Does repulsion-type directional preference in chemotactic migration continue to regularize Keller-Segel systems when coupled to the Navier-Stokes equations?

Michael Winkler
Institut für Mathematik, Universität Paderborn,
Warburger Str. 100, 33098 Paderborn, Germany
michael.winkler@math.uni-paderborn.de

Abstract

The repulsive Keller-Segel-Navier-Stokes system

$$\begin{cases} n_t + u\cdot\nabla n &=& \Delta n + \nabla\cdot(n\nabla c), & & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &=& \Delta c - c + n, & & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u &=& \Delta u + \nabla P + n\nabla\Phi, & \nabla\cdot u = 0, & x\in\Omega,\ t>0,\end{cases} \qquad (\star)$$

is considered in smoothly bounded planar domains, where $\Phi \in W^{2,\infty}(\Omega)$ is given.

It is well-known that the corresponding fluid-free analogue, when posed under homogeneous no-flux boundary conditions, admits global classical solutions for arbitrarily large initial data, thus substantially differing from the classical two-dimensional Keller-Segel system featuring chemoattraction-driven finite-time blow-up for some initial data. The literature on such chemorepulsion systems, however, strongly relies on the presence of an associated energy structure which is apparently destroyed by the fluid interaction mechanism in $(\star)$.

By making use of appropriate functional inequalities involving certain logarithmic expressions arising due to the planarity of the considered setting, it is shown that nevertheless an initial-boundary value problem for $(\star)$ admits globally defined classical solutions for all reasonably regular initial data.

Key words: chemotaxis; repulsion; Navier-Stokes; global existence
MSC (2010): 92C17 (primary); 35Q35, 35Q92, 35K55 (secondary)

1 Introduction

Directional information about chemotactic motion has been playing a key role in essential parts of the literature concerned with the analysis of Keller-Segel type systems. A well-known example is constituted by the detection of critical mass phenomena for the spatially two-dimensional version of the classical Keller-Segel system, as obtained on letting $\chi = 1$ in

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\chi\nabla c), \\ c_t = \Delta c - c + n, \end{cases} \tag{1.1}$$

which indeed has been found to enforce genuine dichotomies with regard to the occurrence of blow-up phenomena: In this prototypical model for self-enhanced and precisely chemoattractive movement, namely, the nonlinear cross-diffusive interaction can mathematically be captured in a convenient and, as it turns out, favorably effective manner through the energy inequality

$$\frac{d}{dt}\Big\{ \int_\Omega n\ln n + \frac{1}{2}\int_\Omega |\nabla c|^2 + \frac{1}{2}\int_\Omega c^2 - \int_\Omega nc \Big\} = -\int_\Omega c_t^2 - \int_\Omega \Big| \frac{\nabla n}{\sqrt{n}} - \sqrt{n}\nabla c\Big|^2 \le 0, \tag{1.2}$$

valid for all suitably regular solutions to Neumann-type boundary value problems for (1.1) in smoothly bounded domains $\Omega \subset \mathbb{R}^2$ ([33]). In fact, although both the Lyapunov functional and the dissipation rate appearing herein link the population density $n = n(x,t)$ and the signal concentration $c = c(x,t)$ in quite an intricate manner, suitable exploitation of this global structural feature revealed that whenever the corresponding initial data $(n_0, c_0) = (n,c)(\cdot,0)$ merely satisfy $\int_\Omega n_0 < 4\pi$, a global bounded classical solution exists ([33]), whereas given any $m \in (4\pi,\infty) \setminus \{4k\pi \mid k \in \mathbb{N}\}$, at least in simply connected planar domains one can find classical solutions (n,c) with $\int_\Omega n(\cdot,0) = m$ for which n becomes unbounded either in finite or in infinite time ([19]); under the additional restriction to radially symmetric settings, even slightly more complete knowledge is available, asserting a corresponding dichotomy between global boundedness for small-mass data on the one hand, and even genuine finite-time blow-up at arbitrary supercritical mass levels on the other, with the threshold value increased to 8π in such settings ([33], [17], [29]; cf. also [31], [32], [2] and [35] for partially even deeper findings for parabolic-elliptic simplifications of (1.1), as well as the surveys [18] and [25]).

Due to an apparent fragility of (1.2) with respect to corresponding model perturbations, however, much less information seems available in situations in which tactic motion does not occur in a purely attractive manner in the sense of being governed by cross-diffusive interaction as in (1.1) with $\chi = 1$ or, more generally, $\chi > 0$. For instance, if in accordance with more recent developments in the modeling literature the possibility of rotational flux components is included by allowing χ in (1.1) to be a 2×2 matrix containing off-diagonal entries ([48]), then the apparent lack of any meaningful analogue of (1.2) seems to go along with significantly reduced knowledge on solution behavior: Findings on global solvability and boundedness then, besides requiring $\int_\Omega n(\cdot,0)$ to remain below some threshold, additionally rely on smallness assumptions on $\int_\Omega |\nabla c(\cdot,0)|^2$ ([4]), while yet more drastically, not any complementing nontrivial blow-up result seems available.

When viewed against this background, it may be regarded as remarkable that independently of the above, (1.1) admits a relevant energy-type inequality also in a second case: Namely, in situations when chemotactic interaction is of precisely repulsive character in that $\chi = -1$ again is diagonal, thus constituting an extreme somewhat opposite to the one determined by the choice $\chi = 1$, an analysis can be based on the observation that then suitably regular solutions satisfy

$$\frac{d}{dt}\Big\{\int_\Omega n\ln n + \frac{1}{2}\int_\Omega |\nabla c|^2\Big\} = -\int_\Omega \frac{|\nabla n|^2}{n} - \int_\Omega |\Delta c|^2 - \int_\Omega |\nabla c|^2 \le 0. \tag{1.3}$$

In fact, corresponding a priori estimates thereby implied could be used to establish an essentially complete theory on global existence and boundedness of classical solutions, as well as their stabilization toward spatially homogeneous equilibria, without any smallness restrictions on the initial data, in smoothly bounded planar domains ([7]); even in the three-dimensional analogue in which the above chemoattractive Keller-Segel system with $\chi = 1$ is known to possess exploding solutions at arbitrarily small mass levels ([42]), suitably utilizing (1.3) has lead to a statement on global existence and stabilization at least within a natural weak solution concept ([7]).

Energy-based analysis of chemotaxis-fluid interaction. Understanding possible effects due to interplay of chemotaxis systems with liquid environments has been forming a substantial branch of the recent literature on cross-diffusion models. Findings in the analytical literature from the past few years indicate that various types of such interaction, being of apparent relevance in several application contexts ([8], [9], [28], [36], [37]), may indeed exert nontrivial influence on chemotaxis systems at least in some particular cases in which the fluid flow can be considered externally given ([21], [22], [23], [15]).

The corresponding knowledge seems much sparser, however, in situations in which the fluid flow itself is potentially affected by cells through buoyancy, and hence forming an additional system variable according to the modeling approach in [37] (cf. also the derivation in [1]). In fact, a considerable part of the literature in this direction is concerned with accordingly obtained chemotaxis-fluid systems in cases in which, in line with the experimental setting addressed and modeled in [37], a chemoattractive signal is consumed by cells, rather than produced as in (1.1). Due to an accordingly increased dissipative character in comparison to (1.1) and associated Keller-Segel-Navier-Stokes counterparts, it turned out that the considered type of fluid interaction does not entirely destroy certain energy-like structures that such systems are known to possess in the absence of fluid flows, thus resulting in rather far-reaching results on global existence, regularity and large time homogenization in two- and even in three-dimensional domains, without size restrictions on the initial data (see [10], [6], [41], [44], [49], [20], [43], [45], and also [5]).

