# The role of superlinear damping in the construction of solutions to drift-diffusion problems with initial data in $L^{1}$ 

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#### Abstract

In bounded $n$-dimensional domains $\Omega$, the Neumann problem for the parabolic equation $$
u_{t}=\nabla \cdot(A(x, t) \cdot \nabla u)+\nabla \cdot(b(x, t) u)-f(x, t, u)+g(x, t)
$$


is considered for sufficiently regular matrix-valued $A$, vector-valued $b$ and real valued $g$, and with $f$ representing superlinear absorption in generalizing the prototypical choice given by $f(\cdot, \cdot, s)=s^{\alpha}$ with $\alpha>1$. Problems of this form arise in a natural manner as sub-problems in several applications such as cross-diffusion systems either of Keller-Segel or of Shigesada-Kawasaki-Teramoto type in mathematical biology, and accordingly a natural space for initial data appears to be $L^{1}(\Omega)$.
The main objective thus consists in examining how far solutions can be constructed for initial data merely assumed to be integrable, with major challenges potentially resulting from the interplay between nonlinear degradation on the one hand, and the possibly destabilizing drift-type action on the other in such contexts. Especially, the applicability of well-established methods such as techniques relying on entropy-like structures available in some particular cases, for instance, seems quite limited in the present setting, as these typically rely on higher initial regularity properties.
The first of the main results shows that in the general framework of $(\star)$, nevertheless certain global very weak solutions can be constructed through a limit process involving smooth solutions to approximate variants thereof, provided that the ingredients of the latter satisfy appropriate assumptions with regard to their stabilization behavior.

The second and seemingly most substantial part of the paper develops a method by which it can be shown, under suitably stregthened hypotheses on the integrability of $b$ and the degradation parameter $\alpha$, that the solutions obtained above in fact form genuine weak solutions in a naturally defined sense. This is achieved by properly exploiting a weak integral inequality, as satisfied by the very weak solution at hand, through a testing procedure that appears to be novel and of potentially independent interest.

To underline the strength of this approach, both these general results are thereafter applied to two specific cross-diffusion systems. Inter alia, this leads to a statement on global solvability in a logistic Keller-Segel system under the assumption $\alpha>\frac{2 n+4}{n+4}$ on the respective degradation rate which seems substantially milder than any previously found condition in the literature. Apart from that, for a Shigesada-Kawasaki-Teramoto system some apparently first results on global solvability for $L^{1}$ initial data are derived.
Key words: rough initial data; generalized solutions; cross-diffusion
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## 1 Introduction

A common feature of numerous evolution equations stemming from population models in mathematical biology is the appearance of superlinear degradation terms. In applications typically interpreted, depending on the respective modeling approach, as accounting for diminution due to competition, or as more generally representing abilities of systems to spontaneously prevent overcrowding, such expressions typically arise in the form of algebraic zero-order absorption terms. In the simplest case combined merely with diffusion and thus resulting in semilinear heat equations such as

$$
\begin{equation*}
u_{t}=\Delta u+\lambda u-\mu|u|^{\alpha-1} u, \quad \alpha>1, \lambda \in \mathbb{R}, \mu>0 \tag{1.1}
\end{equation*}
$$

degradation mechanisms of this type usually provide additional dissipation resulting in accordingly enhanced relaxation features. A favorable mathematical effect thereof is that despite their nonlinear character, such absorptive nonlinearities do not essentially counteract existence theories; in fact, sufficiently elaborate analysis shows that the superlinear damping in (1.1) can be used to even expand the well-known solution theory for the heat equation so as to construct solutions even for very singular initial data with regularity properties far below integrability (see [28, 4, 43, 29] and the detailed discussion in the latter, for instance).

That this situation may substantially change when such absorption interacts with further and possibly destabilizing mechanisms is indicated by findings on extensions of (1.1) to systems involving cross-diffusion, such as the logistic Keller-Segel system ([15])

$$
\left\{\begin{array}{l}
u_{t}=\Delta u-\nabla \cdot(u \nabla v)+\lambda u-\mu u^{\alpha}  \tag{1.2}\\
\tau v_{t}=\Delta v-v+u
\end{array}\right.
$$

or the Shigesada-Kawasaki-Teramoto system ([25])

$$
\left\{\begin{array}{l}
u_{t}=d_{1} \Delta u+a_{11} \Delta u^{2}+a_{12} \Delta(u v)+\mu_{1} u\left(1-u-a_{1} v\right)  \tag{1.3}\\
v_{t}=d_{2} \Delta v+a_{22} \Delta v^{2}+a_{21} \Delta(u v)+\mu_{2} v\left(1-v-a_{2} u\right)
\end{array}\right.
$$

Indeed, the solution theories for both these systems are much less developed than that for e.g. (1.1), which may be viewed as partially reflecting a certain singularity-supporting potential of the respective transport processes therein; drastic caveats in this direction are provided by studies reporting the taxis-driven occurrence of large densities in several versions of (1.2) for $\alpha=2$ ([16, 14, 39, 41]), and even detecting finite-time blow-up of some solutions to (1.2) in $n$-dimensional balls with $n \geq 3$, for $\tau=0$ and $\alpha \in\left(1, \alpha_{0}(n)\right)$ with some $\alpha(n) \in(1,2)$, even for smooth initial data ([42], cf. also [38]).
Apart from accordingly implied natural limitations, the construction of global solutions to both (1.2) and (1.3) in the literature has been confronted with significant additional and possibly technical challenges, and thus in successful cases been strongly relying on the presence of particular global dissipative features expressed in corresponding energy or at least quasi-energy inequalities. For instance, the discovery of an appropriate Lyapunov-like functional has given rise to a breakthrough in the existence theory, within suitably weak solution concepts, for (1.3) with widely arbitrary parameters therein ([7]), thus complementing and extending results on global solvability in classes of smooth functions but under various types of more or less restrictive assumptions on the system ingredients ([12, 21, 8, 19, 22]).

Similarly, the use of certain quasi-energy structures in (1.2) has formed an essential fundament for the construction of global bounded solutions in suitable parameter regimes and in presence of sufficiently regular initial data ([23, 30, 36]).
Beyond the evident circumstance that such structures are commonly quite sensitive with respect to changes in the system ingredients, an apparent application-relevant restraint stems from the observation that a corresponding analysis usually requires the initial data to be regular enough so as to have the associated energy be finite at the initial instant. In the context of (1.3), this leads to the requirement, apparently underrun nowhere in the literature, that $u_{0}:=\left.u\right|_{t=0}$ at least be an element of an $L \log L$-type Zygmund class; as for (1.2), most works even assume continuity of the initial data. Up to one single exception addressing global existence of certain generalized solutions to (1.2) in the simple case $\tau=0$ with $\alpha>2-\frac{1}{n}$, however, the literature does not provide any result on solvability in parabolic drift-diffusion systems of the form (1.2) or (1.3), to say nothing of providing a generalizing or even unifying point of view, in situations when initial data are merely assumed to be integrable, and thus to comply with essentially minimal requirements meaningful in the context of applications in which $\int u_{0}$ usually plays the role of a total population size.
Main objective: Construction of generalized solutions with initial data in $L^{1}$. Methodologically, the main challenges going along with the treatment of less regular initial data seem to be linked to the derivation of appropriate compactness properties of the respective superlinear reaction terms, thereby allowing for suitable limit procedures in conveniently designed approximate problems. Here we especially emphasize that due to the presence of additional drift-type mechanisms therein, the accessibility of cross-diffusion systems like (1.2) and (1.3) to compactness-revealing techniques based on duality arguments, as recently developed to quite a comprehensive extent in frameworks of certain pure reaction-diffusion systems generalizing (1.1) to corresponding multi-component problems ( $[24,5]$ ), seems very limited.
Accordingly, a common characteristic feature of virtually all precedent solution constructions for (1.2) and (1.3) consists in asserting equi-integrability properties of the nonlinearities in question by tracking the time evolution of convex functionals of the crucial unknown $u$, with $\int u \ln u$ consituting the most frequently seen representative. Due to the absorptive character of degradation, namely, the associated testing procedures, essentially involving increasing functions of $u$ as test functions in the respective first equations, yield favorably signed contributions that involve functionals of $u$ with conveniently fast growth as $u \rightarrow \infty$. Indeed, corresponding multiplication by $\ln u$, e.g. in (1.2) resulting in space-time $L^{1}$ estimates for $u^{\alpha} \ln u$ and hence implying suitable (equi-)integrability features of $u^{\alpha}$, has been at the core of various existence proofs in (1.2) as well as in several related taxis-type systems ([17, 44, 27]); through their mere nature, however, such techniques seem restricted to cases in which, again, not only $u_{0}$ but even some superlinear functional of $u_{0}$ is integrable.
The purpose of the present work is to develop an apparently alternative approach toward the construction of generalized solutions, firstly mild enough with regard to the initial data so as to be applicable to data merely belonging to $L^{1}$, and secondly sufficiently robust in not relying on fragile structures like entropies. We shall accordingly be concerned with a rather general class of systems involving superlinear degradation, possibly furthermore perturbed by drift terms, by subsequently considering
the no-flux type parabolic problem

$$
\begin{cases}u_{t}=\nabla \cdot(A(x, t) \cdot \nabla u)+\nabla \cdot(b(x, t) u)-f(x, t, u)+g(x, t), & x \in \Omega, t \in(0, T)  \tag{1.4}\\ (A(x, t) \cdot \nabla u) \cdot \nu+b(x, t) u \cdot \nu=0, & x \in \partial \Omega, t \in(0, T) \\ u(x, 0)=u_{0}(x), & x \in \Omega\end{cases}
$$

where $T \in(0, \infty]$ and $\Omega \subset \mathbb{R}^{n}$ is a bounded domain with smooth boundary. Here we assume throughout that the diffusion operator generalizes the Laplacian in that with some positive constants $k_{A}$ and $K_{A}$,

$$
\left\{\begin{array}{l}
A=\left(A_{i j}\right)_{i, j \in\{1, \ldots, n\}} \in L^{\infty}\left(\Omega \times(0, T) ; \mathbb{R}^{n \times n}\right) \text { is such that }  \tag{1.5}\\
A_{i j}(x, t)=A_{j i}(x, t) \text { for all }(x, t) \in \Omega \times(0, T) \text { and } i, j \in\{1, \ldots, n\} \text { with } \\
(A(x, t) \cdot \xi) \cdot \xi \geq k_{A}|\xi|^{2} \quad \text { for all }(x, t, \xi) \in \Omega \times(0, T) \times \mathbb{R}^{n} \quad \text { and } \\
\left|A_{i j}(x, t)\right| \leq K_{A} \quad \text { for all }(x, t) \in \Omega \times(0, T) \text { and } i, j \in\{1, \ldots, n\}
\end{array}\right.
$$

that the drift coefficient satisfies the crucial square integrability condition

$$
\begin{equation*}
b \in L_{l o c}^{2}\left(\bar{\Omega} \times[0, T) ; \mathbb{R}^{n}\right) \tag{1.6}
\end{equation*}
$$

that the nonlinear part of the reaction term,

$$
\begin{equation*}
f \in C^{1}(\bar{\Omega} \times[0, T) ; \times[0, \infty)) \tag{1.7}
\end{equation*}
$$

essentially represents power-type superlinear absorption of the style in (1.2) and (1.3) in satisfying

$$
\begin{equation*}
k_{f} s^{\alpha} \leq f(x, t, s) \leq K_{f} s^{\alpha} \quad \text { for all }(x, t) \in \Omega \times(0, T) \text { and each } s \geq s_{0} \tag{1.8}
\end{equation*}
$$

with some $k_{f}>0, K_{f}>0$ and $\alpha>1$, and that moreover

$$
\begin{equation*}
g \in L_{l o c}^{1}(\bar{\Omega} \times[0, T)) \tag{1.9}
\end{equation*}
$$

and

$$
\begin{equation*}
u_{0} \in L^{1}(\Omega) \text { is nonnegative. } \tag{1.10}
\end{equation*}
$$

Main results I: Constructing very weak solutions without need for $L^{1}$ compactness properties of $u^{\alpha}$. In view of the above observations on precedent studies, our first objective will consist in examining how far solutions can be obtained even despite possibly lacking estimates ensuring compactness features that allow for standard limit passages in classical weak formulations associated with (1.4). For this purpose, in a first step we shall further develop an approach from [40] by resorting to a solution concept which in its most crucial part concentrates on the function $\ln (u+1)$ and merely requires this quantity to satisfy an integral inequality reflecting a certain supersolution property of $\ln (u+1)$ with respect to its parabolic problem formally corresponding to (1.4); along with a suitable additional mass control from above, this yields a concept which for smooth functions is indeed consistent with classical solvability. The main advantage of this relaxation consists in the circumstance that in comparison to standard notions of weak solvability, such as formulated e.g. in Definition 3.1 below, with respect to the decisive nonlinear parts this will here require significantly reduced integrability and compactness properties only, which we will see to indeed be available in quite a general framework.
More precisely, in this first part we shall adapt a concept originally introduced in [40] for a particular chemotaxis problem, and later on extended to various relatives thereof (see e.g. [34, 3]), in the following manner.

