# Boundedness in a chemotaxis-May-Nowak model for virus dynamics with mildly saturated chemotactic sensitivity 

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#### Abstract

This work is concerned with an extension of the May-Nowak ODE model for virus dynamics to cases in which diffusive motion of cells, as well as cross-diffusive movement of healthy cells toward infected individuals becomes relevant. Specifically, the parabolic system $$
\left\{\begin{array}{l} u_{t}=\Delta u-\nabla \cdot(u f(u) \nabla v)-u w+\kappa-u \\ v_{t}=\Delta v-v+u w \\ w_{t}=\Delta w-w+v \end{array}\right.
$$ is considered for $\kappa \geq 0$ and $f \in C^{2}([0, \infty))$ generalizing the prototypical chemotactic sensitivity function given by $f(s)=(1+s)^{-\alpha}, s \geq 0$, for suitable $\alpha \in \mathbb{R}$. The main results assert global existence of bounded classical solutions to a corresponding no-flux initial-boundary value problem in smoothly bounded $n$-dimensional domains whenever $n \leq 3$ and $$
\alpha> \begin{cases}-1 & \text { if } n=1 \\ \frac{n-2}{n-1} & \text { if } n \in\{2,3\}\end{cases}
$$

In particular, this shows that in the case $n \leq 2$, a respective condition on $\alpha$, known to be essentially optimal with regard to global boundedness for classical Keller-Segel systems with corresponding taxis saturation, remains unchanged even in the context of the superlinear - and hence potentially more destabilizing - signal production mechanism in ( $\star$ ).


Key words: May-Nowak model; virus dynamics; global existence
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## 1 Introduction

As is well known from various studies comparing theory with experiments, the dynamics of virus infections can quite adequately be described with regard to fundamental features by the renowned May-Nowak ODE system ([14])

$$
\left\{\begin{array}{l}
u_{t}=-d_{1} u-\beta u w+\kappa  \tag{1.1}\\
v_{t}=-d_{2} v+\beta u w \\
w_{t}=-d_{3} w+k v
\end{array}\right.
$$

for the unknown population sizes $u=u(t)$ and $v=v(t)$ of healthy and infected cells, respectively, and and the total number $w=w(t)$ of virus particles. In fact, at a spatially global level already this model captures some essential characteristics of infections, such as the dependence of their occurrence, in the sense of convergence to a unique positive equilibrium, on the size of the so-called basic reproduction number $\frac{\beta k \kappa}{d_{1} d_{2} d_{3}}$ relative to the threshold value 1 therefor ([14], cf. also [15]).
Beyond this, however, spatial effects such as the formation of infection hotspots, have been found to be relevant in several biological contexts; here besides random diffusion, certain directed migration mechanisms, especially attraction of target cells by concentration gradients of cytokines from inflammations at sites of infection, seem to play a major role and have accordingly inspired refined modeling ([12], [6], [13], [11]). In line with this, the present work is concerned with a class of correspondingly adapted spatio-temporal variants of (1.1), following the approach in [16] and thus retaining the substantial features of the kinetics in (1.1) but additionally accounting for diffusion in all components as well as a taxis-type cross-diffusive motion of healthy individuals toward increasing concentrations of infected cells. More precisely, we shall be concerned with the initial-boundary value problem

$$
\begin{cases}u_{t}=\Delta u-\nabla \cdot(u f(u) \nabla v)-u w+\kappa-u, & x \in \Omega, t>0  \tag{1.2}\\ v_{t}=\Delta v-v+u w, & x \in \Omega, t>0 \\ w_{t}=\Delta w-w+v, & x \in \Omega, t>0 \\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=\frac{\partial w}{\partial \nu}=0, & x \in \partial \Omega, t>0 \\ u(x, 0)=u_{0}(x), \quad v(x, 0)=v_{0}(x), \quad w(x, 0)=w_{0}(x), & x \in \Omega\end{cases}
$$

in a bounded domain $\Omega \subset \mathbb{R}^{n}, n \geq 1$, where the parameter $\kappa$ is primarily thought of as being positive, but may actually be any nonnegative number, where moreover

$$
\begin{equation*}
f \in C^{2}([0, \infty)) \tag{1.3}
\end{equation*}
$$

is such that

$$
\begin{equation*}
|f(s)| \leq K_{f}(1+s)^{-\alpha} \quad \text { for all } s \geq 0 \tag{1.4}
\end{equation*}
$$

and where

$$
\begin{equation*}
u_{0} \in C^{0}(\bar{\Omega}), v_{0} \in W^{1, \infty}(\Omega) \text { and } w_{0} \in C^{0}(\bar{\Omega}) \quad \text { are nonnegative. } \tag{1.5}
\end{equation*}
$$

Here it should be noted that with regard to the cross-diffusive interaction, (1.2) on the one hand shares some essential properties with the classical Keller-Segel model for bacterial chemotaxis ([10]), but on the other hand seems to incorporate some even stronger destabilizing potential than the latter: In
fact, in contrast to the corresponding version of the Keller-Segel system, with the evolution equations therein given by

$$
\left\{\begin{array}{lll}
u_{t}=\Delta u-\nabla \cdot(u f(u) \nabla v), & & x \in \Omega, t>0  \tag{1.6}\\
v_{t}=\Delta v-v+u, & & x \in \Omega, t>0
\end{array}\right.
$$

the respective mechanism in (1.2) is potentially enhanced through the presence of the additional factor $w$, not necessarily known to be bounded from any obvious a priori information, in the production term $u w$ in the second equation. Such nonlinear signal production mechanisms, especially when being of essentially superlinear type as in the present situation, have apparently been addressed only quite rudimentarily in the literature yet; providing some further step toward their understanding, here in the application-oriented context of the particular model (1.2), forms a main motivation for the present study.
In order to put our precise question in perspective, let us recall that e.g. in the case when $f \equiv 1$ in (1.6), the seemingly most striking feature of the so-called minimal Keller-Segel system thereby obtained consists in its ability to enforce spontaneous formation of singular spatial structures in the spirit of finite-time blow-up of some solutions whenever the spatial dimension satisfies $n \geq 2$ ([7], [21]). In fact, similar features are shared by (1.6) even for some more general chemotactic sensitivity functions $f$, provided that possible saturation effects of tactic migration thereby included are sufficiently weak in the sense that $f(s)$ does not decay too fast as $s \rightarrow \infty$. In the prototypical case when $f(s)=(1+s)^{-\alpha}$ with some $\alpha \in \mathbb{R}$, for instance, it is known from e.g. [3], [20] and [8] that whenever $n \geq 2$ and $\alpha<\frac{n-2}{n}$, the Neumann problem for (1.6) in balls possesses some radial unbounded solutions; for a corresponding parabolic-elliptic simplification of (1.6), this result can actually be extended so as to be valid for arbitrary $n \geq 1$ ([4]). That the exponent $\frac{n-2}{n}$ indeed marks a genuine threshold is indicated by complementing results on global existence of bounded classical solutions for widely arbitrary initial data if

$$
\begin{equation*}
\alpha>\frac{n-2}{n} \tag{1.7}
\end{equation*}
$$

and $n \geq 1$ ([8], [17], [4]).
Main results. The question to which extent the condition (1.7) for global existence and boundedness, thus essentially optimal for Keller-Segel systems of the form (1.6), is affected by passing over to the system (1.2), and especially by replacing the linear signal production from (1.6) with the superlinear mechanism in the latter, has recently been addressed in [9]: Still for $f(s)=(1+s)^{-\alpha}, s \geq 0$, the conditions

$$
\alpha> \begin{cases}\frac{1}{2}+\frac{n^{2}}{6 n+4} & \text { when } 1 \leq n \leq 4  \tag{1.8}\\ \frac{n}{4} & \text { when } n \geq 5\end{cases}
$$

respectively, have there been found to ensure global existence and boundedness in (1.2) for suitably regular initial data.
The purpose of the present work consists in developing an approach which substantially differs from that pursued in [9] and also from that in the work [2] addressing a variant of (1.2) with the crucial term $u w$ replaced by $\frac{u w}{1+u+w}$, and aims at reducing the size of the considerable gap between (1.8) and (1.7). Indeed, at least in all physically relevant frameworks it thereby becomes possible to achieve some improvement; inter alia, our results will show that in the case $n \leq 2$, (1.7) will remain sufficient to rule out any blow-up phenomenon:

Theorem 1.1 Let $n \in\{1,2,3\}$ and $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with smooth boundary, let $\kappa \geq 0$, and suppose that (1.3) and (1.4) are valid with some $K_{f}>0$ and $\alpha \in \mathbb{R}$ such that

$$
\alpha> \begin{cases}-1 & \text { if } n=1  \tag{1.9}\\ \frac{n-2}{n-1} & \text { if } n \in\{2,3\}\end{cases}
$$

Then for any choice of $\left(u_{0}, v_{0}, w_{0}\right)$ complying with (1.5), there exist nonnegative functions $u, v$ and $w$ on $\bar{\Omega} \times[0, \infty)$, uniquely determined by the inclusions

$$
\left\{\begin{array}{l}
u \in C^{0}(\bar{\Omega} \times[0, \infty)) \cap C^{2,1}(\bar{\Omega} \times(0, \infty)) \\
v \in \bigcap_{q>n} C^{0}\left([0, \infty) ; W^{1, q}(\Omega)\right) \cap C^{2,1}(\bar{\Omega} \times(0, \infty)) \quad \text { and } \\
w \in C^{0}(\bar{\Omega} \times[0, \infty)) \cap C^{2,1}(\bar{\Omega} \times(0, \infty))
\end{array}\right.
$$

which form a global classical solution of (1.2). Moreover, for each $q>n$ there exists $C(q)>0$ such that

$$
\begin{equation*}
\|u(\cdot, t)\|_{L^{\infty}(\Omega)}+\|v(\cdot, t)\|_{W^{1, q}(\Omega)}+\|w(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C(q) \quad \text { for all } t>0 \tag{1.10}
\end{equation*}
$$

An interesting question left open here concerns optimality of (1.9) with regard to the conclusion under consideration. Due to the significantly more complex structure of (1.2) as compared to (1.6), apparently not allowing for any meaningful energy-like feature in the style of those forming powerful ingredients to the corresponding analysis for (1.6), addressing this issue through the construction of unbounded solutions for suitably small $\alpha$ will apparently require the design of substantially novel approaches and thereby go beyond the scope of the present work.
Plan of the paper. Besides a standard result on local solvability, up to a maximal existence time $T_{\max } \leq \infty$, Section 2 will contain the very basic observation, already used in [9], that due to the nonlinear absorption term in the first equation therein, as a weak but unconditional a priori regularity feature the system (1.2) enjoys some boundedness property for the total mass of all its solution components. Through appropriate interpolation, the accordingly obtained $L^{1}$ bounds especially for $u$ and $v$ will thereafter be seen in Section 3 to imply, in the case $n \geq 2$, some inequalities relating certain powers of the quantities

$$
\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{p}(\Omega)}, \quad \sup _{s \in(0, t)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)} \quad \text { and } \quad \sup _{s \in(0, t)}\|w(\cdot, s)\|_{L^{\infty}(\Omega)}, \quad t \in\left(0, T_{\max }\right)
$$

to each other, provided that the parameters $p$ and $q$ therein lie within appropriate ranges. In Section 4 we shall then see that when $n \in\{2,3\}$, these inequalities can suitably be combined so as to yield uniform bounds for all these quantities under the respective condition in (1.9), thus implying global extensibility and boundedness in the claimed flavor. The case $n=1$ will finally be seen to be successfully treatable in essentially the same but, in comparison to the above, quite direct manner.

## 2 Preliminaries: Local existence and $L^{1}$ bounds

The following basic statement on local solvability can be derived by straightforward adaptation of well-established techniques to the present setting; details in corresponding results concerning related taxis-type problems can be found e.g. in [8] and in [19] and thus may be omitted here (cf. also the general theory developed in [1]).

Lemma 2.1 Let $n \geq 1$ and $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with smooth boundary, and assume that $f, u_{0}, v_{0}$ and $w_{0}$ satisfy (1.3) and (1.5). Then there exist $T_{\max } \in(0, \infty]$ and a uniquely determined triple $(u, v, w)$ of nonnegative functions

$$
\left\{\begin{array}{l}
u \in C^{0}\left(\bar{\Omega} \times\left[0, T_{\max }\right)\right) \cap C^{2,1}\left(\bar{\Omega} \times\left(0, T_{\max }\right)\right), \\
v \in \bigcap_{q>n} C^{0}\left(\left[0, T_{\max }\right) ; W^{1, q}(\Omega)\right) \cap C^{2,1}\left(\bar{\Omega} \times\left(0, T_{\max }\right)\right) \quad \text { and } \\
w \in C^{0}\left(\bar{\Omega} \times\left[0, T_{\max }\right)\right) \cap C^{2,1}\left(\bar{\Omega} \times\left(0, T_{\max }\right)\right),
\end{array}\right.
$$

such that $(u, v, w)$ is a classical solution of (1.2) in $\Omega \times\left(0, T_{\max }\right)$, and that
either $T_{\max }<\infty \quad$ or $\quad \limsup _{t \nearrow T_{\max }}\left\{\|u(\cdot, t)\|_{L^{\infty}(\Omega)}\|v(\cdot, t)\|_{W^{1, q}(\Omega)}+\|w(\cdot, t)\|_{L^{\infty}(\Omega)}\right\}=\infty \quad$ for all $q>n$.

A first elementary but important structural property of (1.2), strongly relying on the absorptive contribution $-u w$ to the first equation therein, asserts boundedness of all three relevant total mass functionals as follows.

Lemma 2.2 Let $n \geq 1$ and $\kappa \geq 0$, and assume (1.3) and (1.5). Then there exists $C>0$ such that

$$
\begin{equation*}
\|u(\cdot, t)\|_{L^{1}(\Omega)}+\|v(\cdot, t)\|_{L^{1}(\Omega)}+\|w(\cdot, t)\|_{L^{1}(\Omega)} \leq C \quad \text { for all } t \in\left(0, T_{\text {max }}\right) \tag{2.2}
\end{equation*}
$$

Proof. Using (1.2), for $t \in\left(0, T_{\max }\right)$ we compute

$$
\frac{d}{d t}\left\{\int_{\Omega} u+\int_{\Omega} v+\frac{1}{2} \int_{\Omega} w\right\}+\frac{1}{2}\left\{\int_{\Omega} u+\int_{\Omega} v+\frac{1}{2} \int_{\Omega} w\right\}=\kappa|\Omega|-\frac{1}{2} \int_{\Omega} u-\frac{1}{4} \int_{\Omega} w \leq \kappa|\Omega|
$$

from which (2.2) readily results upon direct integration.

## 3 Relationships between bounds for $u, \nabla v$ and $w$ in the case $n \geq 2$

Next approaching the core of our analysis, by applying appropriate smoothing estimates for the Neumann heat semigroup to each of the three evolution problems in (1.2) separately, in this section we will establish some relationships between the quantities

$$
\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{p}(\Omega)}, \quad \sup _{s \in(0, t)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)} \quad \text { and } \quad \sup _{s \in(0, t)}\|w(\cdot, s)\|_{L^{\infty}(\Omega)}, \quad t \in\left(0, T_{\max }\right)
$$

for suitably wide ranges of the free parameters $p \in(1, \infty]$ and $q \in(1, \infty)$ when $n \geq 2$. In the subsequent section, the accordingly obtained a priori estimates will be applied twice, namely firstly for certain suitably small values of $p$ and $q$ to ensure boundedness of $w$, and thereafter to $p:=\infty$ and large $q$ so as to provide $L^{\infty}$ bounds for $u$ and corresponding $L^{q}$ estimates for $\nabla v$.
To begin with, let us draw the following conclusion on presupposed $L^{q}$ regularity of $\nabla v$ on $L^{\infty}$ bounds for $w$, relying on the spatial dimension not only through the regularizing action of the heat kernel but also through a Gagliardo-Nirenberg interpolation involving the $L^{1}$ bound for $v$ available due to Lemma 2.2.