As compared to this, buoyancy-induced fluid interaction of Keller-Segel systems accounting for signal production seems much less understood. The literature in this field seems to concentrate on establishing results on global solvability and boundedness in the purely attractive version of (1.1) and certain variants thereof, mainly concerned with cases which are conveniently subcritical with respect to possible explosion-supporting potential, and which thus allow for regularity arguments independent of vulnerable structural features such as that in (1.2) ([24], [39], [40], [3], [26], [50], [38], [46], [47]).

Main results. The question how far such couplings to fluid flows may affect solution behavior in chemo*repulsion* systems of Keller-Segel type, however, seems essentially unaddressed so far; in particular, it appears to be unknown how far spatially planar versions of such systems retain their explosion-free and essentially diffusion-dominated character when couplings to the Navier-Stokes equations are introduced.

The purpose of the present work is to undertake a first step in this direction by establishing a result on global classical solvability in a correspondingly obtained two-dimensional chemorepulsion-Navier-Stokes system without imposing any restriction on the size of the initial data. More precisely, we shall be concerned with the initial-boundary problem

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n + \nabla\cdot(n\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &= \Delta c - c + n, & x\in\Omega,\ t>0,\\ u_t + (u\cdot\nabla)u &= \Delta u + \nabla P + n\nabla\Phi, \qquad \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad u=0, & & x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad u(x,0)=u_0(x), & & x\in\Omega, \end{cases} \tag{1.4}$$

in a bounded domain $\Omega\subset\mathbb{R}^2$ with smooth boundary, where

$$\Phi\in W^{2,\infty}(\Omega), \tag{1.5}$$

and where

$$\begin{cases} n_0\in C^0(\overline{\Omega}) \quad \text{is nonnegative with } \overline{n}_0>0, \qquad \text{and}\\ c_0\in W^{1,\infty}(\Omega) \quad \text{is nonnegative,} \qquad \text{and where}\\ u_0\in W^{2,2}(\Omega;\mathbb{R}^2)\cap W^{1,2}_{0,\sigma}(\Omega), \end{cases} \tag{1.6}$$

as usual writing $\overline{\varphi} := \frac{1}{|\Omega|}\int_\Omega\varphi$ for $\varphi\in L^1(\Omega)$, and letting $W^{1,2}_{0,\sigma}(\Omega) := W^{1,2}_0(\Omega;\mathbb{R}^2)\cap L^2_\sigma(\Omega)$, with $L^2_\sigma(\Omega) := \{\varphi\in L^2(\Omega;\mathbb{R}^2)\ |\ \nabla\cdot\varphi = 0 \text{ in } \mathcal{D}(\Omega)\}$ denoting the space of all divergence-free vector fields in $L^2(\Omega;\mathbb{R}^2)$.

In line with the reminiscence of the fact that in the case $u\equiv 0$ the subtle structural information encoded in (1.3) has formed the basis for the discovery of complete blow-up suppression from [7], a major challenge underlying our analysis will consist in examining how far a comparable property may persist also in the presence of nontrivial fluid flows. Our key step in this direction, based on a consequence of the two-dimensional Moser-Trudinger inequality (Lemma 3.3) and a basic interpolation inequality relating logarithmic entropies to mass functionals up to certain logarithmic corrections (Lemma 3.6), will rely on the observation that the evolution of a relative of the energy functional from (1.3), namely of the quantity given by

$$y(t) := \int_\Omega n\ln\frac{n}{\overline{n}_0} + \frac12\int_\Omega|\nabla c|^2 + e, \tag{1.7}$$

can be favorably linked to the standard Navier-Stokes energy functional

$$z(t) := \int_\Omega |u|^2. \tag{1.8}$$

In fact, we shall see that with some $b>0$ and $C>0$, the inequality

$$y'(t) + \frac12\int_\Omega|\Delta c|^2 \le Cy(t)\ln y(t) - by(t)z'(t) \tag{1.9}$$

holds throughout the maximal existence interval of a locally existing smooth solution (Lemma 3.7). Thanks to the weakly superlinear growth of the first expression on the right-hand side herein with respect to y, and due to the favorable sign of the rightmost summand, this will enable us to derive bounds for both y and z, as well as a spatio-temporal L^2 estimate for Δc (Lemma 3.8). As seen in Section 4, this information will form a starting point sufficient for a bootstrap procedure finally providing regularity features strong enough so as to allow for global extension of the considered solution.

As a consequence, we will obtain our main result on global classical solvability which confirms absence of any finite-time explosion in (1.4) for arbitrarily large initial data. Here and below, we let $A = -\mathcal{P}\Delta$ denote the realization of the Stokes operator in $L^2(\Omega;\mathbb{R}^2)$, with its domain given by $D(A) = W^{2,2}(\Omega;\mathbb{R}^2) \cap W^{1,2}_{0,\sigma}(\Omega)$, and with $\mathcal{P}$ denoting the Helmholtz projection on $L^2(\Omega;\mathbb{R}^2)$, and for $\alpha > 0$ we let A^α represent the corresponding sectorial fractional powers.

Theorem 1.1 *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, that (1.5) holds, and that n_0, c_0 and u_0 satisfy (1.6). Then there exist functions n, c and u, uniquely determined by the inclusions*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c \in \bigcap_{q>2} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \textit{and} \\ u \in \bigcap_{\alpha \in (\frac{1}{2},1)} C^0([0,\infty); D(A^\alpha)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty);\mathbb{R}^2), \end{cases}$$

such that $n > 0$ and $c \ge 0$ in $\overline{\Omega} \times (0,\infty)$, and that (1.4) is satisfied in the classical sense with some $P \in C^{1,0}(\Omega \times (0,\infty))$.

2 Local existence and extensibility

The following basic result on local existence of mass-preserving solutions, along with a convenient extensibility criterion, can be obtained by straightforward adaptation of well-established arguments, as detailed in quite closely related settings e.g. in [41].

Lemma 2.1 *If (1.5) and (1.6) hold, then there exist $T_{max} \in (0,\infty]$ and a unique triple (n,c,u) of functions*

$$\begin{cases} n \in C^0(\overline{\Omega} \times [0,T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})), \\ c \in \bigcap_{q>2} C^0([0,T_{max}); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})) \qquad \textit{and} \\ u \in \bigcap_{\alpha \in (\frac{1}{2},1)} C^0([0,T_{max}); D(A^\alpha)) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max});\mathbb{R}^2), \end{cases}$$

such that $n > 0$ and $c \ge 0$ in $\overline{\Omega} \times (0,T_{max})$, that (1.4) holds in $\Omega \times (0,T_{max})$ with some $P \in C^{1,0}(\Omega \times (0,T_{max}))$, and that

if $T_{max} < \infty$, then

$$\limsup_{t \nearrow T_{max}} \Big\{ \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \Big\} = \infty \ \textit{for all } q > 2 \textit{ and } \alpha \in (\tfrac{1}{2},1). \tag{2.1}$$

Moreover,

$$\int_\Omega n(\cdot,t) = \int_\Omega n_0 \qquad \textit{for all } t \in (0,T_{max}). \tag{2.2}$$

3 Linking the evolution of logarithmic entropies to the Navier-Stokes energy inequality

Constituting a basic step toward an analysis of the functions y and z from (1.7) and (1.8), let us state two results of standard testing procedures applied to the first two equations from (1.4). In fact, by incompressibility of the considered fluid flow the evolution of the logarithmic entropy appearing in (1.3) is essentially unaffected by the fluid velocity field:

Lemma 3.1 *We have*

$$\frac{d}{dt}\int_\Omega n\ln\frac{n}{\overline{n}_0} + \int_\Omega \frac{|\nabla n|^2}{n} = -\int_\Omega \nabla n\cdot\nabla c \qquad \textit{for all } t\in(0,T_{max}). \tag{3.1}$$

Proof. This follows by straightforward computation using the first equation in (1.4) together with the solenoidality of u. □

A compensation of the interaction term on the right-hand side of (3.1) can be achieved by adding the following identity which now, however, involves nontrivial contributions due to interplay with the fluid flow.