Definition 1.1 Let $T \in(0, \infty]$, and suppose that (1.5), (1.6), (1.7), (1.9) and (1.10) hold with some $k_{A}>0$ and $K_{A}>0$. Then a nonnegative function $u \in L_{l o c}^{1}(\bar{\Omega} \times[0, T))$ will be called $a$ very weak solution of (1.4) in $\Omega \times(0, T)$ if $\frac{f(\cdot, \cdot, u)}{u+1} \in L_{l o c}^{1}(\bar{\Omega} \times[0, T))$ and

$$
\begin{equation*}
\nabla \ln (u+1) \in L_{l o c}^{2}\left(\bar{\Omega} \times[0, T) ; \mathbb{R}^{n}\right) \tag{1.11}
\end{equation*}
$$

if the inequality

$$
\begin{align*}
&-\int_{0}^{T} \int_{\Omega} \ln (u+1) \varphi_{t}-\int_{\Omega} \ln \left(u_{0}+1\right) \varphi(\cdot, 0) \\
& \geq \int_{0}^{T} \int_{\Omega}\{(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \varphi-\int_{0}^{T} \int_{\Omega}(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \varphi \\
&+\int_{0}^{T} \int_{\Omega} \frac{u}{u+1}(b(x, t) \cdot \nabla \ln (u+1)) \varphi-\int_{0}^{T} \int_{\Omega} \frac{u}{u+1} b(x, t) \cdot \nabla \varphi \\
&-\int_{0}^{T} \int_{\Omega} \frac{f(x, t, u)}{u+1} \varphi+\int_{0}^{T} \int_{\Omega} \frac{g(x, t)}{u+1} \varphi \tag{1.12}
\end{align*}
$$

is valid for each nonnegative $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, T))$, and if

$$
\begin{equation*}
\int_{\Omega} u\left(x, t_{0}\right) d x+\int_{0}^{t_{0}} \int_{\Omega} f(x, t, u(x, t)) d x d t \leq \int_{\Omega} u_{0}(x) d x+\int_{0}^{t_{0}} \int_{\Omega} g(x, t) d x d t \quad \text { for a.e. } t_{0} \in(0, T) . \tag{1.13}
\end{equation*}
$$

Indeed, by straightforward modification of the arguments from [40, Lemma 2.1] and [18, Lemma 2.5], one can readily verify that this concept is consistent with that of classical solvability in the sense that if $A, f, g$ and $u$ are suitably smooth and $u$ solves (1.4) in the very weak sense described below, then in fact $u$ already must be a classical solution.
Now to substantiate our approach toward solvability in the context of a convenient approximation to (1.4), let us further specify our setting by imposing the hypothesis, forming a standing assumption in this general part, that from whatever source we are given nonnegative classical solutions $u_{\varepsilon} \in$ $C^{0}(\bar{\Omega} \times[0, T)) \cap C^{2,1}(\bar{\Omega} \times(0, T))$ to the regularized variants of (1.4) specified by

$$
\begin{cases}u_{\varepsilon t}=\nabla \cdot\left(A_{\varepsilon}(x, t) \cdot \nabla u_{\varepsilon}\right)+\nabla \cdot\left(b_{\varepsilon}(x, t) u_{\varepsilon}\right)-f\left(x, t, u_{\varepsilon}\right)+g_{\varepsilon}(x, t), & x \in \Omega, t \in(0, T)  \tag{1.14}\\ \left(A_{\varepsilon}(x, t) \cdot \nabla u_{\varepsilon}\right) \cdot \nu+b_{\varepsilon}(x, t) u_{\varepsilon} \cdot \nu=0, & x \in \partial \Omega, t \in(0, T) \\ u_{\varepsilon}(x, 0)=u_{0 \varepsilon}(x), & x \in \Omega,\end{cases}
$$

where $\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ with some sequence $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \subset(0,1)$ fulfilling $\varepsilon_{j} \searrow 0$ as $j \rightarrow \infty$. As for the ingredients herein, in line with the above we will assume that with positive constants $k_{A}, K_{A}, k_{f}, K_{F}$ and $s_{0}$, without loss of generality coinciding with those introduced above, we have

$$
\left\{\begin{array}{l}
A_{\varepsilon} \in C^{1}\left(\bar{\Omega} \times(0, T) ; \mathbb{R}^{n \times n}\right) \quad \text { is such that }  \tag{1.15}\\
\left(A_{\varepsilon}\right)_{i j}(x, t)=\left(A_{\varepsilon}\right)_{j i}(x, t) \quad \text { for all }(x, t) \in \Omega \times(0, T) \text { and } i, j \in\{1, \ldots, n\} \quad \text { with } \\
\left(A_{\varepsilon}(x, t) \cdot \xi\right) \cdot \xi \geq k_{A}|\xi|^{2} \quad \text { for all }(x, t, \xi) \in \Omega \times(0, T) \times \mathbb{R}^{n} \quad \text { and } \\
\left|\left(A_{\varepsilon}\right)_{i j}(x, t)\right| \leq K_{A} \quad \text { for all }(x, t) \in \Omega \times(0, T) \text { and } i, j \in\{1, \ldots, n\}
\end{array}\right.
$$

with

$$
\begin{equation*}
A_{\varepsilon} \rightarrow A \quad \text { a.e. in } \Omega \times(0, T) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0, \tag{1.16}
\end{equation*}
$$

that

$$
\begin{equation*}
b_{\varepsilon} \in C^{1}\left(\bar{\Omega} \times(0, T) ; \mathbb{R}^{n}\right) \cap L_{l o c}^{2}\left(\bar{\Omega} \times[0, T) ; \mathbb{R}^{n}\right) \tag{1.17}
\end{equation*}
$$

approaches $b$ in the sense that

$$
\begin{equation*}
b_{\varepsilon} \rightarrow b \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, T)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0, \tag{1.18}
\end{equation*}
$$

and that the functions

$$
\begin{equation*}
g_{\varepsilon} \in C^{1}(\bar{\Omega} \times(0, T)) \cap L_{l o c}^{1}(\bar{\Omega} \times[0, T)) \tag{1.19}
\end{equation*}
$$

satisfy

$$
\begin{equation*}
g_{\varepsilon} \rightarrow g \quad \text { in } L_{l o c}^{1}(\bar{\Omega} \times[0, T)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 . \tag{1.20}
\end{equation*}
$$

Finally, the initial data in (1.14) will be subject to the assumptions that

$$
\begin{equation*}
u_{0 \varepsilon} \in C^{0}(\bar{\Omega}) \quad \text { is nonnegative } \tag{1.21}
\end{equation*}
$$

with

$$
\begin{equation*}
u_{0 \varepsilon} \rightarrow u_{0} \quad \text { in } L^{1}(\Omega) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{1.22}
\end{equation*}
$$

The first of our main results, to be achieved in Section 2, then asserts that these approximation properties, and especially the crucial $L^{2}$ convergence requirement in (1.18), ensure solvability in the considered very weak framework, indeed assuming no more regularity of $u_{0}$ than merely integrability:

Theorem 1.2 Suppose that (1.5), (1.6), (1.7), (1.8), (1.9) and (1.10) hold for some $T \in(0, \infty]$, $k_{A}>0, K_{A}>0, k_{f}>0, K_{f}>0, s_{0}>0$ and $\alpha>1$, and for $\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ with some sequence $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \subset(0,1)$ such that $\varepsilon_{j} \searrow 0$ as $j \rightarrow \infty$, assume that $u_{\varepsilon} \in C^{0}(\bar{\Omega} \times[0, T)) \cap C^{2,1}(\bar{\Omega} \times(0, T))$ is a classical solution of (1.14) with certain $A_{\varepsilon}, b_{\varepsilon}, g_{\varepsilon}$ and $u_{0 \varepsilon}$ satisfying (1.15), (1.16), (1.17), (1.18), (1.19), (1.20), (1.21) and (1.22). Then there exist a subsequence $\left(\varepsilon_{j_{k}}\right)_{k \in \mathbb{N}}$ and a very weak solution $u$ of (1.4) in $\Omega \times(0, T)$, in the sense of Definition 1.1 below, such that

$$
\begin{array}{lcc}
u_{\varepsilon} \rightarrow u \quad \text { in } L_{l o c}^{1}(\bar{\Omega} \times[0, T)) & \text { and a.e. in } \Omega \times(0, T), \\
u_{\varepsilon} \rightharpoonup u \quad \text { in } L_{l o c}^{\alpha}(\bar{\Omega} \times[0, T)) & \text { and } \\
\nabla \ln \left(u_{\varepsilon}+1\right) \rightharpoonup \nabla \ln (u+1) & \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, T)) \tag{1.25}
\end{array}
$$

as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$.
Main results II: Construction of genuine weak solutions by turning weak into strong $L^{\alpha}$ convergence for sufficiently regular $b$. The major step in our analysis thereafter consists in investigating how far despite the mentioned obstacles the solution gained above in fact solves (1.4) in the standard weak sense. In view of (1.8), this essentially amounts to identifying conditions under which the weak convergence statement in (1.24) can be turned into a corresponding strong compactness property, where in accordance with the above discussion, our ambition to avoid further regularity requirements on the initial data apparently reduces the availability of well-established techniques which in related situations have provided equi-integrability features of, say, some family $\left(h_{j}\right)_{j \in \mathbb{N}}$ by
deriving $L^{1}$ bounds for $\left(\Psi\left(h_{j}\right)\right)_{j \in \mathbb{N}}$ with certain superlinearly growing $\Psi: \mathbb{R} \rightarrow \mathbb{R}([7,24,17,27])$.
In our key step toward circumventing this, we will purely concentrate on the weak supersolution property satisfied by the limit function $u$ due to Theorem 1.2 , and the main challenge here will be to create an appropriate testing procedure in the corresponding integral inequality which allows for a rigorous justification of the mass evolution relation

$$
\begin{equation*}
\int_{\Omega} u\left(\cdot, t_{0}\right)+\int_{0}^{t_{0}} \int_{\Omega} f(x, t, u) \geq \int_{\Omega} u_{0}+\int_{0}^{t_{0}} \int_{\Omega} g(x, t), \tag{1.26}
\end{equation*}
$$

as formally associated with (1.4) even as an identity. Combined with (1.8) and (2.2) this will readily imply that $\int_{0}^{t_{0}} \int_{\Omega} u^{\alpha} \geq \liminf _{\varepsilon=\varepsilon_{j} \searrow 0} \int_{0}^{t_{0}} \int_{\Omega} u_{\varepsilon}^{\alpha}$, and that hence $f\left(\cdot, \cdot, u_{\varepsilon}\right) \rightarrow f(\cdot, \cdot, u)$ in $L^{1}\left(\Omega \times\left(0, t_{0}\right)\right)$, for suitably many $t_{0} \in(0, T)$. We underline already here that developing (1.26) from the inequality (1.12) will go along with considerable efforts, especially due to the circumstance that (1.12) addresses $\ln (u+1)$ rather than $u$ itself, and that according to the poor regularity information available for $u$, quite restrictive requirements for the corresponding test functions are in order.
It will turn out in Section 3, however, that under slightly sharpened assumptions on $\alpha$ and the integrability properties of $b$ this can successfully be accomplished, thus leading to the following result.

Theorem 1.3 Suppose that the assumptions from Theorem 1.2 hold, and that furthermore $\alpha \geq 2$ and

$$
\begin{equation*}
b \in L_{\text {loc }}^{q}\left(\bar{\Omega} \times[0, T) ; \mathbb{R}^{n}\right) \quad \text { with some } q \geq \frac{2 \alpha}{\alpha-1} \tag{1.27}
\end{equation*}
$$

Then the limit function obtained in Theorem 1.2 is a weak solution of (1.4) in the sense of Definition 3.1 below.

Application to logistic Keller-Segel systems. To indicate how the above general theory can be employed in the construction of solutions to concrete cross-diffusion systems involving couplings to further quantities, in Sections 4 and 5 we will focus on the two examples (1.2) and (1.3) introduced above; in order to avoid to become too extensive here, we only mention that further applications to several models of biological relevance are possible, including chemotaxis-haptotaxis systems for tumor invasion or coupled chemotaxis-fluid systems, for instance ([6, 2]).
Let us firstly consider the Neumann problem for the relative of (1.2) given by

$$
\begin{cases}u_{t}=\Delta u-\nabla \cdot(u \nabla v)+F(u), & x \in \Omega, t>0  \tag{1.28}\\ v_{t}=\Delta v-v+u, & x \in \Omega, t>0 \\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0, & x \in \partial \Omega, t>0 \\ u(x, 0)=u_{0}(x), \quad v(x, 0)=v_{0}(x), & x \in \Omega,\end{cases}
$$

where

$$
\begin{equation*}
F \in C^{1}([0, \infty)) \quad \text { is such that } \quad-k_{F} s^{\alpha} \geq F(s) \geq-K_{F} s^{\alpha} \text { for all } s \geq s_{0} \tag{1.29}
\end{equation*}
$$

with some $k_{F}>0, K_{F}>0, s_{0}>0$ and $\alpha>1$, and where $u_{0} \in L^{1}(\Omega)$ and $v_{0} \in L^{2}(\Omega)$ are nonnegative, with a particular representative constituted by the classical logistic Keller-Segel system with quadratic
degradation, as given by

$$
\begin{cases}u_{t}=\Delta u-\nabla \cdot(u \nabla v)+\lambda u-\mu u^{2}, & x \in \Omega, t>0  \tag{1.30}\\ v_{t}=\Delta v-v+u, & x \in \Omega, t>0 \\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0, & x \in \partial \Omega, t>0 \\ u(x, 0)=u_{0}(x), \quad v(x, 0)=v_{0}(x), & x \in \Omega\end{cases}
$$

for $\lambda \in \mathbb{R}$ and $\mu>0$. It is known from the literature that for initial data additionally satisfying $u_{0} \in C^{0}(\bar{\Omega})$ and $v_{0} \in W^{1, \infty}(\Omega)$, the latter problem admits global classical solutions when either $n \leq 2$ and $\mu>0$ is arbitrary ([23]), or $n \geq 3$ and $\mu>\mu_{0}(\lambda, \Omega)$ with some $\mu_{0}(\lambda, \Omega)>0([36])$; for arbitrary values of $\mu>0$ and suitably regular data, global weak solutions have been obtained in [17]. Analytic studies focusing on solvability issues in presence of smaller powers $\alpha$ in the degradation term $F$ from (1.28) and (1.29) apparently go back to [35] where some global generalized solutions could be constructed for a parabolic-elliptic relative under the assumption that

$$
\begin{equation*}
\alpha>2-\frac{1}{n} \tag{1.31}
\end{equation*}
$$

with a recent extension to the fully parabolic case (1.28) for smooth initial data achieved in [33].
Now based on an application of Theorem 1.2, some considerable relaxation with regard to both the condition (1.31) and the initial regularity becomes possible, thus leading to a result on solvability in the fully parabolic problem (1.28) not only for initial data merely belonging to $L^{1} \times L^{2}$, but apart from that also for a range of degradation parameters $\alpha$ apparently not addressed by any existence result in the literature so far:

Theorem 1.4 Let $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with smooth boundary, let $F$ satisfy (1.29) with some positive constants $k_{F}>0, K_{F}>0$ and

$$
\begin{equation*}
\alpha>\frac{2 n+4}{n+4} \tag{1.32}
\end{equation*}
$$

and suppose that $u_{0} \in L^{1}(\Omega)$ and $v_{0} \in L^{2}(\Omega)$ are nonnegative. Then there exist nonnegative functions defined on $\Omega \times(0, \infty)$ which for all $T>0$ have the properties that

$$
\left\{\begin{array}{l}
u \in L^{\infty}\left((0, T) ; L^{1}(\Omega)\right) \cap L^{\alpha}(\Omega \times(0, T)) \quad \text { and }  \tag{1.33}\\
v \in L^{\infty}\left((0, T) ; L^{2}(\Omega)\right) \cap L^{2}\left((0, T) ; W^{1,2}(\Omega)\right) \cap L^{\frac{2 n+4}{n}}(\Omega \times(0, T))
\end{array}\right.
$$

and that $(u, v)$ forms a very weak solution of (1.28) in $\Omega \times(0, \infty)$ in the sense that $u$ is a very weak solution on (1.4) in the style of Definition 1.1 with $A_{i j}=\delta_{i j}, i, j \in\{1, \ldots, n\}, b:=-\nabla v$, $f(\cdot, \cdot, s):=-F(s), s \geq 0$, and $g:=0$, and that

$$
\begin{equation*}
-\int_{0}^{\infty} \int_{\Omega} v \varphi_{t}-\int_{\Omega} v_{0} \varphi(\cdot, 0)=-\int_{0}^{\infty} \int_{\Omega} \nabla v \cdot \nabla \varphi-\int_{0}^{\infty} \int_{\Omega} v \varphi+\int_{0}^{\infty} \int_{\Omega} u \varphi \tag{1.34}
\end{equation*}
$$

for all $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty))$. This solution can be obtained as the limit of classical solutions $\left(u_{\varepsilon}, v_{\varepsilon}\right)$ to (4.3) below in the sense that there exists $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \subset(0,1)$ such that $\varepsilon_{j} \searrow 0$ as $j \rightarrow \infty$ and that $u_{\varepsilon} \rightarrow u$ and $v_{\varepsilon} \rightarrow v$ a.e. in $\Omega \times(0, \infty)$ as $\varepsilon=\varepsilon_{j} \searrow 0$.