Lemma 3.1 Assume that $n \geq 2$ and $\kappa \geq 0$, that (1.3) and (1.5) hold, and that $q>\max \left\{1, \frac{n}{3}\right\}$. Then for any $\varepsilon>0$ one can find $C(\varepsilon, q)>0$ such that

$$
\begin{equation*}
\|w(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C(\varepsilon, q)+C(\varepsilon, q) \cdot\left\{\sup _{s \in(0, t)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)}\right\}^{\frac{n-2}{n+1-\frac{\pi}{q}}+\varepsilon} \quad \text { for all } t \in\left(0, T_{\text {max }}\right) . \tag{3.1}
\end{equation*}
$$

Proof. Since $q>\frac{n}{3}$, without loss of generality we may assume that apart from $\left(n+1-\frac{n}{q}\right) \varepsilon<2$ the inequality $\left(n+1-\frac{n}{q}\right) q \varepsilon<3 q-n$ holds, so that

$$
r \equiv r(\varepsilon, q):=\frac{n}{2-\left(n+1-\frac{n}{q}\right) \varepsilon}
$$

is a positive number satisfying $r>\frac{n}{2} \geq 1$ as well as

$$
\frac{(n-q) r}{n}=\frac{n-q}{2-\left(n+1-\frac{n}{q}\right) \varepsilon}<\frac{n-q}{2-\frac{3 q-n}{q}}=q .
$$

Therefore, the Gagliardo-Nirenberg inequality provides $c_{1}=c_{1}(\varepsilon, q)>0$ such that with $a \equiv a(\varepsilon, q):=$ $\frac{n-\frac{n}{r}}{n+1-\frac{n}{q}} \in(0,1)$ we have

$$
\begin{equation*}
\|\varphi\|_{L^{r}(\Omega)} \leq c_{1}\|\nabla \varphi\|_{L^{q}(\Omega)}^{a}\|\varphi\|_{L^{1}(\Omega)}^{1-a}+c_{1}\|\varphi\|_{L^{1}(\Omega)} \quad \text { for all } \varphi \in W^{1, q}(\Omega) \tag{3.2}
\end{equation*}
$$

and moreover we can employ well-known smoothing estimates for the Neumann heat semigroup $\left(e^{t \Delta}\right)_{t \geq 0}$ on $\Omega([18])$ to find $c_{2}=c_{2}(\varepsilon, q)>0$ fulfilling

$$
\begin{equation*}
\left\|e^{t \Delta} \varphi\right\|_{L^{\infty}(\Omega)} \leq c_{2}\left(1+t^{-\frac{n}{2 r}}\right)\|\varphi\|_{L^{r}(\Omega)} \quad \text { for all } t>0 \text { and any } \varphi \in L^{r}(\Omega) \tag{3.3}
\end{equation*}
$$

As Lemma 2.2 implies that with some $c_{3}>0$ we have $\|v(\cdot, t)\|_{L^{1}(\Omega)} \leq c_{3}$ for all $t \in\left(0, T_{\text {max }}\right)$, based on a variation-of-constants representation we can combine (3.3) with (3.2) to see that thanks to the maximum principle,

$$
\begin{aligned}
\|w(\cdot, t)\|_{L^{\infty}(\Omega)}= & \left\|e^{t(\Delta-1)} w_{0}+\int_{0}^{t} e^{(t-s)(\Delta-1)} v(\cdot, s) d s\right\|_{L^{\infty}(\Omega)} \\
\leq & e^{-t}\left\|w_{0}\right\|_{L^{\infty}(\Omega)}+c_{2} \int_{0}^{t}\left(1+(t-s)^{-\frac{n}{2 r}}\right) e^{-(t-s)}\|v(\cdot, s)\|_{L^{r}(\Omega)} d s \\
\leq & \left\|w_{0}\right\|_{L^{\infty}(\Omega)}+c_{1} c_{2} \int_{0}^{t}\left(1+(t-s)^{-\frac{n}{2 r}}\right) e^{-(t-s)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)}^{a}\|v(\cdot, s)\|_{L^{1}(\Omega)}^{1-a} d s \\
& +c_{1} c_{2} \int_{0}^{t}\left(1+(t-s)^{-\frac{n}{2 r}}\right) e^{-(t-s)}\|v(\cdot, s)\|_{L^{1}(\Omega)} d s \\
\leq & \left\|w_{0}\right\|_{L^{\infty}(\Omega)}+\left\{c_{1} c_{2} c_{3}^{1-a}\|\nabla v\|_{L^{\infty}\left((0, t) ; L^{q}(\Omega)\right)}^{a}+c_{1} c_{2} c_{3}\right\} \cdot \int_{0}^{t}\left(1+(t-s)^{-\frac{n}{2 r}}\right) e^{-(t-s)} d s
\end{aligned}
$$

for all $t \in\left(0, T_{\max }\right)$. Since

$$
\int_{0}^{t}\left(1+(t-s)^{-\frac{n}{2 r}}\right) e^{-(t-s)} d s \leq c_{4} \equiv c_{4}(\varepsilon, q):=\int_{0}^{\infty}\left(1+\sigma^{-\frac{n}{2 r}}\right) e^{-\sigma} d \sigma \quad \text { for all } t>0
$$

with $c_{4}$ being finite according to the inequality $r>\frac{n}{2}$, this already entails (3.1) due to the fact that

$$
a=\frac{n-\left[2-\left(n+1-\frac{n}{q}\right) \varepsilon\right]}{n+1-\frac{n}{q}}=\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon
$$

by definition of $a$ and $r$.
An argument of a similar flavor yields the following yet quite general statement on regularity of $\nabla v$ in dependence on $L^{p}$ bounds for $u$ and $L^{\infty}$ bounds for $w$.

Lemma 3.2 Let $n \geq 2$ and $\kappa \geq 0$, assume (1.3) and (1.5), and let $p \in(1, \infty]$ and $q>\frac{n}{n-1}$ be such that $(n-p) q<n p$. Then for each $\varepsilon>0$ one can find $C(\varepsilon, p, q)>0$ with the property that

$$
\begin{equation*}
\|\nabla v(\cdot, t)\|_{L^{q}(\Omega)} \leq C(\varepsilon, p, q)+C(\varepsilon, p, q) \cdot\left\{1+\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{q}(\Omega)}\right\}^{\frac{n-1-\frac{n}{q}}{n\left(1-\frac{1}{p}\right)}+\varepsilon} \cdot \sup _{s \in(0, t)}\|w(\cdot, s)\|_{L^{\infty}(\Omega)} \tag{3.4}
\end{equation*}
$$

for all $t \in\left(0, T_{\max }\right)$.
Proof. Since $(n-p) q<n p$ and thus $\frac{1}{q}+\frac{1}{n}-\frac{1}{p}>0$, by diminishing $\varepsilon$ if necessary we may assume that besides the inequality $\left(1-\frac{1}{p}\right) \varepsilon<\frac{1}{n}, \varepsilon$ satisfies $\left(1-\frac{1}{p}\right) \varepsilon<\frac{1}{q}+\frac{1}{n}-\frac{1}{p}$. Here the former property ensures that

$$
\lambda \equiv \lambda(\varepsilon, p, q):=\frac{1}{\frac{1}{q}+\frac{1}{n}-\left(1-\frac{1}{p}\right) \varepsilon}
$$

is a well-defined positive number satisfying $\lambda<q$, and thanks to the latter we moreover know that $\lambda<p$. As furthermore

$$
\begin{equation*}
\lambda>\frac{1}{\frac{1}{q}+\frac{1}{n}}>1 \tag{3.5}
\end{equation*}
$$

due to the condition $q>\frac{n}{n-1}$, we may invoke known smoothing properties of the Neumann heat semigroup to fix $c_{1}=c_{1}(q)>0$ and $c_{2}=c_{2}(\varepsilon, p, q)>0$ such that for all $t>0$, in addition to

$$
\left\|\nabla e^{t \Delta} \varphi\right\|_{L^{q}(\Omega)} \leq c_{1}\|\varphi\|_{W^{1, \infty}(\Omega)} \quad \text { for all } \varphi \in W^{1, \infty}(\Omega)
$$

we also have

$$
\left\|\nabla e^{t \Delta} \varphi\right\|_{L^{q}(\Omega)} \leq c_{2}\left(1+t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\lambda}-\frac{1}{q}\right)}\right)\|\varphi\|_{L^{\lambda}(\Omega)} \quad \text { for all } \varphi \in L^{\lambda}(\Omega)
$$