Lemma 3.2 *We have*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 + \int_\Omega |\nabla c|^2 = \int_\Omega \nabla n\cdot\nabla c - \int_\Omega \nabla c\cdot(\nabla u\cdot\nabla c) \qquad \textit{for all } t\in(0,T_{max}). \tag{3.2}$$

Proof. According to the second equation in (1.4),

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 + \int_\Omega |\nabla c|^2 = \int_\Omega \nabla n\cdot\nabla c + \int_\Omega (u\cdot\nabla c)\Delta c \qquad \text{for all } t\in(0,T_{max}),$$

where two further integrations by parts show that

$$\begin{aligned}\int_\Omega (u\cdot\nabla c)\Delta c &= -\int_\Omega \nabla c\cdot(\nabla u\cdot\nabla c) - \int_\Omega u(D^2c\cdot\nabla c)\\ &= -\int_\Omega \nabla c\cdot(\nabla u\cdot\nabla c) - \frac{1}{2}\int_\Omega u\cdot\nabla|\nabla c|^2\\ &= -\int_\Omega \nabla c\cdot(\nabla u\cdot\nabla c) \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

because $\nabla\cdot u = 0$ and $u|_{\partial\Omega\times(0,T_{max})} = 0$. □

To prepare an appropriate treatment of the rightmost summand in (3.2) by means of a suitable analysis of the Navier-Stokes subsystem of (1.4), let us recall from [47, Lemma 2.2] the following functional inequality (cf. also [33]).

Lemma 3.3 *For all $\varepsilon>0$ there exists $M=M(\varepsilon,\Omega)>0$ such that if $0\not\equiv\phi\in C^0(\overline{\Omega})$ is nonnegative and $\psi\in W^{1,2}(\Omega)$, then for each $a>0$,*

$$\int_\Omega \phi|\psi| \le \frac{1}{a}\int_\Omega \phi\ln\frac{\phi}{\overline{\phi}} + \frac{(1+\varepsilon)a}{8\pi}\cdot\Big\{\int_\Omega\phi\Big\}\cdot\int_\Omega|\nabla\psi|^2 + Ma\cdot\Big\{\int_\Omega\phi\Big\}\cdot\Big\{\int_\Omega|\psi|\Big\}^2 + \frac{M}{a}\int_\Omega\phi. \tag{3.3}$$

The latter indeed enables us to to relate the evolution of the standard Navier-Stokes energy to the logarithmic entropy of the population distribution as follows.

Lemma 3.4 *There exists $C > 0$ such that*

$$\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \le C\int_\Omega n\ln\frac{n}{\overline{n}_0} + C \qquad \textit{for all } t\in(0,T_{max}). \tag{3.4}$$

PROOF. Testing the third equation in (1.4) shows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 &= \int_\Omega n(u\cdot\nabla\Phi) \\ &\le C_1\int_\Omega n|u| \\ &\le C_1\int_\Omega n\cdot(|u_1|+|u_2|) \qquad \text{for all } t\in(0,T_{max}), \end{aligned} \tag{3.5}$$

where $C_1 := \|\nabla\Phi\|_{L^\infty(\Omega)}$, and where $u=(u_1,u_2)$. Here taking $C_2>0$ such that in accordance with a Poincaré inequality we have

$$\int_\Omega |u|^2 \le C_2\int_\Omega |\nabla u|^2 \qquad \text{for all } t\in(0,T_{max}),$$

we apply Lemma 3.3 to $\varepsilon := 1$ and to

$$a := \frac{1}{(\frac{1}{2\pi}+2C_2M|\Omega|)\cdot C_1 m},$$

with $M := M(1,\Omega)$ as correspondingly obtained there, and with $m := \int_\Omega n_0$. Due to (2.2) and the Cauchy-Schwarz inequality, this shows that thanks to this choice of a,

$$\begin{aligned} & C_1\int_\Omega n(|u_1|+|u_2|) \\ &\le \frac{2C_1}{a}\int_\Omega n\ln\frac{n}{\overline{n}_0} + \frac{C_1ma}{4\pi}\int_\Omega|\nabla u|^2 + C_1mMa\cdot\left\{\left\{\int_\Omega|u_1|\right\}^2 + \left\{\int_\Omega|u_2|\right\}^2\right\} + \frac{2mM}{a} \\ &\le \frac{2C_1}{a}\int_\Omega n\ln\frac{n}{\overline{n}_0} + \frac{C_1ma}{4\pi}\int_\Omega|\nabla u|^2 + C_1mMa\cdot\left\{|\Omega|\int_\Omega|u|^2\right\} + \frac{2mM}{a} \\ &\le \frac{2C_1}{a}\int_\Omega n\ln\frac{n}{\overline{n}_0} + \left\{\frac{C_1ma}{4\pi} + C_1C_2mMa|\Omega|\right\}\cdot\int_\Omega|\nabla u|^2 + \frac{2mM}{a} \\ &= \frac{2C_1}{a}\int_\Omega n\ln\frac{n}{\overline{n}_0} + \frac{1}{2}\int_\Omega|\nabla u|^2 + \frac{2mM}{a} \qquad \text{for all } t\in(0,T_{max}), \end{aligned}$$

and that hence (3.4) results from (3.5). $\square$

Now bearing in mind that the expression $z' \equiv \frac{d}{dt}\int_\Omega |u|^2$ from (1.8) enters the desired inequality (1.9) in a way nonlinearly coupled to y, in order to finally establish (1.9) we need to estimate the right-hand side in (3.4) against a logarithm-type function of y, instead of e.g. a multiple of y itself.

This will be achieved on the basis of the following elementary inequality.

Lemma 3.5 *Let $a > 0$. Then*

$$\ln \xi \le a\xi + \ln \frac{1}{a} \qquad \textit{for all } \xi > 0. \tag{3.6}$$

Proof. We only need to observe that $\Psi(\xi) := \ln \xi - a\xi$, $\xi > 0$, attains its maximum at $\xi = \frac{1}{a}$ with $\Psi(\frac{1}{a}) = \ln \frac{1}{a} - 1 < \ln \frac{1}{a}$. □

The latter, namely, will entail the following interpolation inequality, valid for nonnegative functions from $L^2(\Omega)$, actually on domains Ω of arbitrary dimension.

Lemma 3.6 *Let $\varphi \in L^2(\Omega)$ be such that $\varphi \ge 0$ a.e. in Ω. Then*

$$\int_\Omega \varphi \ln \varphi \le \Big\{ \int_\Omega \varphi + 1 \Big\} \cdot \ln \Big\{ \int_\Omega \varphi^2 + e \Big\}. \tag{3.7}$$

Proof. For $\varepsilon > 0$, an application of Lemma 3.5 to

$$a := \frac{\ln \Big\{ \int_\Omega (\varphi+\varepsilon)^2 + e \Big\}}{\int_\Omega (\varphi+\varepsilon)^2 + e} > 0$$

shows that

$$\begin{aligned}
\int_\Omega \varphi \ln(\varphi+\varepsilon) &\le a \int_\Omega \varphi(\varphi+\varepsilon) + \Big(\ln \frac{1}{a}\Big) \cdot \int_\Omega \varphi \\
&\le a \cdot \Big\{ \int_\Omega (\varphi+\varepsilon)^2 + e \Big\} + \Big(\ln \frac{1}{a}\Big) \cdot \int_\Omega \varphi \\
&= \ln \Big\{ \int_\Omega (\varphi+\varepsilon)^2 + e \Big\} + \ln \Big\{ \int_\Omega (\varphi+\varepsilon)^2 + e \Big\} \cdot \int_\Omega \varphi \\
&\quad - \ln\ln \Big\{ \int_\Omega (\varphi+\varepsilon)^2 + e \Big\} \cdot \int_\Omega \varphi.
\end{aligned}$$