Under slightly stronger assumptions on $\alpha$ and the initial regularity of $v$, yet retaining the mere requirement $u_{0} \in L^{1}(\Omega)$, we shall next derive from Theorem 1.3 the following result on genuine weak solvability. Here and below, we let $A$ denote the setorial realization of $-\Delta+1$ under homogeneous Neumann boundary conditions in $L^{2}(\Omega)$ with its domain of definition accordingly given by $D(A)=\left\{\phi \in W^{2,2}(\Omega) \left\lvert\, \frac{\partial \phi}{\partial \nu}=0\right.\right.$ on $\left.\partial \Omega\right\}$.

Theorem 1.5 Let $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with smooth boundary, and let (1.29) be valid with some $k_{F}>0, K_{F}>0$ and

$$
\begin{equation*}
\alpha \geq \frac{n+2}{2} \tag{1.35}
\end{equation*}
$$

Then given any nonnegative functions $u_{0}: \Omega \rightarrow \mathbb{R}$ and $v_{0}: \Omega \rightarrow \mathbb{R}$ fulfilling

$$
u_{0} \in L^{1}(\Omega) \quad \text { and } \quad v_{0} \in D\left(A^{\beta}\right) \quad \text { with some } \quad \begin{cases}\beta \in\left(\frac{n+2}{4 \alpha}, \frac{1}{2}\right] & \text { if } \alpha>\frac{n+2}{2}  \tag{1.36}\\ \beta=\frac{1}{2} & \text { if } \alpha=\frac{n+2}{2}\end{cases}
$$

one can find nonnegative functions $u$ and $v$ defined on $\Omega \times(0, \infty)$ which are such that for all $T>0$,

$$
\left\{\begin{array}{l}
u \in L^{\infty}\left((0, T) ; L^{1}(\Omega)\right) \cap L^{\alpha}(\Omega \times(0, T)) \cap L^{1}\left((0, T) ; W^{1,1}(\Omega)\right) \quad \text { and }  \tag{1.37}\\
v \in L^{\infty}\left((0, T) ; D\left(A^{\beta}\right)\right) \cap L^{q}\left((0, T) ; W^{1, q}(\Omega)\right) \quad \begin{cases}\text { for each } q \in\left[1, \frac{2(n+2)}{n+2-4 \beta}\right) & \text { if } \beta<\frac{1}{2} \\
\text { for } q=\frac{2(n+2)}{n} & \text { if } \beta=\frac{1}{2}\end{cases}
\end{array}\right.
$$

and which form a weak solution of (1.28) in $\Omega \times(0, \infty)$ in the sense that (1.34) holds and that $u$ solves (1.4) with $A, b, f$ and $g$ as specified in Theorem 1.4; in particular,

$$
\begin{equation*}
-\int_{0}^{\infty} \int_{\Omega} u \varphi_{t}-\int_{\Omega} u_{0} \varphi(\cdot, 0)=-\int_{0}^{\infty} \int_{\Omega} \nabla u \cdot \nabla \varphi+\int_{0}^{\infty} \int_{\Omega} u \nabla v \cdot \nabla \varphi+\int_{0}^{\infty} \int_{\Omega} F(u) \varphi \tag{1.38}
\end{equation*}
$$

holds for all $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty))$.
In the particular context of the system (1.30) with quadratic degradation, the latter implies the following.

Corollary 1.6 Let $n=2, \lambda \in \mathbb{R}$ and $\mu>0$, and suppose that $0 \leq u_{0} \in L^{1}(\Omega)$ and $0 \leq v_{0} \in W^{1,2}(\Omega)$. Then there exist nonnegative functions $u$ and $v$ on $\Omega \times(0, \infty)$ such that for any $T>0$ we have

$$
\left\{\begin{array}{l}
u \in L^{\infty}\left((0, T) ; L^{1}(\Omega)\right) \cap L^{2}(\Omega \times(0, T)) \cap L^{1}\left((0, T) ; W^{1,1}(\Omega)\right) \quad \text { and } \\
v \in L^{\infty}\left((0, T) ; W^{1,2}(\Omega)\right) \cap L^{4}\left((0, T) ; W^{1,4}(\Omega)\right)
\end{array}\right.
$$

and that $(u, v)$ solves (1.30) in the weak sense specified in Theorem 1.5.
Application to a Shigesada-Kawasaki-Teramoto type system. Finally, we briefly address a specific version of the comprehensive model (1.3), reducing the full complexity therein by resorting to a tridiagonal case in which cross-diffusion enters only one of the equations. Up to the exceptional approach based on exploiting global entropies ([7]), such simplifications have been an essential prerequisite in most previous studies on global solvability in the context of (1.3), mainly in frameworks of smooth solutions for smooth initial data ([32, 12, 21, 8, 19, 22]).

Specifically, we will focus on the system

$$
\begin{cases}u_{t}=d_{1} \Delta u+a_{12} \Delta(u v)+\mu_{1} u\left(1-u-a_{1} v\right), & x \in \Omega, t>0  \tag{1.39}\\ v_{t}=d_{2} \Delta v+a_{22} \Delta v^{2}+\mu_{2} v\left(1-v-a_{2} u\right), & x \in \Omega, t>0 \\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0, & x \in \partial \Omega, t>0 \\ u(x, 0)=u_{0}(x), \quad v(x, 0)=v_{0}(x), & x \in \Omega\end{cases}
$$

and firstly derive from Theorem 1.2 the following existence result for data in $L^{1} \times L^{\infty}$.
Theorem 1.7 Let $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with smooth boundary, let $d_{1}, d_{2}$ and $\mu_{1}$ be positive and $a_{12}, a_{22}, \mu_{2}, a_{1}$ and $a_{2}$ be nonnegative, and let $0 \leq u_{0} \in L^{1}(\Omega)$ and $0 \leq v_{0} \in L^{\infty}(\Omega)$. Then one can find nonnegative functions $u$ and $v$ on $\Omega \times(0, \infty)$ such that for all $T>0$,

$$
\left\{\begin{array}{l}
u \in L^{\infty}\left((0, T) ; L^{1}(\Omega)\right) \cap L^{2}(\Omega \times(0, T)) \quad \text { and }  \tag{1.40}\\
v \in L^{\infty}(\Omega \times(0, T)) \cap L^{2}\left((0, T) ; W^{1,2}(\Omega)\right),
\end{array}\right.
$$

and such that $u$ is a very weak solution of (1.4) in $\Omega \times(0, \infty)$ in the sense of Definition 1.1 with $A_{i j}(x, t):=\left(d_{1}+a_{12} v(x, t)\right) \delta_{i j}, i, j \in\{1, \ldots, n\}, b(x, t):=a_{12} \nabla v(x, t), f(x, t, s):=\mu_{1} s-\mu_{1} s^{2}$ and $g(x, t):=-\mu_{1} a_{1} u(x, t) v(x, t)$ for $(x, t) \in \Omega \times(0, \infty)$ and $s \geq 0$, and that

$$
\begin{align*}
-\int_{0}^{\infty} \int_{\Omega} v \varphi_{t}-\int_{\Omega} v_{0} \varphi(\cdot, 0)= & -d_{2} \int_{0}^{\infty} \int_{\Omega} \nabla v \cdot \nabla \varphi-2 a_{22} \int_{0}^{\infty} \int_{\Omega} v \nabla v \cdot \nabla \varphi \\
& +\mu_{2} \int_{0}^{\infty} \int_{\Omega} v \varphi-\mu_{2} \int_{0}^{\infty} \int_{\Omega} v^{2} \varphi-\mu_{2} a_{2} \int_{0}^{\infty} \int_{\Omega} u v \varphi \tag{1.41}
\end{align*}
$$

for all $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty))$. Furthermore, letting $\left(u_{\varepsilon}, v_{\varepsilon}\right)$ denote classical solutions of the approximate problem (4.3) below for $\varepsilon \in(0,1)$, with initial data fulfilling (5.2), then with some $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \subset(0,1)$ satisfying $\varepsilon_{j} \searrow 0$ as $j \rightarrow \infty$ we have $u_{\varepsilon} \rightarrow u$ and $v_{\varepsilon} \rightarrow v$ a.e. in $\Omega \times(0, \infty)$ as $\varepsilon=\varepsilon_{j} \searrow 0$.
In order to identify this very weak solution as an actually weak solution by means of Theorem 1.3, we here only need to invest the additional hypothesis that $v_{0}$ belong to $W^{1,2}(\Omega)$.
Theorem 1.8 Let $\Omega \subset \mathbb{R}^{n}$ be a bounded domain with smooth boundary, and let $d_{1}>0, d_{2}>0$ and $\mu_{1}>0$ as well as $a_{12}, a_{22}, \mu_{2}, a_{1}$ and $a_{2}$ be nonnegative. Then whenever $u_{0} \in L^{1}(\Omega)$ and $v_{0} \in$ $W^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ are nonnegative, there exist nonnegative functions $u$ and $v$ defined on $\Omega \times(0, \infty)$ which for all $T>0$ satisfy (1.40) as well as

$$
\left\{\begin{array}{l}
u \in L^{1}\left((0, T) ; W^{1,1}(\Omega)\right) \quad \text { and }  \tag{1.42}\\
v \in L^{\infty}\left((0, T) ; W^{1,2}(\Omega)\right) \cap L^{4}\left((0, T) ; W^{1,4}(\Omega)\right)
\end{array}\right.
$$

and which constitute a weak solution of (1.28) in $\Omega \times(0, \infty)$ in that (1.40) holds for all $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times$ $[0, \infty)$ ), and that $u$ is a weak solution of (1.4) in the sense of Definition 3.1 with $A, b, f$ and $g$ as specified in Theorem 1.7; in particular, we have

$$
\begin{align*}
-\int_{0}^{\infty} \int_{\Omega} u \varphi_{t}-\int_{\Omega} u_{0} \varphi(\cdot, 0)= & -d_{1} \int_{0}^{\infty} \int_{\Omega} \nabla u \cdot \nabla \varphi-a_{12} \int_{0}^{\infty} \int_{\Omega} v \nabla u \cdot \nabla \varphi-a_{12} \int_{0}^{\infty} \int_{\Omega} u \nabla v \cdot \nabla \varphi \\
& +\mu_{1} \int_{0}^{\infty} \int_{\Omega} u \varphi-\mu_{1} \int_{0}^{\infty} \int_{\Omega} u^{2} \varphi-\mu_{1} a_{1} \int_{0}^{\infty} \int_{\Omega} u v \varphi \tag{1.43}
\end{align*}
$$

for all $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty))$.

## 2 Solvability despite lacking strong compactness. Proof of Theorem 1.2

In order to construct a very weak solution by means of a limit procedure involving supposedly given classical solutions of the regularized problems (1.14), let us assume throughout this section that (1.5), (1.6), (1.7), (1.8), (1.9) and (1.10) hold for some $T>0, k_{A}>0, K_{A}>0, k_{f}>0, K_{f}>0, s_{0}>0$ and $\alpha>1$, and that furthermore the boundedness and approximation properties formulated in (1.15), (1.17), (1.19), (1.21) and (1.22) are satisfied.

Then a basic but important property can immediately be seen.
Lemma 2.1 For each $\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$, we have

$$
\begin{equation*}
\int_{\Omega} u_{\varepsilon}(\cdot, t)+\int_{0}^{t_{0}} \int_{\Omega} f\left(x, t, u_{\varepsilon}\right)=\int_{\Omega} u_{0 \varepsilon}+\int_{0}^{t_{0}} \int_{\Omega} g_{\varepsilon}(x, t) \quad \text { for all } t_{0} \in(0, T) \tag{2.1}
\end{equation*}
$$

Proof. Thanks to the no-flux boundary condition in (1.14), integrating the first equation therein yields

$$
\frac{d}{d t} \int_{\Omega} u_{\varepsilon}=-\int_{\Omega} f\left(x, t, u_{\varepsilon}\right)+\int_{\Omega} g_{\varepsilon}(x, t) \quad \text { for all } t \in(0, T)
$$

which directly leads to (2.1).
As a consequence of (1.8), under an additional assumption on the positive part of $g_{\varepsilon}$, actually weaker than our hypothesis (1.19) on $L^{1}$ convergence needed later on, Lemma 2.1 entails a first set of yet quite basic a priori estimates.