Using a Duhamel representation, for all $t \in\left(0, T_{\max }\right)$ we can therefore estimate

$$
\begin{align*}
\|\nabla v(\cdot, t)\|_{L^{q}(\Omega)}= & \left\|\nabla e^{t(\Delta-1)} v_{0}+\int_{0}^{t} \nabla e^{(t-s)(\Delta-1)} u(\cdot, s) w(\cdot, s) d s\right\|_{L^{q}(\Omega)} \\
\leq & c_{1} e^{-t}\left\|v_{0}\right\|_{W^{1, \infty}(\Omega)} \\
& +c_{2} \int_{0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\lambda}-\frac{1}{q}\right)}\right) e^{-(t-s)}\|u(\cdot, s) w(\cdot, s)\|_{L^{\lambda}(\Omega)} d s \tag{3.6}
\end{align*}
$$

where by the Hölder inequality, applicable since $\lambda<p$,

$$
\begin{aligned}
\|u(\cdot, s) w(\cdot, s)\|_{L^{\lambda}(\Omega)} & \leq\|u(\cdot, s)\|_{L^{p}(\Omega)}^{a}\|u(\cdot, s)\|_{L^{1}(\Omega)}^{1-a}\|w(\cdot, s)\|_{L^{\infty}(\Omega)} \\
& \leq c_{3}^{1-a}\|u(\cdot, s)\|_{L^{p}(\Omega)}^{a}\|w(\cdot, s)\|_{L^{\infty}(\Omega)} \quad \text { for all } s \in\left(0, T_{\text {max }}\right)
\end{aligned}
$$

with $a \equiv a(\varepsilon, p, q):=\frac{1-\frac{1}{\lambda}}{1-\frac{1}{p}} \in(0,1)$, and with $c_{3}:=\sup _{t \in\left(0, T_{\text {max }}\right)}\|u(\cdot, t)\|_{L^{1}(\Omega)}$ being finite according to Lemma 2.2.
In consequence, (3.6) thus shows that for all $t \in\left(0, T_{\max }\right)$,

$$
\begin{aligned}
\|\nabla v(\cdot, t)\|_{L^{q}(\Omega)} \leq c_{1}\left\|v_{0}\right\|_{W^{1, \infty}(\Omega)}+c_{2} c_{3}^{1-a} & \cdot\left\{\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{p}(\Omega)}\right\}^{a} \cdot\left\{\sup _{s \in(0, t)}\|w(\cdot, s)\|_{L^{\infty}(\Omega)}\right\} \times \\
& \times \int_{0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\lambda}-\frac{1}{q}\right)}\right) e^{-(t-s)} d s,
\end{aligned}
$$

whence noting that for all $t>0$ we have

$$
\int_{0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\lambda}-\frac{1}{q}\right)}\right) e^{-(t-s)} d s \leq c_{4} \equiv c_{4}(\varepsilon, p, q):=\int_{0}^{\infty}\left(1+\sigma^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\lambda}-\frac{1}{q}\right)}\right) e^{-\sigma} d \sigma
$$

that $c_{4}<\infty$ due to the inequality $\frac{1}{\lambda}<\frac{1}{q}+\frac{1}{n}$ contained in (3.5), and that moreover

$$
a=\frac{1-\left\{\frac{1}{q}+\frac{1}{n}-\left(1-\frac{1}{p}\right) \varepsilon\right\}}{1-\frac{1}{p}}=\frac{n-1-\frac{n}{q}}{n\left(1-\frac{1}{p}\right)}+\varepsilon
$$

we may conclude as intended.
Combining the previous two lemmata allows us to eliminate the dependence on $w$ in (3.4) as follows.
Lemma 3.3 Assume that $n \geq 2$, that $\kappa \geq 0$, and that (1.3) and (1.5) hold, and suppose that $p \in(1, \infty]$ and $q>\frac{n}{n-1}$ satisfy $q>\frac{n}{3}$ and $(n-p) q<n p$. Then for all $\varepsilon>0$ there exists $C(\varepsilon, p, q)>0$ such that

$$
\begin{equation*}
\|\nabla v(\cdot, t)\|_{L^{q}(\Omega)} \leq C(\varepsilon, p, q) \cdot\left\{1+\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{p}(\Omega)}\right\}^{\frac{\left(n+1-\frac{n}{q}\right)\left(n-1-\frac{n}{q}\right)}{n\left(1-\frac{1}{p}\right)\left(3-\frac{n}{q}\right)}+\varepsilon} \quad \text { for all } t \in\left(0, T_{\text {max }}\right) \tag{3.7}
\end{equation*}
$$

Proof. Let us firstly note that $n+1-\frac{n}{q}>n-2$ due to our assumption that $q>\frac{n}{3}$, and that $n-1-\frac{n}{q}>0$ since $q>\frac{n}{n-1}$. Accordingly, there exists $\varepsilon_{\star}=\varepsilon_{\star}(p, q)>0$ such that

$$
\theta\left(\varepsilon_{1}\right):=\left\{\frac{n-1-\frac{n}{q}}{n\left(1-\frac{1}{p}\right)}+\varepsilon_{1}\right\} \cdot \frac{n+1-\frac{n}{q}}{\left(n+1-\frac{n}{q}\right)\left(1-\varepsilon_{1}\right)-(n-2)}
$$

is well-defined for all $\varepsilon_{1} \in\left(0, \varepsilon_{\star}\right)$, with

$$
\theta\left(\varepsilon_{1}\right) \rightarrow \theta_{0}:=\frac{\left(n-1-\frac{n}{q}\right)\left(n+1-\frac{n}{q}\right)}{n\left(1-\frac{1}{p}\right)\left(3-\frac{n}{q}\right)} \quad \text { as } \varepsilon_{1} \searrow 0 \text {. }
$$

Given $\varepsilon>0$, we can thus find $\varepsilon_{1}=\varepsilon_{1}(\varepsilon, p, q) \in\left(0, \varepsilon_{\star}\right)$ such that

$$
\begin{equation*}
\theta\left(\varepsilon_{1}\right) \leq \theta_{0}+\varepsilon, \tag{3.8}
\end{equation*}
$$

and then from Lemma 3.1 and Lemma 3.2 infer the existence of $c_{1}=c_{1}(\varepsilon, q)>0$ and $c_{2}=c_{2}(\varepsilon, p, q)>0$ such that

$$
L(t):=1+\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{p}(\Omega)}, \quad t \in\left(0, T_{\text {max }}\right),
$$

and

$$
M(t):=\sup _{s \in(0, t)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)}, \quad t \in\left(0, T_{\max }\right),
$$

as well as

$$
N(t):=\sup _{s \in(0, t)}\|w(\cdot, s)\|_{L^{\infty}(\Omega)}, \quad t \in\left(0, T_{\max }\right),
$$

satisfy

$$
\begin{equation*}
N(t) \leq c_{1}+c_{1} M^{\frac{n-2}{n+1-\frac{\pi}{q}}+\varepsilon_{1}}(t) \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
M(t) \leq c_{2}+c_{2} L^{\frac{n-1-\frac{n}{q}}{n\left(1-\frac{1}{p}\right)}+\varepsilon_{1}}(t) \cdot N(t) \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{3.10}
\end{equation*}
$$