As

$$\ln\ln \Big\{ \int_\Omega (\varphi+\varepsilon)^2 + e \Big\} \cdot \int_\Omega \varphi \ge \ln\ln e = 0,$$

this implies that

$$\int_\Omega \varphi \ln(\varphi+\varepsilon) \le \Big\{ \int_\Omega \varphi + 1 \Big\} \cdot \ln \Big\{ \int_\Omega (\varphi+\varepsilon)^2 + e \Big\} \qquad \text{for all } \varepsilon > 0,$$

from which (3.7) follows upon taking $\varepsilon \searrow 0$ and twice using Beppo Levi's theorem. □

Combining this with the observations from Lemma 3.1 and Lemma 3.2 and the outcome of Lemma 3.4, we can indeed establish an inequality of the form in (1.9):

Lemma 3.7 *Let*

$$y(t) := \int_\Omega n(\cdot,t)\ln\frac{n(\cdot,t)}{\overline{n}_0} + \frac{1}{2}\int_\Omega |\nabla c(\cdot,t)|^2 + e, \qquad t\in[0,T_{max}), \tag{3.8}$$

and

$$z(t) := \int_\Omega |u(\cdot,t)|^2, \qquad t\in[0,T_{max}). \tag{3.9}$$

Then there exist $b>0$ and $C>0$ such that

$$y'(t) + \frac{1}{2}\int_\Omega |\Delta c(\cdot,t)|^2 \le Cy(t)\ln y(t) - by(t)z'(t) \qquad \text{for all } t\in(0,T_{max}). \tag{3.10}$$

Proof. According to the Gagliardo-Nirenberg inequality and standard elliptic regularity theory ([14]), we can fix $C_1>0$ such that

$$\|\varphi\|^2_{L^4(\Omega)} \le C_1\|\Delta\varphi\|_{L^2(\Omega)}\|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi\in W^{2,2}(\Omega),$$

whence applying the Cauchy-Schwarz inequality and Young's inequality to (3.2) we see that for all $t\in(0,T_{max})$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla c|^2 + \int_\Omega|\Delta c|^2 + \int_\Omega|\nabla c|^2 &\le \int_\Omega \nabla n\cdot\nabla c + \|\nabla u\|_{L^2(\Omega)}\|\nabla c\|^2_{L^4(\Omega)}\\
&\le \int_\Omega \nabla n\cdot\nabla c + C_1\|\nabla u\|_{L^2(\Omega)}\|\Delta c\|_{L^2(\Omega)}\|\nabla c\|_{L^2(\Omega)}\\
&\le \int_\Omega \nabla n\cdot\nabla c + \frac{1}{2}\int_\Omega|\Delta c|^2 + \frac{C_1^2}{2}\cdot\left\{\int_\Omega|\nabla u|^2\right\}\cdot\int_\Omega|\nabla c|^2\\
&\le \int_\Omega \nabla n\cdot\nabla c + \frac{1}{2}\int_\Omega|\Delta c|^2 + C_1^2\cdot\left\{\int_\Omega|\nabla u|^2\right\}\cdot y(t),
\end{aligned}$$

and that thus

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla c|^2 + \frac{1}{2}\int_\Omega|\Delta c|^2 \le \int_\Omega\nabla n\cdot\nabla c + C_1^2\cdot\left\{\int_\Omega|\nabla u|^2\right\}\cdot y(t) \qquad \text{for all } t\in(0,T_{max}). \tag{3.11}$$

To further estimate the rightmost summand herein, we invoke Lemma 3.4 to pick $C_2>0$ such that due to (2.2), writing $m:=\int_\Omega n_0$ we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega|u|^2 + \int_\Omega|\nabla u|^2 &\le C_2\int_\Omega n\ln\frac{n}{\overline{n}_0} + C_2\\
&= C_2\int_\Omega n\ln n + C_2 - C_2 m\ln\overline{n}_0 \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

where by Lemma 3.6, and again by (2.2),

$$C_2\int_\Omega n\ln n \le C_3\ln\left\{\int_\Omega n^2 + e\right\} \qquad \text{for all } t\in(0,T_{max})$$

with $C_3 := C_2 \cdot \left\{ \int_\Omega n_0 + 1 \right\}$, so that

$$\begin{aligned}
\int_\Omega |\nabla u|^2 &\le C_3 \ln \left\{ \int_\Omega n^2 + e \right\} + C_2 - z'(t) \\
&\le C_4 \ln \left\{ \int_\Omega n^2 + e \right\} - z'(t) \qquad \text{for all } t \in (0, T_{max}) \qquad (3.12)
\end{aligned}$$

if we let $C_4 := C_3 + \max\{0\,,\, C_2 - C_2 m \ln \overline{n}_0\}$. Now to appropriately estimate the first summand on the right-hand side herein in terms of the dissipated quantity in (3.1), we once more employ the Gagliardo-Nirenberg inequality to find $C_5 > 0$ such that

$$\|\varphi\|_{L^4(\Omega)}^4 \le C_5 \|\nabla\varphi\|_{L^2(\Omega)}^2 \|\varphi\|_{L^2(\Omega)}^2 + C_5 \|\varphi\|_{L^2(\Omega)}^4 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

which, again by (2.2), entails that

$$\begin{aligned}
\int_\Omega n^2 &= \|\sqrt{n}\|_{L^4(\Omega)}^4 \\
&\le C_5 \|\nabla\sqrt{n}\|_{L^2(\Omega)}^2 \|\sqrt{n}\|_{L^2(\Omega)}^2 + C_5 \|\sqrt{n}\|_{L^2(\Omega)}^4 \\
&\le C_5 m \|\nabla\sqrt{n}\|_{L^2(\Omega)}^2 + C_5 m^2 \\
&= \frac{C_5 m}{4} \int_\Omega \frac{|\nabla n|^2}{n} + C_5 m^2 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

Therefore,

$$\int_\Omega \frac{|\nabla n|^2}{n} \ge \frac{4}{C_5 m} \int_\Omega n^2 - 4m \qquad \text{for all } t \in (0, T_{max}),$$

so that (3.1) implies the inequality

$$\frac{d}{dt} \int_\Omega n \ln \frac{n}{\overline{n}_0} + \frac{4}{C_5 m} \int_\Omega n^2 \le - \int_\Omega \nabla n \cdot \nabla c + 4m \qquad \text{for all } t \in (0, T_{max})$$

and thus, together with (3.11) and (3.12), that for all $t \in (0, T_{max})$,

$$y'(t) + \frac{4}{C_5 m} \int_\Omega n^2 + \frac{1}{2} \int_\Omega |\Delta c|^2 \le C_6 \ln \left\{ \int_\Omega n^2 + e \right\} \cdot y(t) - C_1^2 y(t) z'(t) + 4m \qquad (3.13)$$

with $C_6 := C_1^2 C_4$. Since a Young-type inequality says that

$$\xi\eta \le a\xi \ln \xi + \frac{a}{e} e^{\frac{\eta}{a}} \qquad \text{for all } \xi > 0, \eta > 0 \text{ and } a > 0,$$

an application to $a := \frac{4e}{C_5 m}$, $\xi := \frac{C_6 y(t)}{a}$ and $\eta := a \ln \left\{ \int_\Omega n^2(\cdot, t) + e \right\}$ for $t \in (0, T_{max})$ shows that for any such t,

$$\begin{aligned}
C_6 \ln \left\{ \int_\Omega n^2 + e \right\} \cdot y(t) &= \left\{ \frac{C_6 y(t)}{a} \right\} \cdot \left\{ a \ln \left\{ \int_\Omega n^2 + e \right\} \right\} \\
&\le a \cdot \left\{ \frac{C_6 y(t)}{a} \right\} \cdot \ln \frac{C_6 y(t)}{a} + \frac{a}{e} \cdot \exp \left\{ \frac{1}{a} \cdot a \ln \left\{ \int_\Omega n^2 + e \right\} \right\} \\
&= C_6 y(t) \ln y(t) + \frac{4}{C_5 m} \cdot \left\{ \int_\Omega n^2 + e \right\} + C_6 \cdot \left\{ \ln \frac{C_5 C_6 m}{4e} \right\} \cdot y(t) \qquad (3.14)
\end{aligned}$$