Lemma 2.2 Assume that beyond the above hypotheses we have

$$
\begin{equation*}
\sup _{\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}} \int_{0}^{T_{0}} \int_{\Omega}\left(g_{\varepsilon}\right)_{+}<\infty \quad \text { for all } T_{0} \in(0, T) \tag{2.2}
\end{equation*}
$$

Then for any $T_{0} \in(0, T)$ there exists $C\left(T_{0}\right)>0$ such that

$$
\begin{equation*}
\int_{\Omega} u_{\varepsilon}(\cdot, t) \leq C\left(T_{0}\right) \quad \text { for all } t \in\left(0, T_{0}\right) \text { and } \varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{T_{0}} \int_{\Omega} u_{\varepsilon}^{\alpha} \leq C\left(T_{0}\right) \quad \text { for all } \varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \tag{2.4}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\int_{0}^{T_{0}} \int_{\Omega}\left(g_{\varepsilon}\right)-\leq C\left(T_{0}\right) \quad \text { for all } \varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \tag{2.5}
\end{equation*}
$$

Proof. To adequately exploit (3.29), on splitting the spatial integral of $f\left(x, t, u_{\varepsilon}\right)$ we use (1.8) to estimate

$$
\begin{aligned}
\int_{\Omega} f\left(x, t, u_{\varepsilon}\right) & =\int_{\left\{u_{\varepsilon}<s_{0}\right\}} f\left(x, t, u_{\varepsilon}\right)+\int_{\left\{u_{\varepsilon} \geq s_{0}\right\}} f\left(x, t, u_{\varepsilon}\right) \\
& \geq-c_{1}\left(T_{0}\right)+k_{f} \int_{\left\{u_{\varepsilon} \geq s_{0}\right\}} u_{\varepsilon}^{\alpha} \\
& =-c_{1}\left(T_{0}\right)+k_{f} \int_{\Omega} u_{\varepsilon}^{\alpha}-k_{f} \int_{\left\{u_{\varepsilon}<s_{0}\right\}} u_{\varepsilon}^{\alpha} \\
& \geq-c_{1}\left(T_{0}\right)+k_{f} \int_{\Omega} u_{\varepsilon}^{\alpha}-c_{2} \quad \text { for all } t \in\left(0, T_{0}\right)
\end{aligned}
$$

with $c_{1}\left(T_{0}\right):=\|f\|_{L^{\infty}\left(\Omega \times\left(0, T_{0}\right) \times\left(0, s_{0}\right)\right)} \cdot|\Omega|$ and $c_{2}:=k_{f} s_{0}^{\alpha}|\Omega|$. Therefore, (3.29) implies that

$$
\begin{aligned}
\int_{\Omega} u_{\varepsilon}(\cdot, t)+k_{f} \int_{0}^{t} \int_{\Omega} u_{\varepsilon}^{\alpha}+\int_{0}^{t} \int_{\Omega}\left(g_{\varepsilon}\right)_{-} & \leq \int_{\Omega} u_{0 \varepsilon}+\left(c_{1}+c_{2}\right) t+\int_{0}^{t} \int_{\Omega}\left(g_{\varepsilon}\right)_{+} \\
& \leq u_{0 \varepsilon}+\left(c_{1}\left(T_{0}\right)+c_{2}\right) T_{0}+\int_{0}^{T_{0}} \int_{\Omega}\left(g_{\varepsilon}\right)_{+} \quad \text { for all } t \in\left(0, T_{0}\right)
\end{aligned}
$$

whence (2.3), (2.4) and (2.5) result in view of (1.22) and (2.2).
To achieve further regularity information, especially on spatial gradients, besides the above we will make substantial use of a boundedness assumption on the flux coefficient functions $b_{\varepsilon}$ which is yet weaker than the hypothesis (1.18) to be imposed in Theorem 1.2 , but which already refers to essentially the same topology as the one addressed therein.

Lemma 2.3 Assume that (2.2) holds, and that

$$
\begin{equation*}
\sup _{\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}} \int_{0}^{T_{0}} \int_{\Omega}\left|b_{\varepsilon}\right|^{2}<\infty \quad \text { for all } T_{0} \in(0, T) \tag{2.6}
\end{equation*}
$$

Then for each $T_{0} \in(0, T)$ there exists $C\left(T_{0}\right)>0$ such that

$$
\begin{equation*}
\int_{0}^{T_{0}} \int_{\Omega}\left|\nabla \ln \left(u_{\varepsilon}+1\right)\right|^{2} \leq C\left(T_{0}\right) \quad \text { for all } \varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \tag{2.7}
\end{equation*}
$$

Proof. On testing (1.14) against $\frac{1}{u_{\varepsilon}+1}$ we see that

$$
\begin{align*}
\frac{d}{d t} \int_{\Omega} \ln \left(u_{\varepsilon}+1\right)= & \int_{\Omega} \frac{1}{\left(u_{\varepsilon}+1\right)^{2}}\left(A_{\varepsilon} \cdot \nabla u_{\varepsilon}\right) \cdot \nabla u_{\varepsilon}+\int_{\Omega} \frac{u_{\varepsilon}}{\left(u_{\varepsilon}+1\right)^{2}} b_{\varepsilon} \cdot \nabla u_{\varepsilon} \\
& -\int_{\Omega} \frac{f\left(x, t, u_{\varepsilon}\right)}{u_{\varepsilon}+1}+\int_{\Omega} \frac{g_{\varepsilon}}{u_{\varepsilon}+1} \tag{2.8}
\end{align*}
$$

and due to (1.15) we know that herein

$$
\int_{\Omega} \frac{1}{\left(u_{\varepsilon}+1\right)^{2}}\left(A_{\varepsilon} \cdot \nabla u_{\varepsilon}\right) \cdot \nabla u_{\varepsilon} \geq k_{A} \int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+1\right)^{2}}
$$

Since Young's inequality warrants that

$$
\begin{aligned}
\left|\int_{\Omega} \frac{u_{\varepsilon}}{\left(u_{\varepsilon}+1\right)^{2}} b_{\varepsilon} \cdot \nabla u_{\varepsilon}\right| & \leq \frac{k_{A}}{2} \int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+1\right)^{2}}+\frac{1}{2 k_{A}} \int_{\Omega} \frac{u_{\varepsilon}^{2}}{\left(u_{\varepsilon}+1\right)^{2}}\left|b_{\varepsilon}\right|^{2} \\
& \leq \frac{k_{A}}{2} \int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+1\right)^{2}}+\frac{1}{2 k_{A}} \int_{\Omega}\left|b_{\varepsilon}\right|^{2} \quad \text { for all } t \in(0, T),
\end{aligned}
$$

and since again writing $c_{1}\left(T_{0}\right):=\|f\|_{L^{\infty}\left(\Omega \times\left(0, T_{0}\right) \times\left(0, s_{0}\right)\right)} \cdot|\Omega|$, by (1.8) we have

$$
\begin{aligned}
\int_{\Omega} \frac{f\left(x, t, u_{\varepsilon}\right)}{u_{\varepsilon}+1} & \leq K_{f} \int_{\left\{u_{\varepsilon} \geq s_{0}\right\}} \frac{u_{\varepsilon}^{\alpha}}{u_{\varepsilon}+1}+c_{1}\left(T_{0}\right) \\
& \leq K_{f} \int_{\Omega} u_{\varepsilon}^{\alpha}+c_{1}\left(T_{0}\right) \quad \text { for all } t \in\left(0, T_{0}\right)
\end{aligned}
$$

and, clearly, also

$$
-\int_{\Omega} \frac{g_{\varepsilon}}{u_{\varepsilon}+1} \leq \int_{\Omega}\left(g_{\varepsilon}\right)_{-} \quad \text { for all } t \in(0, T)
$$

from (2.8) it follows that

$$
\begin{aligned}
\int_{\Omega} \ln \left(u_{0 \varepsilon}+1\right)+\frac{k_{A}}{2} \int_{0}^{t} \int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{u_{\varepsilon}+1} \leq & \int_{\Omega} \ln \left(u_{\varepsilon}(\cdot, t)+1\right)+\frac{1}{2 k_{A}} \int_{0}^{t} \int_{\Omega}\left|b_{\varepsilon}\right|^{2} \\
& +K_{f} \int_{0}^{t} \int_{\Omega} u_{\varepsilon}^{\alpha}+c_{1}\left(T_{0}\right) t+\int_{0}^{t} \int_{\Omega}\left(g_{\varepsilon}\right)_{-} \quad \text { for all } t \in\left(0, T_{0}\right) .
\end{aligned}
$$

As evidently $\int_{\Omega} \ln \left(u_{0 \varepsilon}+1\right) \geq 0$ and $\int_{\Omega} \ln \left(u_{\varepsilon}(\cdot, t)+1\right) \leq \int_{\Omega} u_{\varepsilon}(\cdot, t)$ for all $t \in(0, T)$, by making use of (2.3), (2.6), (2.4) and (2.5) we immediately infer (2.7) from this.

Together with Lemma 2.2, this also entails some regularity in time of $\ln \left(u_{\varepsilon}+1\right)$ :
Lemma 2.4 If (2.2) and (2.6) hold, then for all $T_{0} \in(0, T)$ and each $m \in \mathbb{N}$ such that $m>\frac{n}{2}$ there exists $C\left(T_{0}, m\right)>0$ such that

$$
\begin{equation*}
\int_{0}^{T_{0}}\left\|\partial_{t} \ln \left(u_{\varepsilon}(\cdot, t)+1\right)\right\|_{\left(W^{m, 2}(\Omega)\right)^{\star}} d t \leq C\left(T_{0}, m\right) \quad \text { for all } \varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \tag{2.9}
\end{equation*}
$$

Proof. For fixed $t \in(0, T)$ and $\phi \in C^{\infty}(\bar{\Omega})$, from (1.14), (1.15) and (1.8) we obtain that

$$
\begin{aligned}
\left|\int_{\Omega} \partial_{t}\left(u_{\varepsilon}(\cdot, t)+1\right) \phi\right|= & \left\lvert\, \int_{\Omega} \frac{1}{\left(u_{\varepsilon}+1\right)^{2}}\left\{\left(A_{\varepsilon} \cdot \nabla u_{\varepsilon}\right) \cdot \nabla u_{\varepsilon}\right\} \phi-\int_{\Omega} \frac{1}{u_{\varepsilon}+1}\left(A_{\varepsilon} \cdot \nabla u_{\varepsilon}\right) \cdot \nabla \phi\right. \\
& +\int_{\Omega} \frac{u_{\varepsilon}}{\left(u_{\varepsilon}+1\right)^{2}}\left(b_{\varepsilon} \cdot \nabla u_{\varepsilon}\right) \phi-\int_{\Omega} \frac{u_{\varepsilon}}{u_{\varepsilon}+1} b_{\varepsilon} \cdot \nabla \phi \\
& \left.-\int_{\Omega} \frac{f\left(x, t, u_{\varepsilon}\right)}{u_{\varepsilon}+1} \phi-\int_{\Omega} \frac{g_{\varepsilon}}{u_{\varepsilon}+1} \phi \right\rvert\, \\
\leq & K_{A} \cdot\left\{\int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+1\right)^{2}}\right\} \cdot\|\phi\|_{L^{\infty}(\Omega)}+K_{A} \cdot\left\{\int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+1\right)^{2}}\right\}^{\frac{1}{2}} \cdot\|\nabla \phi\|_{L^{2}(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
& +\left\{\int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+1\right)^{2}}\right\}^{\frac{1}{2}} \cdot\left\{\int_{\Omega}\left|b_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\|\phi\|_{L^{\infty}(\Omega)}+\left\{\int_{\Omega}\left|b_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\|\nabla \phi\|_{L^{2}(\Omega)} \\
& +\left\{\|f\|_{L^{\infty}\left(\Omega \times(0, T) \times\left(0, s_{0}\right)\right)} \cdot|\Omega|+K_{f} \int_{\left\{u_{\varepsilon} \geq s_{0}\right\}} u_{\varepsilon}^{\alpha}\right\} \cdot\|\phi\|_{L^{\infty}(\Omega)} \\
& +\left\{\int_{\Omega}\left(g_{\varepsilon}\right)_{+}+\int_{\Omega}\left(g_{\varepsilon}\right)_{-}\right\} \cdot\|\phi\|_{L^{\infty}(\Omega)} .
\end{aligned}
$$

As $W^{m, 2}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ by assumption on $m$, by using Young's inequality we therefore see that with some $c_{1}>0$ we have

$$
\begin{aligned}
\left\|\partial_{t} \ln \left(u_{\varepsilon}(\cdot, t)+1\right)\right\|_{\left(W^{m, 2}(\Omega)\right)^{\star}} \leq & c_{1} \int_{\Omega} \frac{\left|\nabla u_{\varepsilon}\right|^{2}}{\left(u_{\varepsilon}+1\right)^{2}}+c_{1} \int_{\Omega}\left|b_{\varepsilon}\right|^{2} \\
& +c_{1} \int_{\Omega} u_{\varepsilon}^{\alpha}+c_{1} \int_{\Omega}\left(g_{\varepsilon}\right)_{+}+c_{1} \int_{\Omega}\left(g_{\varepsilon}\right)_{-} \\
& +c_{1} \quad \text { for all } t \in(0, T) \text { and each } \varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}},
\end{aligned}
$$

whence (2.9) results upon integrating and applying Lemma 2.3, (2.6), (2.4), (2.2) and (2.5).
Now the extraction of suitably converging subsequences essentially reduces to applying an Aubin-Lions lemma.

Lemma 2.5 Assume (2.2) and (2.6). Then one can find a subsequence $\left(\varepsilon_{j_{k}}\right)_{k \in \mathbb{N}}$ of $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ and a nonnegative function $u \in L_{\text {loc }}^{1}(\bar{\Omega} \times[0, T))$ such that (1.23), (1.24) and (1.25) hold as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$.
Proof. From Lemma 2.3 and Lemma 2.2 it follows that $\left(\ln \left(u_{\varepsilon}+1\right)\right)_{\left.\varepsilon \in\left(\varepsilon_{j}\right)\right)_{j \in \mathbb{N}}}$ is bounded in $L^{2}\left(\left(0, T_{0}\right) ; W^{1,2}(\Omega)\right)$ for all $T_{0} \in(0, T)$, while for any fixed integer $m>\frac{n}{2}$, Lemma 2.4 states boundedness of $\left(\partial_{t} \ln \left(u_{\varepsilon}+1\right)\right)_{\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}}$ in $L^{1}\left(\left(0, T_{0}\right) ;\left(W^{m, 2}(\Omega)\right)^{\star}\right)$ for any such $T_{0}$. Therefore, employing an appropriate Aubin-Lions lemma $([31])$ yields precompactness of $\left(\ln \left(u_{\varepsilon}+1\right)\right)_{\left.\varepsilon \in\left(\varepsilon_{j}\right)\right)_{j \in \mathbb{N}}}$ in $L^{2}\left(\Omega \times\left(0, T_{0}\right)\right)$ for all $T_{0} \in(0, T)$, whence extracting a suitable subsequence $\left(\varepsilon_{j_{k}}\right)_{k \in \mathbb{N}}$ of $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ we obtain a nonnegative function $u: \Omega \times(0, T) \rightarrow \mathbb{R}$ for which both (1.25) and, by strict monotonicity of $0 \leq \xi \mapsto \ln (\xi+1)$, also $u_{\varepsilon} \rightarrow u$ a.e. in $\Omega \times(0, T)$ hold as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$. As $\alpha>1$ and $\left(u_{\varepsilon}\right)_{\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}}$ is bounded in $L^{\alpha}\left(\Omega \times\left(0, T_{0}\right)\right)$ for all $T_{0} \in(0, T)$ by Lemma 2.2, it firstly follows from Egorov's theorem that also (1.24) is valid along this subsequence, and secondly we may conclude from the Vitali convergence theorem that moreover $u_{\varepsilon} \rightarrow u$ in $L^{1}\left(\Omega \times\left(0, T_{0}\right)\right)$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$ for all $T_{0} \in(0, T)$.
It thus remains to be shown that the obtained limit function solves (1.4) in the spirit of Definition 1.1). By arguments based on Fatou's lemma and lower semicontinuity of Hilbert space norms with respect to weak convergence, however, the properties asserted by Lemma 2.5 can indeed be identified as sufficient for guaranteeing the integral inequalities (1.12) and (1.13):
Proof of Theorem 1.2. Taking $\left(\varepsilon_{j_{k}}\right)_{k \in \mathbb{N}}$ and $u$ as provided by Lemma 2.5 , from the latter we directly obtain that (1.23), (1.24) and (1.25) hold, and that in view of (1.8) also the regularity requirements imposed in Definition 1.1 are satisfied.
For the verification of (1.13), according to (1.23) and the Fubini-Tonelli theorem we fix a null set $N \subset(0, T)$ such that for all $t_{0} \in(0, T) \backslash N$ we have $u_{\varepsilon}\left(\cdot, t_{0}\right) \rightarrow u\left(\cdot, t_{0}\right)$ a.e. in $\Omega$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$, whence
by Fatou's lemma and Lemma 2.1,