In the case when $t \in\left(0, T_{\text {max }}\right)$ is such that $M(t) \geq 1$, from (3.9) we thus obtain that

$$
N(t) \leq 2 c_{1} M^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon_{1}}(t)
$$

and that thus, by (3.10),

$$
\begin{aligned}
& M(t) \leq c_{2}+2 c_{1} c_{2} L^{\frac{n-1-\frac{n}{q}}{n\left(1-\frac{1}{p}\right)}+\varepsilon_{1}}(t) M^{\frac{n-2}{n+1-\frac{n}{q}}+\varepsilon_{1}}(t) \\
& \leq\left(c_{2}+2 c_{1} c_{2}\right) L^{\frac{n-1-\frac{n}{q}}{n\left(1-\frac{p}{p}\right)}} \varepsilon_{1} \\
&(t) M^{\frac{n-2}{n+1-\frac{\pi}{q}}+\varepsilon_{1}}(t),
\end{aligned}
$$

because $L(t) \geq 1$ by definition. Since for any such $t$ we therefore have

$$
M^{1-\varepsilon_{1}-\frac{n-2}{n+1-\frac{n}{q}}}(t) \leq\left(c_{2}+2 c_{1} c_{2}\right) L^{\frac{n-1-\frac{n}{n}}{n\left(1-\frac{1}{p}\right)}+\varepsilon_{1}}(t),
$$

and since

$$
1-\varepsilon_{1}-\frac{n-2}{n+1-\frac{n}{q}}=\frac{\left(n+1-\frac{n}{q}\right)\left(1-\varepsilon_{1}\right)-(n-2)}{n+1-\frac{n}{q}}>0
$$

by positivity of $\theta\left(\varepsilon_{1}\right)$, from this we readily infer that actually for arbitrary $t \in\left(0, T_{\text {max }}\right)$, regardless of the sign of $M(t)-1$,

$$
M(t) \leq c_{3} L^{\theta\left(\varepsilon_{1}\right)}(t)
$$

with $c_{3} \equiv c_{3}(\varepsilon, p, q):=\max \left\{1,\left(c_{2}+2 c_{1} c_{2}\right)^{\frac{n+1-\frac{n}{q}}{\left(n+1-\frac{n}{q}\right)\left(1-\varepsilon_{1}\right)-(n-2)}}\right\}>0$. Once again since $L(t) \geq 0$ for all $t \in\left(0, T_{\text {max }}\right)$, in view of (3.8) this establishes (3.7).
Independently of the previous three lemmata, again utilizing parabolic smoothing but now also relying on (1.4) we can quantify the influence of supposedly available $L^{q}$ bounds for $\nabla v$ on $L^{p}$ regularity of $u$, provided that $p$ lies above some threshold.

Lemma 3.4 Let $n \geq 2$ and $\kappa \geq 0$, and suppose that (1.3), (1.4) and (1.5) are valid with some $K_{f}>0$ and $\alpha \in(0,1)$. Then whenever $p \in\left(\frac{n}{n-1}, \infty\right]$ and

$$
\begin{equation*}
q>\frac{1}{\frac{1}{n}+\frac{\alpha}{p}}, \tag{3.11}
\end{equation*}
$$

for all $\varepsilon>0$ there exists $C(\varepsilon, p, q)>0$ such that

$$
\begin{equation*}
\|u(\cdot, t)\|_{L^{p}(\Omega)} \leq C(\varepsilon, p, q)+C(\varepsilon, p, q) \cdot\left\{\sup _{s \in(0, t)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)}\right\}^{\frac{1-\frac{1}{p}}{\alpha+\frac{1}{n}-\frac{1}{q}}+\varepsilon} \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{3.12}
\end{equation*}
$$

Proof. We first observe that (3.11) entails that necessarily $q>1$, because $\frac{1}{n}+\frac{\alpha}{p}<\frac{1}{n}+\frac{1}{p}<1$ due to the assumptions that $\alpha<1$ and that $p>\frac{n}{n-1}$. Apart from that, (3.11) in particular warrants that $\frac{1}{q}<\frac{1}{n}+\frac{\alpha}{p}<\frac{1}{n}+\frac{1}{p}$, so that the interval $J_{1}:=\left(\frac{1}{q}, \frac{1}{n}+\frac{1}{p}\right]$ is not empty and

$$
\phi_{1}(\xi):=\frac{1-\alpha}{\xi-\frac{1}{q}}, \quad \xi \in J_{1}
$$

defines a positive function $\phi_{1}$ on $J_{1}$ which, again by (3.11), satisfies

$$
\begin{equation*}
\frac{\phi_{1}\left(\frac{1}{n}+\frac{1}{p}\right)}{p}=\frac{\frac{1}{p}-\frac{\alpha}{p}}{\frac{1}{n}+\frac{1}{p}-\frac{1}{q}}<\frac{\frac{1}{p}-\frac{\alpha}{p}}{\frac{1}{n}+\frac{1}{p}-\left(\frac{1}{n}+\frac{\alpha}{p}\right)}=1 \tag{3.13}
\end{equation*}
$$

Next, since (3.11) together with the inequality $p \geq 1$ ensures that $\frac{1}{q}<\frac{1}{n}+\alpha$ and hence $\frac{1}{p}+\frac{1}{q}-\alpha<\frac{1}{n}+\frac{1}{p}$, it similarly follows that also $J_{2}:=\left(\frac{1}{p}+\frac{1}{q}-\alpha, \frac{1}{n}+\frac{1}{p}\right] \neq \emptyset$, and that

$$
\phi_{2}(\xi):=\frac{1-\frac{1}{p}}{\alpha-\frac{1}{p}-\frac{1}{q}+\xi}, \quad \xi \in J_{2},
$$

is well-defined and nonnegative with

$$
\begin{equation*}
\phi_{2}\left(\frac{1}{n}+\frac{1}{p}\right)=\frac{1-\frac{1}{p}}{\alpha+\frac{1}{n}-\frac{1}{q}} . \tag{3.14}
\end{equation*}
$$

In view of (3.13), (3.14) and the continuity of $\phi_{1}$ and $\phi_{2}$, we thus see that for any $\varepsilon>0$ it is possible to pick $\xi=\xi(\varepsilon, p, q) \in J_{1} \cap J_{2}$ such that $\xi<\frac{1}{n}+\frac{1}{p}$ and that $\phi_{1}(\xi)<p$ as well as $\phi_{2}(\xi) \leq \frac{1-\frac{1}{p}}{\alpha+\frac{1}{n}-\frac{1}{q}}+\varepsilon$,
where we can clearly moreover achieve that $\xi>\frac{1}{p}$.
Letting $\mu \equiv \mu(\varepsilon, p, q):=\frac{1}{\xi}$, we have thereby found a positive number $\mu$ simultaneously fulfilling

$$
\begin{equation*}
\mu<p, \quad \mu<q, \quad \frac{1}{\mu}>\frac{1}{p}+\frac{1}{q}-\alpha \quad \text { and } \quad \frac{1}{\mu}<\frac{1}{n}+\frac{1}{p} \tag{3.15}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\frac{q \mu(1-\alpha)}{q-\mu}<p \tag{3.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1-\frac{1}{p}}{\alpha-\frac{1}{p}-\frac{1}{q}+\frac{1}{\mu}} \leq \frac{1-\frac{1}{p}}{\alpha+\frac{1}{n}-\frac{1}{q}}+\varepsilon \tag{3.17}
\end{equation*}
$$