Now letting $C_7 := \max\Big\{0\,,\,C_6 \cdot \ln \frac{C_5 C_6 m}{4e}\Big\}$ and using that $y(t) \ge e$ and hence $\ln y(t) \ge 1$ for all $t \in (0, T_{max})$, by combining (3.14) with (3.13) we infer that

$$\begin{aligned} y'(t) + \frac{1}{2}\int_\Omega |\Delta c|^2 &\le C_6 y(t) \ln y(t) + C_7 y(t) + \frac{4e}{C_5 m} + 4m - C_1^2 y(t) z'(t) \\ &\le C_6 y(t) \ln y(t) + \Big(C_7 + \frac{4}{C_5 m} + \frac{4m}{e}\Big) \cdot y(t) - C_1^2 y(t) z'(t) \\ &\le \Big(C_6 + C_7 + \frac{4}{C_5 m} + \frac{4m}{e}\Big) \cdot y(t) \ln y(t) - C_1^2 y(t) z'(t) \qquad \text{for all } t \in (0, T_{max}) \end{aligned}$$

and hence conclude upon defining $b := C_1^2$ and $C := C_6 + C_7 + \frac{4}{C_5 m} + \frac{4m}{e}$. $\square$

Among the regularity features thereby implied upon integration, we only note those which are relevant to our subsequent analysis:

Lemma 3.8 *Suppose that $T_{max} < \infty$. Then*

$$\int_0^{T_{max}} \int_\Omega |\Delta c|^2 < \infty, \tag{3.15}$$

and there exists $C > 0$ such that

$$\int_\Omega |\nabla c(\cdot, t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{3.16}$$

and

$$\int_\Omega |u(\cdot, t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{3.17}$$

Proof. By Lemma 3.7, we can fix $C_1 > 0$ and $b > 0$ such that with y and z as defined in (3.8) and (3.9) we have

$$y'(t) + \frac{1}{2}\int_\Omega |\Delta c|^2 \le C_1 y(t) \ln y(t) - b y(t) z'(t) \qquad \text{for all } t \in (0, T_{max}), \tag{3.18}$$

which upon letting $h(t) := \frac{1}{2y(t)} \cdot \int_\Omega |\Delta c(\cdot, t)|^2$, $t \in (0, T_{max})$, means that

$$(\ln y)'(t) + h(t) \le C_1 \ln y(t) - b z'(t) \qquad \text{for all } t \in (0, T_{max})$$

and that thus, after integrating,

$$\ln y(t) + \int_0^t e^{C_1(t-s)} h(s) ds \le \Big\{\ln y(0)\Big\} \cdot e^{C_1 t} - b \int_0^t e^{C_1(t-s)} z'(s) ds \qquad \text{for all } t \in (0, T_{max}). \tag{3.19}$$

Here an integration by parts reveals that since z is nonnegative,

$$\begin{aligned} -b \int_0^t e^{C_1(t-s)} z'(s) ds &= -b z(t) + b z(0) e^{C_1 t} - b C_1 \int_0^t e^{C_1(t-s)} z(s) ds \\ &\le -b z(t) + b z(0) e^{C_1 t} \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

whence (3.19) entails that

$$\ln y(t) + bz(t) + \int_0^t e^{C_1(t-s)} h(s)ds \le C_2 := \Big\{ \ln y(0) \Big\} \cdot e^{C_1 T_{max}} + bz(0) e^{C_1 T_{max}} \qquad \text{for all } t \in (0, T_{max}), \tag{3.20}$$

with C_2 being finite according to our hypothesis on T_{max}. In particular, from this we infer that

$$y(t) \le e^{C_2} \qquad \text{for all } t \in (0, T_{max}),$$

and that thus, by definition of h,

$$\int_0^t e^{C_1(t-s)} h(s)ds \ge \int_0^t h(s)ds \ge \frac{1}{2e^{C_2}} \int_0^t \int_\Omega |\Delta c|^2 \qquad \text{for all } t \in (0, T_{max}),$$

so that (3.20) implies that not only

$$\int_\Omega |\nabla c|^2 \le 2e^{C_2} \qquad \text{for all } t \in (0, T_{max})$$

and

$$\int_\Omega |u|^2 \le \frac{C_2}{b} \qquad \text{for all } t \in (0, T_{max}),$$

but that also

$$\int_0^t \int_\Omega |\Delta c|^2 \le 2C_2 e^{C_2} \qquad \text{for all } t \in (0, T_{max}),$$

whereby the proof becomes complete. □

4 Higher regularity properties. Proof of Theorem 1.1

Having the bounds from Lemma 3.8 at hand, once more relying on the two-dimensionality of the considered setting we can successively increase the available information on regularity, firstly asserting the following.

Lemma 4.1 *If $T_{max} < \infty$, then there exists $C > 0$ such that*

$$\int_\Omega n^2(\cdot, t) \le C \qquad \textit{for all } t \in (0, T_{max}), \tag{4.1}$$

and, moreover,

$$\int_0^{T_{max}} \int_\Omega |\nabla n|^2 < \infty. \tag{4.2}$$

PROOF. The argument is quite standard, essentially following the precedent in [33]: According to the first equation in (1.4), the Cauchy-Schwarz inequality and Young's inequality, we see that with some $C_1 > 0$ we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega n^2 + \int_\Omega |\nabla n|^2 &= -\int_\Omega n\nabla n \cdot \nabla c \\
&= \frac{1}{2}\int_\Omega n^2 \Delta c \\
&\le \frac{1}{2}\|\Delta c\|_{L^2(\Omega)}\|n\|^2_{L^4(\Omega)} \\
&\le C_1\|\Delta c\|_{L^2(\Omega)} \cdot \Big\{\|\nabla n\|_{L^2(\Omega)}\|n\|_{L^2(\Omega)} + \|n\|^2_{L^2(\Omega)}\Big\} \\
&\le \frac{1}{2}\|\nabla n\|^2_{L^2(\Omega)} + \frac{C_1^2}{2}\|\Delta c\|^2_{L^2(\Omega)}\|n\|^2_{L^2(\Omega)} + C_1\|\Delta c\|_{L^2(\Omega)}\|n\|^2_{L^2(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |\nabla n|^2 + g(t)\int_\Omega n^2 \qquad \text{for all } t \in (0, T_{max})
\end{aligned}$$

and hence

$$\frac{d}{dt}\int_\Omega n^2 + \int_\Omega |\nabla n|^2 \le 2g(t)\int_\Omega n^2 \qquad \text{for all } t \in (0, T_{max}),$$

where

$$g(t) := C_1^2 \int_\Omega |\Delta c(\cdot,t)|^2 + 1, \qquad t \in (0, T_{max}).$$

As Lemma 3.8 guarantees that $\int_0^{T_{max}} g(t)dt$ is finite, integrating this readily leads to both (4.1) and (4.2). $\Box$