$$
\begin{align*}
\int_{\Omega} u\left(\cdot, t_{0}\right)+\int_{0}^{t_{0}} \int_{\Omega} f(x, t, u)= & \int_{\Omega} u\left(\cdot, t_{0}\right)+\int_{0}^{t_{0}} \int_{\Omega} f_{+}(x, t, u)-\int_{0}^{t_{0}} \int_{\Omega} f_{-}(x, t, u) \\
\leq & \liminf _{\varepsilon=\varepsilon_{j_{k}} \backslash 0}\left\{\int_{\Omega} u_{\varepsilon}\left(\cdot, t_{0}\right)+\int_{0}^{t_{0}} \int_{\Omega} f_{+}\left(x, t, u_{\varepsilon}\right)\right\} \\
& -\int_{0}^{t_{0}} \int_{\Omega} f_{-}(x, t, u) \\
= & \liminf _{\varepsilon=\varepsilon_{j_{k}} \searrow 0}\left\{\int_{\Omega} u_{\varepsilon}\left(\cdot, t_{0}\right)+\int_{0}^{t_{0}} \int_{\Omega} f\left(x, t, u_{\varepsilon}\right)+\int_{0}^{t_{0}} \int_{\Omega} f_{-}\left(x, t, u_{\varepsilon}\right)\right\} \\
& -\int_{0}^{t_{0}} \int_{\Omega} f_{-}(x, t, u) \\
= & \liminf _{\varepsilon=\varepsilon_{j_{k}} \searrow 0}\left\{\int_{\Omega} u_{0 \varepsilon}+\int_{0}^{t_{0}} \int_{\Omega} g_{\varepsilon}+\int_{0}^{t_{0}} \int_{\Omega} f_{-}\left(x, t, u_{\varepsilon}\right)\right\} \\
& -\int_{0}^{t_{0}} \int_{\Omega} f_{-}(x, t, u) \quad \text { for all } t_{0} \in(0, T) \backslash N . \tag{2.10}
\end{align*}
$$

Here by (1.22) and (1.20),

$$
\int_{\Omega} u_{0 \varepsilon} \rightarrow \int_{\Omega} u_{0} \quad \text { and } \quad \int_{0}^{t_{0}} \int_{\Omega} g_{\varepsilon} \rightarrow \int_{0}^{t_{0}} \int_{\Omega} g \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0
$$

and combining (1.23) with the continuity of $f_{-}$we find that

$$
\int_{0}^{t_{0}} \int_{\Omega} f_{-}\left(x, t, u_{\varepsilon}\right) \rightarrow \int_{0}^{t_{0}} \int_{\Omega} f_{-}(x, t, u) \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0
$$

by the dominated convergence theorem, because $f_{-}$is bounded in $\Omega \times\left(0, t_{0}\right) \times(0, \infty)$ thanks to (1.8). Therefore, (1.13) is a consequence of (2.10), so that it remains to derive (1.12).
To this end, we fix a nonnegative $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, T))$ and then obtain from (1.14) that the identity

$$
\begin{align*}
&-\int_{0}^{T} \int_{\Omega} \ln \left(u_{\varepsilon}+1\right) \varphi_{t}-\int_{\Omega} \ln \left(u_{0 \varepsilon}+1\right) \varphi(\cdot, 0) \\
&= \int_{0}^{T} \int_{\Omega}\left\{\left(A_{\varepsilon} \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right) \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right\} \varphi-\int_{0}^{T} \int_{\Omega}\left(A_{\varepsilon} \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right) \cdot \nabla \varphi \\
&+\int_{0}^{T} \int_{\Omega} \frac{u_{\varepsilon}}{u_{\varepsilon}+1}\left(b_{\varepsilon} \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right) \varphi-\int_{0}^{T} \int_{\Omega} \frac{u_{\varepsilon}}{u_{\varepsilon}+1} b_{\varepsilon} \cdot \nabla \varphi \\
&-\int_{0}^{T} \int_{\Omega} \frac{f\left(x, t, u_{\varepsilon}\right)}{u_{\varepsilon}+1} \varphi+\int_{0}^{T} \int_{\Omega} \frac{g_{\varepsilon}}{u_{\varepsilon}+1} \varphi \tag{2.11}
\end{align*}
$$

is valid for each $\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$. Here since $\left|\ln \left(\xi_{1}+1\right)-\ln \left(\xi_{2}+1\right)\right| \leq\left|\xi_{1}-\xi_{2}\right|$ for all $\xi_{1} \geq 0$ and $\xi_{2} \geq 0$, from (1.23) and (1.22) it follows that

$$
\begin{equation*}
-\int_{0}^{T} \int_{\Omega} \ln \left(u_{\varepsilon}+1\right) \varphi_{t}-\int_{\Omega} \ln \left(u_{0 \varepsilon}+1\right) \varphi(\cdot, 0) \rightarrow-\int_{0}^{T} \int_{\Omega} \ln (u+1) \varphi_{t}-\int_{\Omega} \ln \left(u_{0}+1\right) \varphi(\cdot, 0) \tag{2.12}
\end{equation*}
$$

as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$, and (1.23) together with (1.20) ensures that furthermore

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} \frac{g_{\varepsilon}}{u_{\varepsilon}+1} \varphi \rightarrow \int_{0}^{T} \int_{\Omega} \frac{g}{u+1} \varphi \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{2.13}
\end{equation*}
$$

because with $T_{0} \in(0, T)$ taken such that $\varphi \equiv 0$ on $\Omega \times\left(T_{0}, T\right)$ we have

$$
\begin{aligned}
\int_{0}^{T_{0}} \int_{\Omega}\left|\frac{g_{\varepsilon}}{u_{\varepsilon}+1}-\frac{g}{u+1}\right| & =\int_{0}^{T_{0}} \int_{\Omega}\left|\frac{g_{\varepsilon}-g}{u_{\varepsilon}+1}-\frac{\left(u_{\varepsilon}+u\right) g}{\left(u_{\varepsilon}+1\right)(u+1)}\right| \\
& \leq \int_{0}^{T_{0}} \int_{\Omega}\left|g_{\varepsilon}-g\right|+\int_{0}^{T_{0}} \int_{\Omega} \frac{\left|u_{\varepsilon}-u\right|}{\left(u_{\varepsilon}+1\right)(u+1)} \cdot|g|
\end{aligned}
$$

for all $\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$, and because the majorization $\frac{\left|u_{\varepsilon}-u\right|}{\left(u_{\varepsilon}+1\right)(u+1)} \leq \frac{u_{\varepsilon}}{\left(u_{\varepsilon}+1\right)(u+1)}+\frac{u}{\left(u_{\varepsilon}+1\right)(u+1)} \leq 2$ along with the dominated convergence theorem warrants that

$$
\int_{0}^{T_{0}} \int_{\Omega} \frac{\left|u_{\varepsilon}-u\right|}{\left(u_{\varepsilon}+1\right)(u+1)} \cdot|g| \rightarrow 0 \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0
$$

Moreover, since

$$
\begin{aligned}
\int_{0}^{T_{0}} \int_{\Omega}\left|\frac{f\left(x, t, u_{\varepsilon}\right)}{u_{\varepsilon}+1}\right|^{\alpha} & \leq \iint_{\left\{u_{\varepsilon}<s_{0}\right\}}|f|^{\frac{\alpha}{\alpha-1}}+K_{f}^{\frac{\alpha}{\alpha-1}} \iint_{\left\{u_{\varepsilon} \geq s_{0}\right\}} u_{\varepsilon}^{\alpha} \\
& \leq\|f\|_{L^{\infty}\left(\Omega \times\left(0, T_{0}\right) \times\left(0, s_{0}\right)\right)}^{\frac{\alpha}{\alpha-1}} \cdot|\Omega| T_{0}+K_{f}^{\frac{\alpha}{\alpha-1}} \int_{0}^{T_{0}} \int_{\Omega} u_{\varepsilon}^{\alpha} \quad \text { for all } \varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}
\end{aligned}
$$

due to (1.8), using Lemma 2.2 and that $\frac{\alpha}{\alpha-1}>1$ we infer from the accordingly implied equi-integrability property of $\left(\frac{f\left(\cdot,, u_{\varepsilon}\right)}{u_{\varepsilon}-1}\right)_{\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}}$ that again thanks to (1.23), $\frac{f\left(\cdot, \cdot, u_{\varepsilon}\right)}{u_{\varepsilon}+1} \rightarrow \frac{f(\cdot,, u)}{u+1}$ in $L^{1}\left(\Omega \times\left(0, T_{0}\right)\right)$ and hence

$$
\begin{equation*}
-\int_{0}^{T} \int_{\Omega} \frac{f\left(x, t, u_{\varepsilon}\right)}{u_{\varepsilon}+1} \varphi \rightarrow-\int_{0}^{T} \int_{\Omega} \frac{f(x, t, u)}{u+1} \varphi \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{2.14}
\end{equation*}
$$

We next rely on (1.25) to firstly see that the second summand in (2.11) satisfies

$$
\begin{equation*}
-\int_{0}^{T} \int_{\Omega}\left(A_{\varepsilon} \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right) \cdot \nabla \varphi \rightarrow-\int_{0}^{T} \int_{\Omega}(A \cdot \nabla \ln (u+1)) \cdot \nabla \varphi \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{2.15}
\end{equation*}
$$

for clearly (1.15) together with (1.16) ensures that $A_{\varepsilon} \rightarrow A$ in $L^{2}\left(\Omega \times\left(0, T_{0}\right)\right)$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$. We secondly combine (1.25) with the fact that

$$
\begin{equation*}
\frac{u_{\varepsilon}}{u_{\varepsilon}+1} b_{\varepsilon} \rightarrow \frac{u}{u+1} b \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, T)) \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{2.16}
\end{equation*}
$$

the latter resulting from (1.18) and the circumstance that $0 \leq \frac{u_{\varepsilon}}{u_{\varepsilon}+1} \leq 1$ and $\frac{u_{\varepsilon}}{u_{\varepsilon}+1} \rightarrow \frac{u}{u+1}$ a.e. in $\Omega \times(0, T)$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$ by (1.23), through a well-known stabilization feature of products involving uniformly bounded and a.e. convergent function sequences as well as strongly $L^{2}$-convergent factors ([40, Lemma 10.4]). By (1.25), namely, (2.16) guarantees that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} \frac{u_{\varepsilon}}{u_{\varepsilon}+1}\left(b_{\varepsilon} \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right) \varphi \rightarrow \int_{0}^{T} \int_{\Omega} \frac{u}{u+1}(b \cdot \nabla \ln (u+1)) \varphi \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{2.17}
\end{equation*}
$$

whereas another application of (2.16) shows that

$$
\begin{equation*}
-\int_{0}^{T} \int_{\Omega} \frac{u_{\varepsilon}}{u_{\varepsilon}+1} b_{\varepsilon} \cdot \nabla \varphi \rightarrow-\int_{0}^{T} \int_{\Omega} \frac{u}{u+1} b \cdot \nabla \varphi \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{2.18}
\end{equation*}
$$

Finally, as the matrices $A_{\varepsilon}$ are symmetric and positive definite, and hence possess self-adjoint square roots $\sqrt{A_{\varepsilon}}$, the limiting behavior of the first summand on the right of (2.11) can be made accessible to a standard argument based on lower semicontinuity with respect to weak convergence: Indeed, from (1.25), (1.23) and (1.15) it follows that also $\left(\sqrt{A_{\varepsilon}} \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right) \sqrt{\varphi} \rightharpoonup(\sqrt{A} \cdot \nabla \ln (u+1)) \sqrt{\varphi}$ in $L^{2}(\Omega \times(0, T))$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$, and that therefore

$$
\begin{aligned}
\int_{0}^{T} \int_{\Omega}\{(A \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \varphi & =\int_{0}^{T} \int_{\Omega}|(\sqrt{A} \cdot \nabla \ln (u+1)) \sqrt{\varphi}|^{2} \\
& \leq \liminf _{\varepsilon=\delta_{j_{k}} \searrow 0} \int_{0}^{T} \int_{\Omega}\left|\left(\sqrt{A_{\varepsilon}} \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right) \sqrt{\varphi}\right|^{2} \\
& =\liminf _{\varepsilon=\delta_{j_{k}} \searrow 0} \int_{0}^{T} \int_{\Omega}\left\{\left(A_{\varepsilon} \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right) \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right\} \varphi .
\end{aligned}
$$

In conjunction with $(2.12),(2.13),(2.14),(2.15),(2.17)$ and (2.18), this shows that (1.12) is a consequence of (2.11).

## 3 Turning weak into strong convergence. Proof of Theorem 1.3

Next approaching the core of our analysis, we intend to make sure that under the assumptions from Theorem 1.3, the very weak solutions obtained above are indeed weak solutions in the natural sense specifies as follows.

Definition 3.1 Let $T \in(0, \infty]$, and let $A, b, f, g$ and $u_{0}$ be such that (1.5), (1.6), (1.7), (1.9) and (1.10) are satisfied with some $k_{A}>0$ and $K_{A}>0$. Then by a weak solution of (1.4) in $\Omega \times(0, T)$ we mean a nonnegative function

$$
\begin{equation*}
u \in L_{l o c}^{1}\left([0, T) ; W^{1,1}(\Omega)\right) \tag{3.1}
\end{equation*}
$$

which is such that

$$
\begin{equation*}
u b \in L_{l o c}^{1}\left(\bar{\Omega} \times[0, T) ; \mathbb{R}^{n}\right) \quad \text { and } \quad f(\cdot, \cdot, u) \in L_{l o c}^{1}(\bar{\Omega} \times[0, T)), \tag{3.2}
\end{equation*}
$$

and that

$$
\begin{align*}
-\int_{0}^{T} \int_{\Omega} u \varphi_{t}-\int_{\Omega} \ln u_{0} \varphi(\cdot, 0)= & -\int_{0}^{T} \int_{\Omega}(A(x, t) \cdot \nabla u) \cdot \nabla \varphi-\int_{0}^{T} \int_{\Omega} u b(x, t) \cdot \nabla \varphi \\
& -\int_{0}^{T} \int_{\Omega} f(x, t, u) \varphi+\int_{0}^{T} \int_{\Omega} g(x, t) \varphi \tag{3.3}
\end{align*}
$$

for all $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, T))$.