where we note that the rightmost property in (3.15) ensures that furthermore $\mu>1$, again because $p>\frac{n}{n-1}$.
Keeping this parameter $\mu$ fixed henceforth, using the first inequality in (3.15) we now again resort to known regularization features of the Neumann heat semigroup ([18, Lemma 1.3], [5, Lemma 3.3]) to pick $c_{1}=c_{1}(\varepsilon, p, q)>0$ satisfying

$$
\left\|e^{t \Delta} \nabla \cdot \varphi\right\|_{L^{p}(\Omega)} \leq c_{1}\left(1-t^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{p}\right)}\right)\|\varphi\|_{L^{\mu}(\Omega)}
$$

for all $t>0$ and each $\varphi \in C^{1}\left(\bar{\Omega} ; \mathbb{R}^{n}\right)$ such that $\varphi \cdot \nu=0$ on $\partial \Omega$,
which when combined with (1.4) shows that for all $t \in\left(0, T_{\max }\right)$,

$$
\begin{align*}
& \int_{0}^{t}\left\|e^{(t-s)(\Delta-1)} \nabla \cdot(u(\cdot, s) f(u(\cdot, s)) \nabla v(\cdot, s))\right\|_{L^{p}(\Omega)} d s \\
& \quad \leq c_{1} \int_{0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{p}\right)}\right) e^{-(t-s)}\|u(\cdot, s) f(u(\cdot, s)) \nabla v(\cdot, s)\|_{L^{\mu}(\Omega)} d s \\
& \quad \leq c_{1} K_{f} \int_{0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{p}\right)}\right) e^{-(t-s)}\left\|u^{1-\alpha}(\cdot, s) \nabla v(\cdot, s)\right\|_{L^{\mu}(\Omega)} d s \tag{3.18}
\end{align*}
$$

Here thanks to the second relation in (3.15), we may employ the Hölder inequality to estimate

$$
\begin{aligned}
\left\|u^{1-\alpha}(\cdot, s) \nabla v(\cdot, s)\right\|_{L^{\mu}(\Omega)} & \leq\left\|u^{1-\alpha}(\cdot, s)\right\|_{L^{\frac{q \mu}{q-\mu}}(\Omega)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)} \\
& =\|u(\cdot, s)\|_{L^{\frac{q \mu(1-\alpha)}{q-\mu}}(\Omega)}^{1-\alpha}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)} \quad \text { for all } s \in\left(0, T_{\max }\right),
\end{aligned}
$$

where using that Lemma 2.2 provides $c_{2}>0$ such that $\|u(\cdot, t)\|_{L^{1}(\Omega)} \leq c_{2}$ for all $t \in\left(0, T_{\max }\right)$, again by the Hölder inequality we see that in the case when $\theta:=\frac{q \mu(1-\alpha)}{q-\mu} \leq 1$,

$$
\begin{equation*}
\left\|u^{1-\alpha}(\cdot, s) \nabla v(\cdot, s)\right\|_{L^{\mu}(\Omega)} \leq|\Omega|^{\frac{(1-\alpha)(1-\theta)}{\theta}} c_{2}^{1-\alpha}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)} \quad \text { for all } s \in\left(0, T_{\max }\right) \tag{3.19}
\end{equation*}
$$

If $\theta>1$, however, then (3.16) asserts that after all $\theta<p$, whence another application of the Hölder inequality shows that again writing $L(t):=1+\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{p}(\Omega)}$ and $M(t):=\sup _{s \in(0, t)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)}$,
$t \in\left(0, T_{\max }\right)$, for any such $t$ we have

$$
\begin{aligned}
\left\|u^{1-\alpha}(\cdot, s) \nabla v(\cdot, s)\right\|_{L^{\mu}(\Omega)} & \leq\|u(\cdot, s)\|_{L^{p}(\Omega)}^{(1-\alpha) a}\|u(\cdot, s)\|_{L^{1}(\Omega)}^{(1-\alpha)(1-a)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)} \\
& \leq c_{2}^{(1-\alpha)(1-a)} L^{(1-\alpha) a}(t) M(t) \quad \text { for all } s \in(0, t)
\end{aligned}
$$

with $a \equiv a(\varepsilon, p, q):=\frac{1}{1-\frac{1}{p}} \cdot \frac{1-\alpha-\frac{1}{\mu}+\frac{1}{q}}{1-\alpha} \in(0,1)$. Combining this with (3.19) shows that in both these cases, due to the fact that $\int_{0}^{\infty}\left(1+\sigma^{-\frac{1}{2}-\frac{n}{2}\left(\frac{1}{\mu}-\frac{1}{p}\right)}\right) e^{-\sigma} d \sigma$ is finite by the first inequality in $(3.15)$, the relation (3.18) implies that with some $c_{3}=c_{3}(\varepsilon, p, q)>0$,

$$
\begin{equation*}
\int_{0}^{t}\left\|e^{(t-s)(\Delta-1)} \nabla \cdot(u(\cdot, s) f(u(\cdot, s)) \nabla v(\cdot, s))\right\|_{L^{p}(\Omega)} d s \leq c_{3} L^{\frac{1-\alpha-\frac{1}{\mu}+\frac{1}{q}}{1-\frac{1}{p}}}(t) M(t) \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{3.20}
\end{equation*}
$$

again because $L(t) \geq 1$ for all $t \in\left(0, T_{\max }\right)$. In order to make appropriate use of this, we observe that according to (1.2),

$$
u_{t} \leq \Delta u-u-\nabla \cdot(u f(u) \nabla v)+\kappa \quad \text { in } \Omega \times\left(0, T_{\max }\right)
$$

so that thanks to the nonnegativity of $u$ and an associated variation-of-constants formula,

$$
\begin{aligned}
\|u(\cdot, t)\|_{L^{p}(\Omega)} \leq & \left\|e^{t(\Delta-1)} u_{0}-\int_{0}^{t} e^{(t-s)(\Delta-1)} \nabla \cdot(u(\cdot, s) f(u(\cdot, s)) \nabla v(\cdot, s)) d s+\int_{0}^{t} e^{(t-s)(\Delta-1)} \kappa d s\right\|_{L^{p}(\Omega)} \\
\leq & e^{-t}\left\|u_{0}\right\|_{L^{p}(\Omega)} \\
& +\int_{0}^{t}\left\|e^{(t-s)(\Delta-1)} \nabla \cdot(u(\cdot, s) f(u(\cdot, s)) \nabla v(\cdot, s))\right\|_{L^{p}(\Omega)} d s \\
& +\kappa \cdot\left(1-e^{-t}\right) \quad \text { for all } t \in\left(0, T_{\max }\right)
\end{aligned}
$$

because $e^{t \Delta}$ is nonexpansive on $L^{p}(\Omega)$ for each $t>0$, and because $e^{t \Delta} \kappa \equiv \kappa$ in $\Omega$ for all $t>0$. In conjunction with (3.20), this entails the existence of $c_{4}=c_{4}(\varepsilon, p, q)>0$ such that

$$
L(t) \leq c_{4}+c_{4} L^{\frac{1-\alpha-\frac{1}{\mu}+\frac{1}{q}}{1-\frac{1}{p}}}(t) M(t) \quad \text { for all } t \in\left(0, T_{\max }\right)
$$

where the third inequality in (3.15) ensures that $\frac{1-\alpha-\frac{1}{\mu}+\frac{1}{q}}{1-\frac{1}{p}}<1$, and that thus Young's inequality applies so as to provide $c_{5}=c_{5}(\varepsilon, p, q)>0$ fulfilling

$$
c_{4} L^{\frac{1-\alpha-\frac{1}{\mu}+\frac{1}{q}}{1-\frac{1}{p}}}(t) M(t) \leq \frac{1}{2} L(t)+c_{5} M^{\frac{1-\frac{1}{p}}{\alpha-\frac{1}{p}-\frac{1}{q}+\frac{1}{\mu}}}(t) \quad \text { for all } t \in\left(0, T_{\max }\right)
$$

In light of (3.17), this yields (3.12).