The latter provides favorable bounds on the forcing term in the Navier-Stokes subsystem of (1.4), in particular implying estimates for u in norms appearing in (2.1). As usual, this will be seen in the course of two steps, the first of which is achieved by means of a standard testing procedure:

Lemma 4.2 *If $T_{max} < \infty$, then there exists $C > 0$ such that*

$$\int_\Omega |\nabla u(\cdot,t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.3}$$

PROOF. Proceeding in a standard manner (see e.g. [41, p. 340]), we test a projected version of the third equation in (1.4) by Au and apply Young's inequality and the Gagliardo-Nirenberg inequality to infer that with $C_1 := \|\nabla\Phi\|^2_{L^\infty(\Omega)}$ and some $C_2 > 0$ we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |Au|^2 &= \int_\Omega Au \cdot \mathcal{P}[(u\cdot\nabla)u] + \int_\Omega Au \cdot \mathcal{P}[n\nabla\Phi] \\
&\le \frac{1}{2}\int_\Omega |Au|^2 + \int_\Omega |\mathcal{P}[(u\cdot\nabla)u]|^2 + \int_\Omega |\mathcal{P}[n\nabla\Phi]|^2 \\
&\le \frac{1}{2}\int_\Omega |Au|^2 + \int_\Omega |(u\cdot\nabla)u|^2 + \int_\Omega |n\nabla\Phi|^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\int_\Omega |Au|^2 + \|u\|^2_{L^\infty(\Omega)}\|\nabla u\|^2_{L^2(\Omega)} + C_1\int_\Omega n^2 \\
&\leq \frac{1}{2}\int_\Omega |Au|^2 + C_2\|Au\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}\|\nabla u\|^2_{L^2(\Omega)} + C_1\int_\Omega n^2 \\
&\leq \int_\Omega |Au|^2 + \frac{C_2^2}{2}\|u\|^2_{L^2(\Omega)}\|\nabla u\|^4_{L^2(\Omega)} + C_1\int_\Omega n^2 \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}
$$

so that writing $g(t) := C_2^2\|u(\cdot,t)\|^2_{L^2(\Omega)}\|\nabla u(\cdot,t)\|^2_{L^2(\Omega)}$ and $h)t) := 2C_1\int_\Omega n^2(\cdot,t)$, $t\in(0,T_{max})$, we obtain that

$$
\frac{d}{dt}\int_\Omega |\nabla u|^2 \leq g(t)\cdot\int_\Omega |\nabla u|^2 + h(t) \qquad \text{for all } t\in(0,T_{max}).
$$

Since $\int_0^{T_{max}} g(t)dt$ and $\int_0^{T_{max}} h(t)dt$ are both finite thanks to Lemma 3.8 and Lemma 4.1, an integration thereof yields (4.3). □

This will in turn enable us to suitably control, besides the forcing term containing n, also the nonlinear convection term from the Navier-Stokes system in a regularity argument based on smoothing properties of the Stokes semigroup:

Lemma 4.3 *Assume that $T_{max}<\infty$. Then given any $\alpha\in(\frac{1}{2},1)$ one can find $C(\alpha)>0$ fulfilling*

$$
\int_\Omega |A^\alpha u(\cdot,t)|^2 \leq C(\alpha) \qquad \textit{for all } t\in(0,T_{max}). \tag{4.4}
$$

In particular,

$$
\sup_{t\in(0,T_{max})}\|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty. \tag{4.5}
$$

Proof. Using that $\alpha<1$, we can pick $p\in(1,2)$ such that $\frac{1}{p}<\frac{3}{2}-\alpha$, and rely on known smoothing properties of the Stokes semigroup $(e^{-tA})_{t\geq 0}$ and continuity features of $\mathcal{P}$ ([13], [12]) to find $C_1>0$ and $C_2>0$ such that

$$
\begin{aligned}
\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} &= \bigg\|A^\alpha e^{-tA}u_0 - \int_0^t A^\alpha e^{-(t-s)A}\mathcal{P}\Big[(u(\cdot,s)\cdot\nabla)u(\cdot,s)\Big]ds \\
&\qquad + \int_0^t A^\alpha e^{-(t-s)A}\mathcal{P}\Big[n(\cdot,s)\nabla\Phi\Big]ds\bigg\|_{L^2(\Omega)} \\
&\leq \|u_0\|_{W^{2,2}(\Omega)} + C_1\int_0^t (t-s)^{-\alpha-\frac{1}{p}+\frac{1}{2}}\Big\|\mathcal{P}\Big[(u(\cdot,s)\cdot\nabla)u(\cdot,s)\Big]\Big\|_{L^p(\Omega)}ds \\
&\quad + C_1\int_0^t (t-s)^{-\alpha}\|n(\cdot,s)\nabla\Phi\|_{L^2(\Omega)}ds \\
&\leq \|u_0\|_{W^{2,2}(\Omega)} + C_2\int_0^t (t-s)^{-\alpha-\frac{1}{p}+\frac{1}{2}}\|u(\cdot,s)\|_{L^{\frac{2p}{2-p}}(\Omega)}\|\nabla u(\cdot,s)\|_{L^2(\Omega)}ds \\
&\quad + C_1\|\nabla\Phi\|_{L^\infty(\Omega)}\int_0^t (t-s)^{-\alpha}\|n(\cdot,s)\|_{L^2(\Omega)}ds \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}
$$

Since $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2p}{2-p}}(\Omega)$, and since Lemma 4.2 and Lemma 4.1 thus warrant boundedness not only of $(\nabla u(\cdot,t))_{t\in(0,T_{max})}$ in $L^2(\Omega)$, but also of $(u(\cdot,t))_{t\in(0,T_{max})}$ in $L^{\frac{2p}{2-p}}(\Omega)$, and of $(n(\cdot,t))_{t\in(0,T_{max})}$ in $L^2(\Omega)$, and since $-\alpha > -\alpha - \frac{1}{p} + \frac{1}{2} > -1$ according to our choice of p, this already yields (4.4). Applying this to an arbitrary fixed $\alpha \in (\frac{1}{2}, 1)$ thereafter implies (4.5) by continuity of the embedding $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$ ([16]). $\Box$

The fluid flow is thereby known to be smooth enough so as to allow for a conclusion on regularity of the signal gradient that goes beyond those from Lemma 3.8.

Lemma 4.4 *Suppose that $T_{max} < \infty$. Then there exists $C > 0$ such that*

$$\int_\Omega |\nabla c(\cdot,t)|^4 \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.6}$$

PROOF. A standard computation on the basis of the second equation in (1.4) ([38, Lemma 7.1]) yields the identity

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla c|^4 + \frac{1}{2}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + \int_\Omega |\nabla c|^2 |D^2 c|^2 + \int_\Omega |\nabla c|^4 \\ = \int_\Omega |\nabla c|^2 \nabla n \cdot \nabla c + \int_\Omega (u\cdot\nabla c)\nabla\cdot(|\nabla c|^2 \nabla c) \\ = +\frac{1}{2}\int_{\partial\Omega} |\nabla c|^2 \frac{\partial |\nabla c|^2}{\partial\nu} \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{4.7}$$