Here a crucial step will consist in passing to the limit $\varepsilon \searrow 0$ in the respective second last summand in (3.3), which in view of (1.8) essentially amounts to turning the weak convergence feature in (1.24) into an appropriate statement on strong convergence. Our method of approaching this is in principle inspired by a strategy already pursued in previous studies (see e.g. [40, 34, 24]), namely intending to derive inequalities of the form

$$
\begin{equation*}
\int_{0}^{t_{0}} \int_{\Omega} u^{\alpha} \geq \liminf _{\varepsilon=\varepsilon_{j_{k}} \searrow 0} \int_{0}^{t_{0}} \int_{\Omega} u_{\varepsilon}^{\alpha}, \quad t_{0} \in(0, T) \tag{3.4}
\end{equation*}
$$

by estimating the left-hand side therein directly through the weak inequality (1.12); in contrast to virtually all precedent cases, however, a major challenge in the present context stems from the circumstance that the integral inequality (1.12) merely addresses $\ln (u+1)$ rather than $u$ itself, which seems to substantially impede appropriate testing procedures.
As a preparation for our main argument in this direction, to be detailed in the proof of Lemma 3.2, let us recall (cf. e.g. [40] for statements quite precisely covering the present situation) the well-known fact that for $T_{0}>0$ and $\psi \in L^{p}\left(\Omega \times\left(-1, T_{0}\right) ; \mathbb{R}^{N}\right)$ with $p \in[1, \infty]$ and $N \in \mathbb{N}$, the Steklov averages $S_{h} \psi \in L^{p}\left(\Omega \times\left(0, T_{0}\right) ; \mathbb{R}^{N}\right), h \in(0,1)$, as defined by letting

$$
\begin{equation*}
\left(S_{h} \psi\right)(x, t):=\frac{1}{h} \int_{t-h}^{t} \psi(x, s) d s, \quad x \in \Omega, t \in\left(0, T_{0}\right), h \in(0,1) \tag{3.5}
\end{equation*}
$$

in the limit $h \searrow 0$ satisfy $S_{h} \psi \rightarrow \psi$ in $L^{p}\left(\Omega \times\left(0, T_{0}\right)\right)$ whenever $p \in[1, \infty)$ and $S_{h} \psi \stackrel{\star}{\rightharpoonup} \psi$ in $L^{\infty}\left(\Omega \times\left(0, T_{0}\right)\right)$ if $p=\infty$, and that clearly $\nabla S_{h} \psi=S_{h}[\nabla \psi]$ a.e. in $\Omega \times\left(0, T_{0}\right)$ for all $h \in(0,1)$ if $\psi \in L^{1}\left(\left(-1, T_{0}\right) ; W^{1,1}(\Omega)\right)$.

By adequately exploiting (1.12) with carefully chosen test functions, we can achieve our main technical step toward th derivation of Theorem 1.3 in the following.

Lemma 3.2 Under the assumptions of Theorem 1.2, there exists a null set $N \subset(0, T)$ such that the function from Theorem 1.2 has the property that for all $t_{0} \in(0, T) \backslash N$,

$$
\begin{align*}
& \int_{\Omega} \frac{u\left(\cdot, t_{0}\right)+1}{1+\frac{u\left(\cdot, t_{0}\right)+1}{k}} \cdot \ln \left(1+\frac{u\left(\cdot, t_{0}\right)+1}{k}\right)+\int_{\Omega} \frac{k}{1+\frac{u\left(\cdot, t_{0}\right)+1}{k}} \cdot \ln \left(1+\frac{u\left(\cdot, t_{0}\right)+1}{k}\right) \\
& -\int_{\Omega} \frac{u_{0}+1}{1+\frac{u_{0}+1}{k}} \cdot \ln \left(1+\frac{u_{0}+1}{k}\right)-\int_{\Omega} \frac{k}{1+\frac{u_{0}+1}{k}} \cdot \ln \left(1+\frac{u_{0}+1}{k}\right) \\
& \geq \int_{0}^{t_{0}} \int_{\Omega} \frac{(u+1)^{2}}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}}\{(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \\
& \quad+\int_{0}^{t_{0}} \int_{\Omega} \frac{u(u+1)}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}} b(x, t) \cdot \nabla \ln (u+1) \\
& \quad-\int_{0}^{t_{0}} \int_{\Omega} \frac{f(x, t, u)}{u+1}+\int_{0}^{t_{0}} \int_{\Omega} \frac{g(x, t)}{u+1} \quad \text { for all } k \in \mathbb{N} . \tag{3.6}
\end{align*}
$$

Proof. Without loss of generality we may assume that $T$ be finite. For $k \in \mathbb{N}$, we then let

$$
\begin{equation*}
\psi_{k}(x, t):=\frac{u(x, t)+1}{1+\frac{u(x, t)+1}{k}}, \quad x \in \Omega, t \in(0, T) \tag{3.7}
\end{equation*}
$$

and using that $\psi_{0 k}:=\frac{u_{0}+1}{1+\frac{u_{0}+1}{k}}$ belongs to $L^{\infty}(\Omega)$ with $0 \leq \psi_{0 k} \leq k$ a.e. in $\Omega$ we can fix $\left(\psi_{0 k l}\right)_{l \in \mathbb{N}} \subset$ $C^{1}(\bar{\Omega})$ such that

$$
\begin{equation*}
0 \leq \psi_{0 k l} \leq 2 k \text { in } \Omega \text { for all } l \in \mathbb{N} \quad \text { and } \quad \psi_{0 k l} \rightarrow \psi_{0 k} \text { a.e. in } \Omega \text { as } l \rightarrow \infty \tag{3.8}
\end{equation*}
$$

and extend $\psi_{k}$ to a function $\psi_{k l}$ defined on all of $\Omega \times \mathbb{R}$ by letting

$$
\psi_{k l}(x, t):= \begin{cases}\psi_{0 k l}(x) & \text { if } x \in \Omega \text { and } t \leq 0  \tag{3.9}\\ \psi_{k}(x, t) & \text { if } x \in \Omega \text { and } t \in(0, T) \\ 0 & \text { if } x \in \Omega \text { and } t \geq T\end{cases}
$$

We furthermore abbreviate

$$
\begin{equation*}
\ell_{k}(\xi):=\ln \frac{\xi}{1-\frac{\xi}{k}}, \quad \xi \in[0, k), k \in \mathbb{N} \tag{3.10}
\end{equation*}
$$

and

$$
\begin{equation*}
L_{k}(\xi):=\int_{0}^{\xi} \ell_{k}(\sigma) d \sigma, \quad \xi \in[0, k), k \in \mathbb{N} \tag{3.11}
\end{equation*}
$$

so that actually $L_{k}$ is explicitly given by

$$
\begin{equation*}
L_{k}(\xi)=\xi \ln \xi+k\left(1-\frac{\xi}{k}\right) \ln \left(1-\frac{\xi}{k}\right) \quad \text { for all } \xi \in[0, k) \text { and } k \in \mathbb{N} \tag{3.12}
\end{equation*}
$$

Then

$$
L_{k}\left(\psi_{k}\right)=\frac{u+1}{1+\frac{u+1}{k}} \cdot \ln \frac{u+1}{1+\frac{u+1}{k}}+\frac{k}{1+\frac{u+1}{k}} \cdot \ln \left(1+\frac{u+1}{k}\right) \quad \text { in } \Omega \times(0, T)
$$

whence using that $u \in L_{l o c}^{1}(\bar{\Omega} \times[0, T))$ and that $\ln (1+\xi) \leq \xi$ for all $\xi \geq 0$ we conclude that besides the inclusion $\ln (u+1) \cdot \psi_{k} \in L_{l o c}^{1}(\bar{\Omega} \times[0, T))$ we also have $L_{k}\left(\psi_{k}\right) \in L_{l o c}^{1}(\bar{\Omega} \times[0, T))$, whereby it becomes possible to find a null set $N \subset(0, T)$ such that

$$
\begin{equation*}
u\left(\cdot, t_{0}\right) \in L^{1}(\Omega) \quad \text { for all } t_{0} \in(0, T) \backslash N \tag{3.13}
\end{equation*}
$$

and that moreover each $t_{0} \in(0, T) \backslash N$ is a common Lebesgue point of all the countably many mappings $(0, T) \ni t \mapsto \int_{\Omega} \ln (u(x, t)+1) \psi_{k}(x) d x$ and $(0, T) \ni t \mapsto \int_{\Omega} L_{k}\left(\psi_{k}(x, t)\right) d x$ for $k \in \mathbb{N}$.
Now given any $t_{0} \in(0, T) \backslash N$, we let

$$
\varphi(x, t) \equiv \varphi_{\delta, h}^{\left(t_{0}\right)}(x, t):=\zeta_{\delta}(t) \cdot\left(S_{h} \psi_{k l}\right)(x, t), \quad x \in \Omega, t \in(0, T), \delta \in\left(0, T-t_{0}\right), h \in(0,1),
$$

with $S_{h}$ as determined through (3.5), and with

$$
\zeta_{\delta}(t) \equiv \zeta_{\delta}^{\left(t_{0}\right)}(t):= \begin{cases}1 & \text { if } t \in\left[0, t_{0}\right]  \tag{3.14}\\ 1-\frac{t-t_{0}}{\delta} & \text { if } t \in\left(t_{0}, t_{0}+\delta\right) \\ 0 & \text { if } t \geq t_{0}+\delta,\end{cases}
$$

noting that then $\zeta_{\delta}$ belongs to $W^{1, \infty}(\mathbb{R})$ and satisfies

$$
\zeta_{\delta}^{\prime}(t)= \begin{cases}0 & \text { if } t \in[0, \infty) \backslash\left[t_{0}, t_{0}+\delta\right]  \tag{3.15}\\ -\frac{1}{\delta} & \text { if } t \in\left(t_{0}, t_{0}+\delta\right)\end{cases}
$$

Then moreover observing that

$$
\begin{equation*}
\nabla \psi_{k}=\frac{u+1}{\left(1+\frac{u+1}{k}\right)^{2}} \nabla \ln (u+1) \quad \text { a.e. in } \Omega \times(0, T) \tag{3.16}
\end{equation*}
$$

on the basis of the regularity property (1.11) one can readily verify that $\varphi$ beongs to $L^{\infty}(\Omega \times(0, T))$ with $\nabla \varphi \in L^{2}\left(\Omega \times(0, T) ; \mathbb{R}^{n}\right)$ and $\varphi_{t} \in L^{2}(\Omega \times(0, T))$, and that $\varphi=0$ a.e. in $\Omega \times\left(t_{0}+\delta, T\right)$. By means of a standard approximation argument, we therefore conclude that the integral inequality in (1.12) extends so as to remain valid for any such $\varphi=\varphi_{\delta, h}^{\left(t_{0}\right)}$, and that accordingly, by (3.15),

$$
\begin{align*}
\frac{1}{\delta} \int_{t_{0}}^{t_{0}+\delta} \int_{\Omega} & \ln (u+1) \cdot S_{h} \psi_{k l}-\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \ln (u+1) \cdot \frac{\psi_{k l}(\cdot, t)-\psi_{k l}(\cdot, t-h)}{h}-\int_{\Omega} \ln \left(u_{0}+1\right) \cdot \psi_{0 k l} \\
\geq & \int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot\{(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \cdot S_{h} \psi_{k l} \\
& -\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot(A(x, t) \cdot \nabla \ln (u+1)) \cdot S_{h}\left[\nabla \psi_{k l}\right] \\
& +\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{u}{u+1}(b(x, t) \cdot \nabla \ln (u+1)) \cdot S_{h} \psi_{k l} \\
& -\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{u}{u+1} \cdot\left(b \cdot S_{h}\left[\nabla \psi_{k l}\right]\right) \\
& -\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{f(x, t, u))}{u+1} \cdot S_{h} \psi_{k l} \\
& +\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{g(x, t)}{u+1} \cdot S_{h} \psi_{k l} \quad \text { for all } \delta \in\left(0, T-t_{0}\right), h \in(0,1) \text { and } l \in \mathbb{N} . \tag{3.17}
\end{align*}
$$

Here since (3.10) and (3.11) ensure that $\ell_{k}$ is increasing and hence $L_{k}$ is convex on $[0, k)$, we obtain the pointwise inequality

$$
\begin{aligned}
\frac{L_{k}\left(\psi_{k l}(x, t)\right)-L_{k}\left(\psi_{k l}(x, t-h)\right)}{h} & \leq L_{k}^{\prime}\left(\psi_{k l}(x, t)\right) \cdot \frac{\psi_{k l}(x, t)-\psi_{k l}(x, t-h)}{h} \\
& =\ell_{k}\left(\psi_{k l}(x, t)\right) \cdot \frac{\psi_{k l}(x, t)-\psi_{k l}(x, t-h)}{h}
\end{aligned}
$$

for a.e. $(x, t) \in \Omega \times(0, T)$, whence on the left-hand side of (3.17) we can estimate

$$
\begin{align*}
- & -\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \ell_{k}\left(\psi_{k l}(\cdot, t)\right) \cdot \frac{\psi_{k l}(\cdot, t)-\psi_{k l}(\cdot, t-h)}{h} \\
\leq & -\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{L_{k}\left(\psi_{k l}(\cdot, t)\right)-L_{k}\left(\psi_{k l}(\cdot, t-h)\right)}{h} \\
& \text { for all } \delta \in\left(0, T-t_{0}\right), h \in(0,1) \text { and } l \in \mathbb{N} . \tag{3.18}
\end{align*}
$$

