. **252** (1), 692-715 (2012)

[38] Tao, Y., Winkler, M.: *Critical mass for infinite-time aggregation in a chemotaxis model with indirect signal production.* J. European Math. Soc., to appear

[39] Winkler, M.: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Equations **248**, 2889-2905 (2010)

[40] Winkler, M.: *Does a 'volume-filling effect' always prevent chemotactic collapse?* Mathematical Methods in the Applied Sciences **33**, 12-24 (2010)

[41] Winkler, M: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* Journal de Mathématiques Pures et Appliquées **100**, 748-767 (2013), `arXiv:1112.4156v1`