Here by means of the Cauchy-Schwarz inequality, the Gagliardo-Nirenberg inequality and (3.16), we can find $C_1 > 0$ and $C_2 > 0$ such that due to Young's inequality,

$$\begin{aligned} \int_\Omega |\nabla c|^2 \nabla n \cdot \nabla c &\le \|\nabla n\|_{L^2(\Omega)} \Big\| |\nabla c|^2 \Big\|_{L^3(\Omega)}^{\frac{3}{2}} \\ &\le C_1 \|\nabla n\|_{L^2(\Omega)} \cdot \Big\{ \Big\|\nabla|\nabla c|^2\Big\|_{L^2(\Omega)}^2 \Big\||\nabla c|^2\Big\|_{L^1(\Omega)} + \Big\||\nabla c|^2\Big\|_{L^1(\Omega)}^3 \Big\}^{\frac{1}{2}} \\ &\le C_2 \|\nabla n\|_{L^2(\Omega)} \cdot \Big\{ \Big\|\nabla|\nabla c|^2\Big\|_{L^2(\Omega)}^2 + 1 \Big\}^{\frac{1}{2}} \\ &\le \frac{1}{4}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + \frac{1}{4} + C_2^2 \int_\Omega |\nabla n|^2 \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{4.8}$$

whereas employing (4.5) we obtain $C_3 > 0$ fulfilling

$$\begin{aligned} \int_\Omega (u\cdot\nabla c)\nabla\cdot(|\nabla c|^2\nabla c) &= \int_\Omega (u\cdot\nabla c)\cdot\Big(2\nabla c\cdot(D^2 c\cdot\nabla c) + |\nabla c|^2 \Delta c\Big) \\ &\le (2+\sqrt{2})\int_\Omega |u|\cdot|\nabla c|^3 |D^2 c| \\ &\le C_3 \int_\Omega |\nabla c|^3 |D^2 c| \\ &\le \int_\Omega |\nabla c|^2 |D^2 c|^2 + \frac{C_3^2}{4}\int_\Omega |\nabla c|^4 \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{4.9}$$

again thanks to Young's inequality. Since $\frac{\partial|\nabla c|^2}{\partial\nu} \le C_4|\nabla c|^2$ on $\partial\Omega\times(0,T_{max})$ with some $C_4>0$ by ([30]), in since thus by continuity of $W^{\frac{1}{2},2}(\Omega)\hookrightarrow L^2(\partial\Omega)$ and by compactness of $W^{1,2}(\Omega)\hookrightarrow W^{\frac{1}{2},2}(\Omega)$ there exists $C_5>0$ such that

$$\frac{1}{2}\int_{\partial\Omega}|\nabla c|^2\frac{\partial|\nabla c|^2}{\partial\nu} \le \frac{C_4}{2}\Big\||\nabla c|^2\Big\|^2_{L^2(\partial\Omega)} \le \frac{1}{4}\int_\Omega\Big|\nabla|\nabla c|^2\Big|^2 + C_5\int_\Omega|\nabla c|^4 \qquad \text{for all } t\in(0,T_{max}),$$

from (4.7)-(4.9) we hence infer that

$$\frac{d}{dt}\int_\Omega|\nabla c|^4 \le 1+4C_2^2\int_\Omega|\nabla n|^2+(C_3^2+4C_5)\int_\Omega|\nabla c|^4 \qquad \text{for all } t\in(0,T_{max}),$$

which due to (4.2) entails (4.6) upon integration. □

Finally, Lemma 4.4 together with Lemma 4.3 imply an L^∞ bound for n through regularization features of the Neumann heat semigroup.

Lemma 4.5 *If $T_{max}<\infty$, then one can find $C>0$ such that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t\in(0,T_{max}). \tag{4.10}$$

Proof. We estimate $K(T):=\sup_{t\in(0,T)}\|n(\cdot,t)\|_{L^\infty(\Omega)}$, $T\in(0,T_{max})$, by using the first equation in (1.4) along with standard smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ on Ω ([11]), the Hölder inequality, (2.2) and the boundedness of $(h(\cdot,t))_{t\in(0,T_{max})}$, as asserted to hold for $h:=\nabla c+u$ in $L^4(\Omega)$ by Lemma 4.4 and Lemma 4.3, to see that with some $C_1>0$ and $C_2>0$,

$$\begin{aligned}
\|n(\cdot,t)\|_{L^\infty(\Omega)} &= \left\|e^{t\Delta}n_0-\int_0^t e^{(t-s)\Delta}\nabla\cdot\Big\{n(\cdot,s)h(\cdot,s)\Big\}ds\right\|_{L^\infty(\Omega)}\\
&\le \|n_0\|_{L^\infty(\Omega)}+C_1\int_0^t(t-s)^{-\frac{5}{6}}\|n(\cdot,s)h(\cdot,s)\|_{L^3(\Omega)}ds\\
&\le \|n_0\|_{L^\infty(\Omega)}+C_1\int_0^t(t-s)^{-\frac{5}{6}}\|n(\cdot,s)\|^{\frac{11}{12}}_{L^\infty(\Omega)}\|n(\cdot,s)\|^{\frac{1}{12}}_{L^1(\Omega)}\|h(\cdot,s)\|_{L^4(\Omega)}ds\\
&\le \|n_0\|_{L^\infty(\Omega)}+C_2\int_0^t(t-s)^{-\frac{5}{6}}\|n(\cdot,s)\|^{\frac{11}{12}}_{L^\infty(\Omega)}ds\\
&\le C_3+C_3M^{\frac{11}{12}}(T) \qquad \text{for all } T\in(0,T_{max}) \text{ and any } t\in(0,T),
\end{aligned}$$

where $C_3:=\max\{\|n_0\|_{L^\infty(\Omega)}\,,\,6C_2T_{max}^{\frac{1}{6}}\}$. Therefore, $K(T)\le C_3+C_3M^{\frac{11}{12}}(T)$ and hence $K(T)\le\max\{1\,,\,(2C_3)^{12}\}$ for all $T\in(0,T_{max})$, which establishes (4.10). □

In summary, we thereby arrive at our main result on global smooth solvability in (1.4):

Proof of Theorem1.1. According to (2.1) and the fact that

$$\int_\Omega c(\cdot,t)\le\max\left\{\int_\Omega n_0\,,\,\int_\Omega c_0\right\} \qquad \text{for all } t\in(0,T_{max})$$

due to a simple integration of the second equation in (1.4), followed by an ODE comparison using (2.2), the claim is a consequence of Lemma 4.5, Lemma 4.4 and Lemma 4.3 when combined with the statements on local existence, regularity and positivity from Lemma 2.1. □

Acknowledgement. The author thanks the anonymous reviewer for several helpful remarks, and he acknowledges support of the *Deutsche Forschungsgemeinschaft* in the context of the project *Emergence of structures and advantages in cross-diffusion systems* (No. 411007140, GZ: WI 3707/5-1).

References

[1] Bellomo, N., Bellouquid, A., Tao, Y., Winkler, M.: *Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues.* Math. Models Methods Appl. Sci. **25**, 1663-1763 (2015)

[2] Biler, P.: *Local and global solvability of some parabolic systems modelling chemotaxis.* Adv. Math. Sci. Appl. **8**, 715-743 (1998)

[3] Black, T.: *Global very weak solutions to a chemotaxis-fluid system with nonlinear diffusion.* SIAM J. Math. Anal. **50**, 4087-4116 (2018)

[4] Cao, X.: *Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces.* Discr. Cont. Dyn. Syst. Ser. A **35**, 1891-1904 (2015)

[5] Cao, X., Lankeit, J.: *Global classical small-data solutions for a three-dimensional chemotaxis Navier-Stokes system involving matrix-valued sensitivities.* Calc. Var. Part. Differ. Eq. **55**, Art. No. 107

[6] Chae, M., Kang, K., Lee, J.: *Global existence and temporal decay in Keller-Segel models coupled to fluid equations.* Comm. Part. Differ. Eq. **39**, 1205-1235 (2014)

[7] Cieślak, T., Morales Rodrigo, C., Laurençot, Ph.: *Global existence and convergence to steady states in a chemorepulsion system.* Parabolic and Navier-Stokes equations. Part 1, 105117, Banach Center Publ. **81**, Part 1, Polish Acad. Sci. Inst. Math., Warsaw (2008)