Since according to our definition (3.9) of $\psi_{k l}$ a substitution shows that

$$
\begin{aligned}
&-\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{L_{k}\left(\psi_{k l}(\cdot, t)\right)-L_{k}\left(\psi_{k l}(\cdot, t-h)\right)}{h} \\
&= \int_{0}^{T} \int_{\Omega} \frac{\zeta_{\delta}(t+h)-\zeta_{\delta}(t)}{h} \cdot L_{k}\left(\psi_{k l}(\cdot, t)\right)+\int_{\Omega} L_{k}\left(\psi_{0 k l}\right) \\
& \text { for all } \delta \in\left(0, T-t_{0}\right), h \in\left(0, \min \left\{1, T-t_{0}-\delta\right\}\right) \text { and } l \in \mathbb{N},
\end{aligned}
$$

by using that clearly $\frac{\zeta_{\delta}(\cdot+h)-\zeta_{\delta}}{h} \stackrel{\star}{\star} \zeta_{\delta}^{\prime}$ in $L^{\infty}((0, \infty))$ as $h \searrow 0$ due to (3.14), we obtain that

$$
\begin{align*}
& \underset{h \searrow 0}{\limsup }\left\{-\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \ln (u+1) \cdot \frac{\psi_{k l}(\cdot, t)-\psi_{k l}(\cdot, t-h)}{h}\right\} \\
& \leq-\frac{1}{\delta} \int_{t_{0}}^{t_{0}+\delta} \int_{\Omega} L_{k}\left(\psi_{k l}\right)+\int_{\Omega} L_{k}\left(\psi_{0 k l}\right) \quad \text { for all } \delta \in\left(0, T-t_{0}\right) \text { and } l \in \mathbb{N} \tag{3.19}
\end{align*}
$$

thanks to (3.15).
Now in the remaining seven integrals in (3.17) we only need to recall that as a consequence of the inclusions $\psi_{k l} \in L^{\infty}(\Omega \times \mathbb{R})$ and $\nabla \psi_{k l} \in L^{2}\left(\Omega \times\left(-1, t_{0}+\delta\right) ; \mathbb{R}^{n}\right)$, as for each fixed $\delta \in\left(0, T-t_{0}\right)$ asserted by (3.7), (3.9) and (3.16), we have $S_{h} \psi_{k l} \stackrel{\star}{\triangle} \psi_{k l} \equiv \psi_{k}$ in $L^{\infty}\left(\Omega \times\left(0, t_{0}+\delta\right)\right)$ and $S_{h}\left[\nabla \psi_{k l}\right] \rightarrow$ $\nabla \psi_{k l} \equiv \nabla \psi_{k}$ in $L^{2}\left(\Omega \times\left(0, t_{0}+\delta\right)\right)$ as $h \searrow 0$. Since
$\left\{\ln (u+1),(A \cdot \nabla \ln (u+1)) \cdot \ln (u+1), \frac{u}{u+1}(b \cdot \nabla \ln (u+1)), \frac{f(\cdot, \cdot, u)}{u+1}, \frac{g}{u+1}\right\} \subset L^{1}\left(\Omega \times\left(0, t_{0}+\delta\right)\right)$,
and since

$$
\begin{equation*}
\left\{A \cdot \nabla \ln (u+1), \frac{u}{u+1} b\right\} \subset L^{2}\left(\Omega \times\left(0, t_{0}+\delta\right) ; \mathbb{R}^{n}\right) \tag{3.20}
\end{equation*}
$$

for any such $\delta$, namely, these properties enable us to take $h \searrow 0$ in the first integral in the left and each of the summands on the right of (3.17) to infer by using (3.19) that

$$
\begin{align*}
\frac{1}{\delta} \int_{t_{0}}^{t_{0}+\delta} \int_{\Omega} & \ln (u+1) \psi_{k}-\frac{1}{\delta} \int_{t_{0}}^{t_{0}+\delta} \int_{\Omega} L_{k}\left(\psi_{k}\right)+\int_{\Omega} L_{k}\left(\psi_{0 k l}\right)-\int_{\Omega} \ln \left(u_{0}+1\right) \psi_{0 k l} \\
\geq & \int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot\{(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \psi_{k} \\
& -\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \psi_{k} \\
& +\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{u}{u+1} \cdot(b(x, t) \cdot \nabla \ln (u+1)) \psi_{k} \\
& -\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{u}{u+1} b(x, t) \cdot \nabla \psi_{k} \\
& -\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{f(x, t, u)}{u+1} \psi_{k} \\
& +\int_{0}^{T} \int_{\Omega} \zeta_{\delta}(t) \cdot \frac{g(x, t)}{u+1} \psi_{k} \quad \text { for all } \delta \in\left(0, T-t_{0}\right) \text { and } l \in \mathbb{N} . \tag{3.22}
\end{align*}
$$

Here the Lebesgue point properties of $t_{0}$ apply so as to guarantee that on the left-hand side we have

$$
\frac{1}{\delta} \int_{t_{0}}^{t_{0}+\delta} \int_{\Omega} \ln (u+1) \psi_{k} \rightarrow \int_{\Omega} \ln \left(u\left(\cdot, t_{0}\right)+1\right) \psi_{k}\left(\cdot, t_{0}\right) \quad \text { as } \delta \searrow 0
$$

and

$$
-\frac{1}{\delta} \int_{t_{0}}^{t_{0}+\delta} \int_{\Omega} L_{k}\left(\psi_{k}\right) \rightarrow-\int_{\Omega} L_{k}\left(\psi_{k}\left(\cdot, t_{0}\right)\right) \quad \text { as } \delta \searrow 0
$$

while on the right-hand side we may use the evident fact that $\zeta_{\delta} \stackrel{\star}{\triangleleft} \zeta$ in $L^{\infty}((0, \infty))$, with $\zeta(t):=1$ for $t \in\left(0, t_{0}\right)$ and $\zeta(t):=0$ for $t \geq t_{0}$, which when combined with (3.20), (3.21) and the inclusion $\nabla \psi_{k} \in L^{2}\left(\Omega \times\left(0, t_{0}+1\right) ; \mathbb{R}^{n}\right)$ ensures that each of the integrals approach their expected limit as $\delta \searrow 0$. In conclusion, (3.22) entails that

$$
\begin{align*}
& \int_{\Omega} \ln \left(u\left(\cdot, t_{0}\right)+1\right) \psi_{k}\left(\cdot, t_{0}\right)-\int_{\Omega} L_{k}\left(\psi_{k}\left(\cdot, t_{0}\right)\right)+\int_{\Omega} L_{k}\left(\psi_{0 k l}\right)-\int_{\Omega} \ln \left(u_{0}+1\right) \psi_{0 k l} \\
& \geq \int_{0}^{t_{0}} \int_{\Omega}\{(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \psi_{k} \\
&-\int_{0}^{t_{0}} \int_{\Omega}(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \psi_{k} \\
&+\int_{0}^{t_{0}} \int_{\Omega} \frac{u}{u+1}(b(x, t) \cdot \nabla \ln (u+1)) \psi_{k} \\
&-\int_{0}^{t_{0}} \int_{\Omega} \frac{u}{u+1} b(x, t) \cdot \nabla \psi_{k} \\
&-\int_{0}^{t_{0}} \int_{\Omega} \frac{f(x, t, u)}{u+1} \psi_{k}+\int_{0}^{t_{0}} \int_{\Omega} \frac{g(x, t)}{u+1} \psi_{k} \quad \text { for all } l \in \mathbb{N} . \tag{3.23}
\end{align*}
$$

In a last limiting step, we recall the approximation property $(3.8)$ of $\left(\psi_{0 k l}\right)_{l \in \mathbb{N}}$, which through two arguments based on the dominated convergence theorem, namely, asserts that

$$
\begin{equation*}
\int_{\Omega} L_{k}\left(\psi_{0 k l}\right) \rightarrow \int_{\Omega} L_{k}\left(\psi_{0 k}\right) \quad \text { as } l \rightarrow \infty \tag{3.24}
\end{equation*}
$$

and that

$$
\begin{equation*}
-\int_{\Omega} \ln \left(u_{0}+1\right) \psi_{0 k l} \rightarrow-\int_{\Omega} \ln \left(u_{0}+1\right) \psi_{0 k} \quad \text { as } l \rightarrow \infty \tag{3.25}
\end{equation*}
$$

because for each fixed $k \in \mathbb{N}$ and all $l \in \mathbb{N}$ we have $0 \leq L_{k}\left(\psi_{0 k l}\right) \leq L_{k}(2 k)$ and $0 \leq \ln \left(u_{0}+1\right) \psi_{0 k l} \leq$ $2 k \ln \left(u_{0}+1\right)$ a.e. in $\Omega$ due to (3.8), with the majorants $L_{k}(2 k)$ and $2 k \ln \left(u_{0}+1\right)$ being integrable thanks to our assumption that $u_{0} \in L^{1}(\Omega)$.
We finally observe that according to (3.7) and the representation (3.16), on the right of (3.23) we can simplify

$$
\int_{0}^{t_{0}} \int_{\Omega}\{(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \psi_{k}-\int_{0}^{t_{0}} \int_{\Omega}(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \psi_{k}
$$

$$
\begin{align*}
= & \int_{0}^{t_{0}} \int_{\Omega} \frac{u+1}{1+\frac{u+1}{k}} \cdot\{(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \\
& -\int_{0}^{t_{0}} \int_{\Omega} \frac{u+1}{\left(1+\frac{u+1}{k}\right)^{2}} \cdot\{(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \\
= & \int_{0}^{t_{0}} \int_{\Omega} \frac{u+1}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}} \cdot\{(A(x, t) \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\} \tag{3.26}
\end{align*}
$$

and

$$
\begin{align*}
& \int_{0}^{t_{0}} \int_{\Omega} \frac{u}{u+1}(b(x, t) \cdot \nabla \ln (u+1)) \psi_{k}-\int_{0}^{t_{0}} \int_{\Omega} \frac{u}{u+1} b(x, t) \cdot \nabla \psi_{k} \\
& =\int_{0}^{t_{0}} \int_{\Omega} \frac{u}{1+\frac{u+1}{k}} b(x, t) \cdot \nabla \ln (u+1)-\int_{0}^{t_{0}} \int_{\Omega} \frac{u}{\left(1+\frac{u+1}{k}\right)^{2}} b(x, t) \cdot \nabla \ln (u+1) \\
& =\int_{0}^{t_{0}} \int_{\Omega} \frac{u(u+1)}{\left(1+\frac{u+1}{k}\right)} b(x, t) \cdot \nabla \ln (u+1) \tag{3.27}
\end{align*}
$$

as well as

$$
\begin{equation*}
-\int_{0}^{t_{0}} \int_{\Omega} \frac{f(x, t, u)}{u+1} \psi_{k}+\int_{0}^{t_{0}} \int_{\Omega} \frac{g(x, t)}{u+1} \psi_{k}=-\int_{0}^{t_{0}} \int_{\Omega} \frac{f(x, t, u)}{1+\frac{u+1}{k}}+\int_{0}^{t_{0}} \int_{\Omega} \frac{g(x, t)}{1+\frac{u+1}{k}} . \tag{3.28}
\end{equation*}
$$

Similarly inserting (3.7) into (3.24) and (3.25), in view of the definition (3.12) of $L_{k}$ we immediately conclude that (3.6) is a consequence of (3.23)-(3.28).

Now if $b$ complies with the regularity assumptions from Theorem 1.3 , then the above can be combined with the convergence statements from Theorem 1.2 to deduce (3.4), and hence the desired strong approximation property, in the following sense.

Lemma 3.3 In addition to the assumptions from Theorem 1.2, suppose that

$$
\begin{equation*}
b \in L_{l o c}^{q}\left(\bar{\Omega} \times[0, T) ; \mathbb{R}^{n}\right) \quad \text { for some } q \geq \frac{2 \alpha}{\alpha-1} \tag{3.29}
\end{equation*}
$$

Then there exists a null set $N_{\star} \subset(0, T)$ such that with $u$ and $\left(\varepsilon_{j_{k}}\right)_{k \in \mathbb{N}}$ as given by Theorem 1.2 we have

$$
\begin{equation*}
u_{\varepsilon} \rightarrow u \quad \text { in } L^{\alpha}\left(\Omega \times\left(0, t_{0}\right)\right) \quad \text { for all } t_{0} \in(0, T) \backslash N_{\star} \tag{3.30}
\end{equation*}
$$

as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$.
Proof. Since Lemma 2.5 especially entails that for a.e. $t_{0} \in(0, T)$ we have

$$
\begin{equation*}
u_{\varepsilon}\left(\cdot, t_{0}\right) \rightarrow u\left(\cdot, t_{0}\right) \quad \text { in } L^{1}(\Omega) \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0, \tag{3.31}
\end{equation*}
$$

according to Lemma 3.2 we can pick a null set $N_{\star} \subset(0, T)$ with the property that both (3.31) and (3.6) hold for each $t_{0} \in(0, T) \backslash N_{\star}$ and all $k \in \mathbb{N}$. Using that (3.31) in particular warrants that for any such $t_{0}$ we know that $u\left(\cdot, t_{0}\right)+1$ belongs to $L^{1}(\Omega)$ and hence is finite a.e. in $\Omega$, we see that

$$
\frac{u\left(\cdot, t_{0}\right)+1}{1+\frac{u\left(\cdot, t_{0}\right)+1}{k}} \cdot \ln \left(1+\frac{u\left(\cdot, t_{0}\right)+1}{k}\right) \rightarrow 0 \quad \text { a.e. in } \Omega \quad \text { as } k \rightarrow \infty,
$$

whereas the validity of

$$
\begin{equation*}
0 \leq \ln (1+\xi) \leq \xi \quad \text { for all } \xi \geq 0 \tag{3.32}
\end{equation*}
$$

asserts the majorization

$$
\begin{aligned}
0 & \leq \frac{u\left(\cdot, t_{0}\right)+1}{1+\frac{u\left(\cdot, t_{0}\right)+1}{k}} \cdot \ln \left(1+\frac{u\left(\cdot, t_{0}\right)+1}{k}\right) \\
& \leq \frac{u\left(\cdot, t_{0}\right)+1}{1+\frac{u\left(\cdot, t_{0}\right)+1}{k}} \cdot \frac{u\left(\cdot, t_{0}\right)+1}{k} \\
& \leq u\left(\cdot, t_{0}\right)+1 \quad \text { a.e. in } \Omega .
\end{aligned}
$$

Therefore, the dominated convergence theorem ensures that

$$
\begin{equation*}
\int_{\Omega} \frac{u\left(\cdot, t_{0}\right)+1}{1+\frac{u\left(\cdot, t_{0}\right)+1}{k}} \cdot \ln \left(1+\frac{u\left(\cdot, t_{0}\right)+1}{k}\right) \rightarrow 0 \quad \text { as } k \rightarrow \infty \quad \text { for all } t_{0} \in(0, T) \backslash N_{\star}, \tag{3.33}
\end{equation*}
$$

and quite a similar reasoning based on (1.22) shows that

$$
\begin{equation*}
\int_{\Omega} \frac{u_{0}+1}{1+\frac{u_{0}+1}{k}} \cdot \ln \left(1+\frac{u_{0}+1}{k}\right) \rightarrow 0 \quad \text { as } k \rightarrow \infty \tag{3.34}
\end{equation*}
$$