## 4 Proof of Theorem 1.1

### 4.1 Boundedness for $\alpha>\frac{n-2}{n-1}$ when $n \in\{2,3\}$

Now in multi-dimensional settings the outcomes of Lemma 3.3 and Lemma 3.4 can fruitfully combined when $n \leq 3$ and $\alpha$ is as accodingly required by Theorem 1.1. This assumption forms a crucial ingredient to the following argument which involves as a first application of the results from the previous section, namely to values of $p$ and $q$ both lying above but suitably close to the number $\frac{n}{n-1}$.

Lemma 4.1 Let $n \in\{2,3\}$ and $\kappa \geq 0$, and assume (1.3), (1.4) and (1.5) with some $K_{f}>0$ and $\alpha>0$ satisfying

$$
\begin{equation*}
\alpha>\frac{n-2}{n-1} . \tag{4.1}
\end{equation*}
$$

Then one can find $C>0$ such that

$$
\begin{equation*}
\|w(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{4.2}
\end{equation*}
$$

Proof. Without loss of generality assuming that $\alpha<1$, we note that the hypothesis $\alpha>\frac{n-2}{n-1}$ ensures that $\frac{n \alpha}{n-2}>\frac{n}{n-1}$, so that we can find $p>\frac{n}{n-1}$ suitably close to $\frac{n}{n-1}$ such that

$$
\begin{equation*}
p<\frac{n \alpha}{n-2} . \tag{4.3}
\end{equation*}
$$

With this value of $p$ fixed, we let

$$
\theta(\xi, \varepsilon):=\left\{\frac{1-\frac{1}{p}}{\alpha+\frac{1}{n}-\xi}+\varepsilon\right\} \cdot\left\{\frac{(n+1-n \xi)(n-1-n \xi)}{n\left(1-\frac{1}{p}\right)(3-n \xi)}+\varepsilon\right\}, \quad \xi \in J:=\left(0, \frac{n-1}{n}\right], \varepsilon>0
$$

noting that $\theta$ is well-defined because $\frac{n-1}{n}<\frac{3}{n}$, and because $\frac{n-1}{n}<\alpha+\frac{1}{n}$ due to the fact that $\alpha>\frac{n-2}{n-1}>\frac{n-2}{n}$. Since evidently $\theta\left(\frac{n-1}{n}, 0\right)=0$, and since apart from that clearly $\frac{1}{p}-\frac{1}{n}<\frac{n-1}{n}$, by means of a continuity argument we can choose $\xi \in J$ and $\varepsilon>0$ such that $\xi<\frac{n-1}{n}$ and

$$
\begin{equation*}
\xi>\frac{1}{p}-\frac{1}{n} \tag{4.4}
\end{equation*}
$$

and that

$$
\begin{equation*}
\theta(\xi, \varepsilon)<1, \tag{4.5}
\end{equation*}
$$

and we observe that then we moreover have $\xi<\frac{1}{n}+\frac{\alpha}{p}$, for from (4.3) we know that $\frac{1}{n}+\frac{\alpha}{p}>$ $\frac{1}{n}+\frac{n-2}{n}=\frac{n-1}{n}$. Writing $q:=\frac{1}{\xi}$, we see that therefore $q>\frac{n}{n-1}$ and $(n-p) q<n p$ as well as $q>\frac{1}{\frac{1}{n}+\frac{\alpha}{p}}$, where the latter relation together with the inequality $p>\frac{n}{n-1}$ enables us to invoke Lemma 3.4, thus inferring the existence of $c_{1}>0$ such that for $L(t):=1+\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{p}(\Omega)}$ and $M(t):=\sup _{s \in(0, t)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)}, t \in\left(0, T_{\text {max }}\right)$, we have

$$
\begin{equation*}
L(t) \leq c_{1}+c_{1} M^{\frac{1-\frac{1}{p}}{\alpha+\frac{1}{n}-\frac{1}{q}}+\varepsilon}(t) \quad \text { for all } t \in\left(0, T_{\max }\right) . \tag{4.6}
\end{equation*}
$$

On the other hand, using that $(n-p) q<n p$ and $q>\frac{n}{n-1}$, and that thus also $q>\frac{n}{3}$, we may employ Lemma 3.3 to find $c_{2}>0$ such that

$$
\begin{equation*}
M(t) \leq c_{2} L^{\frac{\left(n+1-\frac{n}{q}\right)\left(n-1-\frac{n}{q}\right)}{n\left(1-\frac{1}{p}\right)\left(3-\frac{n}{q}\right)}+\varepsilon}(t) \quad \text { for all } t \in\left(0, T_{\max }\right) . \tag{4.7}
\end{equation*}
$$

Combined with (4.6), this implies that

$$
L(t) \leq c_{1}+c_{1} c_{2}^{\frac{1-\frac{1}{p}}{\alpha+\frac{1}{n}-\frac{1}{q}}+\varepsilon} L^{\theta\left(\frac{1}{q}, \varepsilon\right)}(t) \quad \text { for all } t \in\left(0, T_{\max }\right)
$$

and thereby shows that with some $c_{3}>0$ we have

$$
L(t) \leq c_{3} \quad \text { for all } t \in\left(0, T_{\max }\right),
$$

because $\theta\left(\frac{1}{q}, \varepsilon\right)<1$ by (4.5). Through (4.7), the latter entails boundedness of $\left(0, T_{\max }\right) \ni t \mapsto$ $\|\nabla v(\cdot, t)\|_{L^{q}(\Omega)}$, so that Lemma 3.1 establishes the claim.
With the above boundedness property of $w$ at hand, we are now in the position to go back to Lemma 3.2 to see that as a consequence thereof, when combined with again Lemma 4.1 but now for $p:=\infty$ and arbitrarily large $q$, in fact the second alternative in the extensibility criterion (2.1) cannot occur in the presently considered framework.
Lemma 4.2 Let $n \in\{2,3\}$ and $\kappa \geq 0$, and suppose that (1.3), (1.4) and (1.5) hold with some $K_{f}>0$ and $\alpha>\frac{n-2}{n}$. Then for all $q>n$ there exists $C(q)>0$ fulfiling

$$
\begin{equation*}
\|u(\cdot, t)\|_{L^{\infty}(\Omega)}+\|\nabla v(\cdot, t)\|_{L^{q}(\Omega)} \leq C(q) \quad \text { for all } t \in\left(0, T_{\max }\right) . \tag{4.8}
\end{equation*}
$$

Proof. Again assuming that $\alpha<1$, using that $\alpha>\frac{n-2}{n-1}>\frac{n-2}{n}$ we see that for each fixed $q>n$,

$$
\frac{n-1-\frac{n}{q}}{n\left(\alpha+\frac{1}{n}-\frac{1}{q}\right)}<\frac{n-1-\frac{n}{q}}{n \cdot\left(\frac{n-2}{n}+\frac{1}{n}-\frac{1}{q}\right)}=1,
$$

whence again by a continuity argument we can pick $\varepsilon=\varepsilon(q)>0$ appropriately small such that still

$$
\theta:=\left\{\frac{1}{\alpha+\frac{1}{n}-\frac{1}{q}}+\varepsilon\right\} \cdot\left\{\frac{n-1-\frac{n}{q}}{n}+\varepsilon\right\}<1 .
$$

Then relying on Lemma 4.1, we may employ Lemma 3.2 with $p:=\infty$ to find $c_{1}=c_{1}(q)>0$ such that $L(t):=1+\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{p}(\Omega)}$ and $M(t):=\sup _{s \in(0, t)}\|\nabla v(\cdot, s)\|_{L^{q}(\Omega)}, t \in\left(0, T_{\text {max }}\right)$, satisfy

$$
\begin{equation*}
M(t) \leq c_{1} L^{\frac{n-1-\frac{n}{q}}{n}+\varepsilon}(t) \quad \text { for all } t \in\left(0, T_{\max }\right), \tag{4.9}
\end{equation*}
$$

which we combine with the outcome of Lemma 3.4, applicable since the inequality $q>n$ asserts (3.11), which namely yields $c_{2}=c_{2}(q)>0$ fulfilling

$$
L(t) \leq c_{2}+c_{2} M^{\frac{1}{\alpha+\frac{1}{n}-\frac{1}{q}}+\varepsilon}(t) \quad \text { for all } t \in\left(0, T_{\max }\right) .
$$

Therefore,

$$
L(t) \leq c_{2}+c_{1}^{\frac{1}{\alpha+\frac{1}{n}-\frac{1}{q}}+\varepsilon} c_{2} L^{\theta}(t) \quad \text { for all } t \in\left(0, T_{\max }\right)
$$

so that the inequality $\theta<1$ warrants boundedness of $L$ and thus, by (4.9), also of $M$ throughout $\left(0, T_{\text {max }}\right)$.