[8] Coll, J., et al.: *Chemical aspects of mass spawning in corals. I. Sperm-attractant molecules in the eggs of the scleractinian coral montipora digitata.* Mar. Biol. **118**, 177-182 (1994)

[9] Deshmane, S.L., Kremlev, S., Amini, S., Sawaya, B.E.: *Monocyte chemoattractant protein-1 (mcp-1): an overview.* J. Interferon Cytokine Res. **29**, 313-326 (2009)

[10] Duan, R.J., Lorz, A., Markowich, P.A.: *Global solutions to the coupled chemotaxis-fluid equations.* Comm. Partial Differ. Equations **35**, 1635-1673 (2010)

[11] Fujie, K., Ito, A., Winkler, M., Yokota, T.: *Stabilization in a chemotaxis model for tumor invasion.* Discrete Cont. Dyn. Syst. **36**, 151-169 (2016)

[12] Giga, Y.: *The Stokes operator in L_r spaces.* Proc. Japan Acad. Ser. A Math. Sci. **2**, 85-89 (1981)

[13] Giga, Y.: *Solutions for Semilinear Parabolic Equations in L_p and Regularity of Weak Solutions of the Navier-Stokes System.* J. Differential Equations **61**, 186-212 (1986)

[14] Gilbarg, D., Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin/Heidelberg 2001

[15] He, S., Tadmor, E.: *Suppressing Chemotactic Blow-Up Through a Fast Splitting Scenario on the Plane.* Arch. Ration. Mech. Anal., to appear

[16] Henry, D.: *Geometric theory of semilinear parabolic equations.* Lecture Notes in Mathematics. 840. Springer, Berlin-Heidelberg-New York, 1981

[17] Herrero, M. A., Velázquez, J. J. L.: *A blow-up mechanism for a chemotaxis model.* Ann. Sc. Norm. Super. Pisa Cl. Sci. **24**, 633-683 (1997)

[18] Horstmann, D.: *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences. I.* Jahresber. Deutsch. Math.-Verein. **105**, 103-165 (2003)

[19] Horstmann, D., Wang, G.: *Blow-up in a chemotaxis model without symmetry assumptions.* European J. Appl. Math. **12**, 159-177 (2001)

[20] Jiang, J., Wu, H., Zheng, S.: *Global existence and asymptotic behavior of solutions to a chemotaxis-fluid system on general bounded domains.* Asymptot. Anal. **92**, 249-258 (2015)

[21] Kiselev, A., Ryzhik, L.: *Biomixing by chemotaxis and enhancement of biological reactions.* Comm. Partial Differential Equations **37** (1-3), 298-318 (2012)

[22] Kiselev, A., Ryzhik, L.: *Biomixing by chemotaxis and efficiency of biological reactions: the critical reaction case.* J. Math. Phys. **53** (11), 115609, 9 p. (2012)

[23] Kiselev, A. Xu, X.: *Suppression of Chemotactic Explosion by Mixing.* Arch. Ration. Mech. Anal. **222**, 1077-1112 (2016)

[24] Kozono, H., Miura, M., Sugiyama, Y.: *Existence and uniqueness theorem on mild solutions to the Keller-Segel system coupled with the Navier-Stokes fluid.* J. Funct. Anal. **270**, 1663-1683 (2016)

[25] Lankeit, J., Winkler, M.: *Facing low regularity in chemotaxis systems.* Preprint

[26] Liu, J., Wang, Y.: *Global weak solutions in a three-dimensional Keller-Segel-Navier-Stokes system involving a tensor-valued sensitivity with saturation.* J. Differential Eq. **262**, 5271-5305 (2017)

[27] Lorz, A.: *Coupled Keller-Segel-Stokes model: global existence for small initial data and blow-up delay.* Comm. Math. Sci. **10**, 555-574 (2012)

[28] Miller, R.L.: *Demonstration of sperm chemotaxis in echinodermata: Asteroidea, holothuroidea, ophiuroidea.* J. Exp. Zool. **234**, 383-414 (1985)

[29] Mizoguchi, N., Winkler, M.: *Blow-up in the two-dimensional parabolic Keller-Segel system.* Preprint

[30] Mizoguchi, N., Souplet, Ph.: *Nondegeneracy of blow-up points for the parabolic Keller-Segel system.* Ann. Inst. H. Poincaré Anal. Non Linéaire **31**, 851-875 (2014)

[31] Nagai, T.: *Blow-up of radially symmetric solutions to a chemotaxis system.* Adv. Math. Sci. Appl. **5**, 581-601 (1995)

[32] Nagai, T.: *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains.* J. Inequal. Appl. **6**, 37-55 (2001)

[33] Nagai, T., Senba, T., Yoshida, K.: *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis.* Funkc. Ekvacioj, Ser. Int. **40**, 411-433 (1997)

[34] Sohr, H.: *The Navier-Stokes Equations. An Elementary Functional Analytic Approach.* Birkhäuser, Basel, 2001

[35] Suzuki, T.: *Free Energy and Self-Interacting Particles.* Birkhäuser, Boston, 2005

[36] Taub, D., Proost, P., Murphy, W., Anver, M., Longo, D., Van Damme, J., Oppenheim, J.: *Monocyte chemotactic protein-1 (mcp-1),-2, and-3 are chemotactic for human t lymphocytes.* J. Clin. Investig. **95**, 1370 (1995)

[37] Tuval, I., Cisneros, L., Dombrowski, C., Wolgemuth, C.W., Kessler, J.O., Goldstein, R.E.: *Bacterial swimming and oxygen transport near contact lines.* Proc. Nat. Acad. Sci. USA **102**, 2277-2282 (2005)

[38] Wang, Y., Winkler, M., Xiang, Z.: *Global classical solutions in a two-dimensional chemotaxis-Navier-Stokes system with subcritical sensitivity.* Ann. Sc. Norm. Super. Pisa Cl. Sci., to appear

[39] Wang, Y., Xiang, Z.: *Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation.* J. Differential Eq. **259**, 7578-7609 (2015)

[40] Wang, Y., Xiang, Z.: *Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation: The 3D case.* J. Differ. Eq. **261**, 4944-4973 (2016)

[41] Winkler, M.: *Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops.* Comm. Partial Differ. Eq. **37**, 319-351 (2012)

[42] Winkler, M: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013)

[43] Winkler, M.: *Stabilization in a two-dimensional chemotaxis-Navier-Stokes system.* Arch. Ration. Mech. Anal. **211** (2), 455-487 (2014)

[44] Winkler, M: *Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system.* Ann. Inst. H. Poincaré Anal. Non Linéaire **33**, 1329-1352 (2016)

[45] Winkler, M.: *How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system?* Trans. Amer. Math. Soc. **369**, 3067-3125 (2017)

[46] Winkler, M.: *Does Fluid Interaction Affect Regularity in the Three-Dimensional KellerSegel System with Saturated Sensitivity?* J. Math. Fluid Mech. **20**, 1889-1909 (2018)

[47] Winkler, M.: *Small-mass solutions in the two-dimensional Keller-Segel system coupled to the Navier-Stokes equations.* Preprint

[48] Xue, C., Othmer, H.: *Multiscale models of taxis-driven patterning in bacterial populations.* SIAM J. Appl. Math. **70**, 133-167 (2009)

[49] Zhang, Q., Li, Y.: *Convergence rates of solutions for a two-dimensional chemotaxis-Navier-Stokes system.* Discr. Cont. Dyn. Syst. B **20**, 2751-2759 (2015)

[50] Zheng, J.: *An optimal result for global existence and boundedness in a three-dimensional Keller-Segel-Stokes system with nonlinear diffusion.* J. Differential Eq. **267**, 2385-2415 (2019)