### 4.2 Boundedness when $\alpha>-1$ in the one-dimensional case

In the corresponding one-dimensional case, our reasoning is essentially simpler and can actually be compressed to an argument essentially consisting of only one step for each solution component:

Lemma 4.3 Let $n=1$ and $\kappa \geq 0$, and assume (1.3), (1.4) and (1.5) to be valid with some $K_{f}>0$ and

$$
\begin{equation*}
\alpha>-1 . \tag{4.10}
\end{equation*}
$$

Then for all $q>1$ there exists $C(q)>0$ such that

$$
\begin{equation*}
\|u(\cdot, t)\|_{L^{\infty}(\Omega)}+\left\|v_{x}(\cdot, t)\right\|_{L^{q}(\Omega)}+\|w(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C(q) \quad \text { for all } t \in\left(0, T_{\max }\right) . \tag{4.11}
\end{equation*}
$$

Proof. Without loss of generality assuming that $\alpha \leq 1$, thanks to (4.10) we can fix $q_{0}>1$ large enough such that $q \alpha>-q+1$ for all $q>q_{0}$, and it is evidently sufficient to verify (4.11) for $q>q_{0}$. To accomplish this, given any such $q$ we can find some $\mu=\mu(q) \in(1, q)$ conveniently close to 1 such that still

$$
\begin{equation*}
q \mu \alpha>-q+\mu . \tag{4.12}
\end{equation*}
$$

Then according to the boundedness of $\left(0, T_{\max }\right) \ni t \mapsto\|v(\cdot, t)\|_{L^{1}(\Omega)}$ asserted by Lemma 2.2, straightforward application of $L^{1}-L^{\infty}$ smoothing estimates for the Neumann heat semigroup in the present one-dimensional situation provide $c_{1}>0$ such that

$$
\begin{equation*}
\|w(\cdot, t)\|_{L^{\infty}(\Omega)} \leq c_{1} \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{4.13}
\end{equation*}
$$

which again due to Lemma 2.2 guarantees boundedness of $\left(0, T_{\max }\right) \ni t \mapsto\|u(\cdot, t) w(\cdot, t)\|_{L^{1}(\Omega)}$. Accordingly, standard $L^{\infty}-W^{1, q}$ regularization properties of $\left(e^{t \Delta}\right)_{t \geq 0}$ ensure the existence of $c_{2}=c_{2}(q)>0$ fulfilling

$$
\begin{equation*}
\left\|v_{x}(\cdot, t)\right\|_{L^{q}(\Omega)} \leq c_{2} \quad \text { for all } t \in\left(0, T_{\max }\right), \tag{4.14}
\end{equation*}
$$

so that it remains to establish an $L^{\infty}$ bound for $u$.
To achieve this, we fix any $q>q_{0}$ and let $\mu=\mu(q)$ be as above, and again combine the maximum principle with a known smoothing feature of the heat semigroup to fix $c_{3}>0$ such that

$$
\begin{align*}
\|u(\cdot, t)\|_{L^{\infty}(\Omega)} \leq & \left\|e^{t(\Delta-1)} u_{0}\right\|_{L^{\infty}(\Omega)}+\int_{0}^{t}\left\|e^{(t-s)(\Delta-1)} \partial_{x}\left(u(\cdot, s) f(u(\cdot, s)) v_{x}(\cdot, s)\right)\right\|_{L^{\infty}(\Omega)} d s \\
& +\int_{0}^{t}\left\|e^{(t-s)(\Delta-1)} \kappa\right\|_{L^{\infty}(\Omega)} d s \\
\leq & e^{-t}\left\|u_{0}\right\|_{L^{\infty}(\Omega)}+c_{3} \int_{0}^{t}\left(1-(t-s)^{-\frac{1}{2}-\frac{1}{2 \mu}}\right) e^{-(t-s)}\left\|u(\cdot, s) f(u(\cdot, s)) v_{x}(\cdot, s)\right\|_{L^{\mu}(\Omega)} d s \\
& +\kappa \cdot\left(1-e^{-t}\right) \quad \text { for all } t \in\left(0, T_{\text {max }}\right), \tag{4.15}
\end{align*}
$$

where by (1.4) and the Hölder inequality, for all $s \in\left(0, T_{\max }\right)$ we can estimate

$$
\begin{aligned}
\left\|u(\cdot, s) f(u(\cdot, s)) v_{x}(\cdot, s)\right\|_{L^{\mu}(\Omega)} & \leq K_{f}\left\|(1+u(\cdot, s))^{1-\alpha} v_{x}(\cdot, s)\right\|_{L^{\mu}(\Omega)} \\
& \leq K_{f}\|1+u(\cdot, s)\|_{L^{\frac{q \mu(1-\alpha)}{q-\alpha}}(\Omega)}^{1-\alpha}\left\|v_{x}(\cdot, s)\right\|_{L^{q}(\Omega)} \\
& \leq K_{f}\|1+u(\cdot, s)\|_{L^{\infty}(\Omega)}^{(1-\alpha) a}\|1+u(\cdot, s)\|_{L^{1}(\Omega)}^{(1-\alpha)(1-a)}\left\|v_{x}(\cdot, s)\right\|_{L^{q}(\Omega)}
\end{aligned}
$$

with $a:=\frac{q \mu(1-\alpha)-q+\mu}{q \mu(1-\alpha)} \in(0,1)$.
In view of (4.14) and (2.2), from (4.15) we thus infer the existence of $c_{4}>0$ such that if now we let $L(t):=1+\sup _{s \in(0, t)}\|u(\cdot, s)\|_{L^{\infty}(\Omega)}, t \in\left(0, T_{\text {max }}\right)$, then

$$
\begin{aligned}
L(t) & \leq c_{4}+c_{4} \cdot\left\{\int_{0}^{t}\left(1+(t-s)^{-\frac{1}{2}-\frac{1}{2 \mu}}\right) e^{-(t-s)} d s\right\} \cdot L^{(1-\alpha) a}(t) \\
& \leq c_{4}+c_{4} c_{5} L^{(1-\alpha) a}(t) \quad \text { for all } t \in\left(0, T_{\text {max }}\right),
\end{aligned}
$$

where $c_{5}:=\int_{0}^{\infty}\left(1+\sigma^{-\frac{1}{2}-\frac{1}{2 \mu}}\right) e^{-\sigma} d \sigma$ is finite since $\mu>1$. As

$$
(1-\alpha) a=1-\alpha-\frac{1}{\mu}+\frac{1}{q}<1-\frac{-q+\mu}{q \mu}-\frac{1}{\mu}+\frac{1}{q}=1
$$

by (4.12), this implies boundedness of $u$ and hence completes the proof.

### 4.3 Proof of Theorem 1.1

Our main result thereby in fact reduces to a mere summary:
Proof of Theorem 1.1. When $n \in\{2,3\}$, the claim follows on combining Lemma 4.2 and Lemma 4.1 with Lemma 2.1, while in the case $n=1$ we similarly conclude by relying on Lemma 4.3.

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