Next, once more relying on (3.31), by means of the l'Hospital rule we readily find that

$$
\frac{k}{1+\frac{u\left(\cdot t_{0}\right)+1}{k}} \cdot \ln \left(1+\frac{u\left(\cdot, t_{0}\right)+1}{k}\right) \rightarrow u\left(\cdot, t_{0}\right)+1 \quad \text { a.e. in } \Omega \quad \text { as } k \rightarrow \infty
$$

while thanks to (3.32),

$$
\begin{aligned}
0 & \leq \frac{k}{1+\frac{u\left(\cdot, t_{0}\right)+1}{k}} \cdot \ln \left(1+\frac{u\left(\cdot, t_{0}\right)+1}{k}\right) \\
& \leq \frac{u\left(\cdot, t_{0}\right)+1}{1+\frac{u\left(\cdot, t_{0}\right)+1}{k}} \\
& \leq u\left(\cdot, t_{0}\right)+1 \quad \text { a.e. in } \Omega .
\end{aligned}
$$

Again by the dominated convergence theorem, we thus obtain that

$$
\begin{equation*}
\int_{\Omega} \frac{k}{1+\frac{u\left(\cdot, t_{0}\right)+1}{k}} \cdot \ln \left(1+\frac{u\left(\cdot, t_{0}\right)+1}{k}\right) \rightarrow \int_{\Omega}\left(u\left(\cdot, t_{0}\right)+1\right) \quad \text { as } k \rightarrow \infty \quad \text { for all } t_{0} \in(0, T) \backslash N_{\star} \tag{3.35}
\end{equation*}
$$

and that, similarly,

$$
\begin{equation*}
\int_{\Omega} \frac{k}{1+\frac{u_{0}+1}{k}} \cdot \ln \left(1+\frac{u_{0}+1}{k}\right) \rightarrow \int_{\Omega}\left(u_{0}+1\right) \quad \text { as } k \rightarrow \infty . \tag{3.36}
\end{equation*}
$$

Now on the right-hand side in (3.6), in order to adequately cope with the second summand we first recall (1.15) and invoke Young's inequality to estimate

$$
\begin{align*}
& \int_{0}^{t_{0}} \int_{\Omega} \frac{(u+1)^{2}}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}} \cdot\{(A \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\}+\int_{0}^{t_{0}} \int_{\Omega} \frac{u(u+1)}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}} b \cdot \nabla \ln (u+1) \\
& \quad \geq k_{A} \int_{0}^{t_{0}} \int_{\Omega} \frac{(u+1)^{2}}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}}|\nabla \ln (u+1)|^{2}+\int_{0}^{t_{0}} \int_{\Omega} \frac{u(u+1)}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}} b \cdot \nabla \ln (u+1) \\
& \geq-\frac{1}{4 k_{A}} \int_{0}^{t_{0}} \int_{\Omega} \frac{u^{2}}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}}|b|^{2} \quad \text { for all } t_{0} \in(0, T) . \tag{3.37}
\end{align*}
$$

Since

$$
\frac{u^{2}}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}} \leq \frac{u^{2}}{k \cdot\left(1+\frac{u+1}{k}\right)}=\frac{u^{2}}{u+1+k} \leq \frac{u^{2}}{u+k} \quad \text { a.e. in } \Omega \times(0, T)
$$

using the Hölder inequality we see that here

$$
\begin{equation*}
\int_{0}^{t_{0}} \int_{\Omega} \frac{u^{2}}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}}|b|^{2} \leq\left\{\int_{0}^{t_{0}} \int_{\Omega}\left(\frac{u^{2}}{u+k}\right)^{\frac{q}{q-2}}\right\}^{\frac{q-2}{q}} \cdot\left\{\int_{0}^{t_{0}} \int_{\Omega}|b|^{q}\right\}^{\frac{2}{q}} \quad \text { for all } t_{0} \in(0, T) \tag{3.38}
\end{equation*}
$$

and observe that the first integrand on the right satisfies $\left(\frac{u^{2}}{u+k}\right)^{\frac{q}{q-2}} \rightarrow 0$ a.e. in $\Omega \times\left(0, t_{0}\right)$ as $k \rightarrow \infty$, and is majorized according to $\left(\frac{u^{2}}{u+k}\right)^{\frac{q}{q-2}} \leq u^{\frac{q}{q-2}}$ a.e. in $\Omega \times\left(0, t_{0}\right)$ with $u^{\frac{q}{q-2}} \in L^{1}\left(\Omega \times\left(0, t_{0}\right)\right)$ due to Lemma 2.5 and the fact that $\frac{q}{q-2}=\frac{1}{1-\frac{2}{q}} \leq \frac{1}{1-2 \cdot \frac{\alpha-1}{2 \alpha}}=\alpha$ by hypothesis. As a further consequence of the dominated convergence theorem, from (3.38) we thus infer that

$$
\int_{0}^{t_{0}} \int_{\Omega} \frac{u^{2}}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}}|b|^{2} \rightarrow 0 \quad \text { as } k \rightarrow \infty \quad \text { for all } t_{0} \in(0, T)
$$

and that hence, by (3.37),

$$
\begin{align*}
& \liminf _{k \rightarrow \infty}\left\{\int_{0}^{t_{0}} \int_{\Omega} \frac{(u+1)^{2}}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}} \cdot\{(A \cdot \nabla \ln (u+1)) \cdot \nabla \ln (u+1)\}\right. \\
&+\left.\int_{0}^{t_{0}} \int_{\Omega} \frac{u(u+1)}{k \cdot\left(1+\frac{u+1}{k}\right)^{2}} b \cdot \nabla \ln (u+1)\right\} \geq 0 \quad \text { for all } t_{0} \in(0, T) \tag{3.39}
\end{align*}
$$

Finally, two further arguments based on dominated convergence show that thanks to (1.9) and the inclusion $f(\cdot, \cdot, u) \in L_{l o c}^{\alpha}(\bar{\Omega} \times[0, T))$, as asserted by (1.24) in view of (1.7) and (1.8),

$$
\begin{equation*}
\int_{0}^{t_{0}} \int_{\Omega} \frac{g(x, t)}{1+\frac{u+1}{k}} \rightarrow \int_{0}^{t_{0}} \int_{\Omega} g(x, t) \quad \text { as } k \rightarrow \infty \quad \text { for all } t_{0} \in(0, T) \tag{3.40}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{t_{0}} \int_{\Omega} \frac{f(x, t, u)}{1+\frac{u+1}{k}} \rightarrow \int_{0}^{t_{0}} \int_{\Omega} f(x, t, u) \quad \text { as } k \rightarrow \infty \quad \text { for all } t_{0} \in(0, T) \tag{3.41}
\end{equation*}
$$

In summary, upon collecting (3.33)-(3.36) and (3.39)-(3.41) we obtain from Lemma 3.2 that

$$
\begin{equation*}
\int_{0}^{t_{0}} \int_{\Omega} f(x, t, u) \geq \int_{0}^{t_{0}} \int_{\Omega} g(x, t)-\int_{\Omega} u\left(\cdot, t_{0}\right)+\int_{\Omega} u_{0} \quad \text { for all } t_{0} \in(0, T) \backslash N_{\star}, \tag{3.42}
\end{equation*}
$$

where now making full use of (3.31) we see that due to (1.20) and (1.22), the right-hand side appears as a limit of the corresponding expressions associated with (1.14) in the sense that for all $t_{0} \in(0, T) \backslash N_{\star}$,

$$
\int_{0}^{t_{0}} \int_{\Omega} g_{\varepsilon}(x, t)-\int_{\Omega} u_{\varepsilon}\left(\cdot, t_{0}\right)+\int_{\Omega} u_{0 \varepsilon} \rightarrow \int_{0}^{t_{0}} \int_{\Omega} g(x, t)-\int_{\Omega} u\left(\cdot, t_{0}\right)+\int_{\Omega} u_{0} \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 .
$$

Since moreover, again by dominated convergence,

$$
\int_{0}^{t_{0}} \int_{\Omega} f_{-}\left(x, t, u_{\varepsilon}\right) \rightarrow \int_{0}^{t_{0}} \int_{\Omega} f_{-}(x, t, u) \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \quad \text { for all } t_{0} \in(0, T)
$$

thanks to (1.23) and the boundedness of $f_{-}$in $\Omega \times\left(0, t_{0}\right) \times[0, \infty)$ for $t_{0} \in(0, T)$, as implied by (1.7) and (1.8), from (3.42) and Lemma 2.1 we infer that

$$
\begin{aligned}
\int_{0}^{t_{0}} \int_{\Omega} f_{+}(x, t, u) & =\int_{0}^{t_{0}} \int_{\Omega} f(x, t, u)+\int_{0}^{t_{0}} \int_{\Omega} f_{-}(x, t, u) \\
& \geq \int_{0}^{t_{0}} \int_{\Omega} g(x, t)-\int_{\Omega} u\left(\cdot, t_{0}\right)+\int_{\Omega} u_{0}+\int_{0}^{t_{0}} \int_{\Omega} f_{-}(x, t, u) \\
& =\lim _{\varepsilon=\varepsilon_{j_{k}} \searrow 0}\left\{\int_{0}^{t_{0}} \int_{\Omega} g_{\varepsilon}(x, t)-\int_{\Omega} u_{\varepsilon}\left(\cdot, t_{0}\right)+\int_{\Omega} u_{0 \varepsilon}+\int_{0}^{t_{0}} \int_{\Omega} f_{-}\left(x, t, u_{\varepsilon}\right)\right\} \\
& =\lim _{\varepsilon=\varepsilon_{j_{k}} \searrow 0}\left\{\int_{0}^{t_{0}} \int_{\Omega} f\left(x, t, u_{\varepsilon}\right)+\int_{0}^{t_{0}} \int_{\Omega} f_{-}\left(x, t, u_{\varepsilon}\right)\right\} \\
& =\lim _{\varepsilon=\varepsilon_{j_{k}} \searrow 0} \int_{0}^{t_{0}} \int_{\Omega} f_{+}\left(x, t, u_{\varepsilon}\right) \quad \text { for all } t_{0} \in(0, T) \backslash N_{\star} .
\end{aligned}
$$

As furthermore $\int_{0}^{t_{0}} \int_{\Omega} f_{+}(x, t, u) \leq \liminf _{\varepsilon=\varepsilon_{j_{k}} \searrow 0} \int_{0}^{t_{0}} \int_{\Omega} f_{+}\left(x, t, u_{\varepsilon}\right)$ for all $t_{0} \in(0, T)$ due to (1.23) and Fatou's lemma, this means that actually

$$
\int_{0}^{t_{0}} \int_{\Omega} f_{+}\left(x, t, u_{\varepsilon}\right) \rightarrow \int_{0}^{t_{0}} \int_{\Omega} f_{+}(x, t, u) \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \quad \text { for all } t_{0} \in(0, T) \backslash N_{\star},
$$

which again in view of (1.23) implies that for any such $t_{0}, f_{+}\left(\cdot, \cdot, u_{\varepsilon}\right) \rightarrow f_{+}(\cdot, \cdot, u)$ in $L^{1}\left(\Omega \times\left(0, t_{0}\right)\right)$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$. Since (1.8) entails that

$$
u_{\varepsilon}^{\alpha} \leq \max \left\{s_{0}^{\alpha}, \frac{f_{+}\left(\cdot, \cdot, u_{\varepsilon}\right)}{k_{f}}\right\} \quad \text { a.e. in } \Omega \quad \text { for all } \varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}
$$

one final application of a dominated convergence principle reveals that again by (1.23),

$$
\int_{0}^{t_{0}} \int_{\Omega} u_{\varepsilon}^{\alpha} \rightarrow \int_{0}^{t_{0}} \int_{\Omega} u^{\alpha} \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \quad \text { for all } t_{0} \in(0, T) \backslash N_{\star}
$$

Together with the weak convergence statement in (1.24), by uniform convexity of $L^{\alpha}\left(\Omega \times\left(0, t_{0}\right)\right)$ for all $t_{0}>0$ this yields (3.30).

### 3.1 Proof of Theorem 1.3

As a last preliminary for Theorem 1.3 , let us state a chain rule type statement which should be essentially well-known, but for which we include a brief argument as we could not find a precise reference in the literature.

Lemma 3.4 Let $w: \Omega \rightarrow \mathbb{R}$ be measurable and nonnegative and such that $e^{w} \in L^{2}(\Omega)$ as well as $\nabla w \in L^{2}\left(\Omega ; \mathbb{R}^{n}\right)$. Then $e^{w} \in W^{1,1}(\Omega)$ with $\nabla e^{w}=e^{w} \nabla w$ a.e. in $\Omega$.

Proof. For $k \in \mathbb{N}$ letting $\rho_{k}(\xi):=\min \left\{e^{\xi}, e^{k}\right\}$, due to the Lipschitz continuity of $\rho_{k}$ we may invoke a well-known version of the chain rule in $W^{1,2}(\Omega)$ to infer from the inclusions $e^{w} \in L^{2}(\Omega)$ and $\nabla w \in L^{2}\left(\Omega ; \mathbb{R}^{n}\right)$ that $\rho_{k}(w)$ belongs to $W^{1,2}(\Omega)$ with

$$
\begin{equation*}
\nabla \rho_{k}(w)=\chi_{\{w<k\}} e^{w} \nabla w \quad \text { a.e. in } \Omega . \tag{3.43}
\end{equation*}
$$

Accordingly, for integers $k$ and $l$ with $l>k$ we can estimate

$$
\int_{\Omega}\left|\nabla \rho_{l}(w)-\nabla \rho_{k}(w)\right|=\int_{\{k \leq w<l+1\}} e^{w}|\nabla w| \leq\left\{\int_{\{w \geq k\}} e^{2 w}\right\}^{\frac{1}{2}} \cdot\left\{\int_{\{w \geq k\}}|\nabla w|^{2}\right\}^{\frac{1}{2}}
$$

whence again by hypothesis we conclude that $\left(\nabla \rho_{k}(w)\right)_{k \in \mathbb{N}}$ forms a Cauchy sequence in $L^{1}\left(\Omega ; \mathbb{R}^{n}\right)$. Since, on the other hand, clearly $\rho_{k}(w) \rightarrow e^{w}$ in $L^{2}(\Omega)$ as $k \rightarrow \infty$ by Beppo Levi's theorem, we thus must have $\nabla \rho_{k}(w) \rightarrow \nabla e^{w}$ in $L^{1}(\Omega)$ as $k \rightarrow \infty$, so that the claim results on observing that an application of the dominated convergence theorem to (3.43) directly shows that $\nabla \rho_{k}(w) \rightarrow e^{w} \nabla w$ in $L^{1}(\Omega)$ as $k \rightarrow \infty$.

We are now in the position to verify our main result on genuine weak solvability in (1.4).
Proof of Theorem 1.3. We fix $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, T))$ and then obtain on integrating by parts in (1.14) that

$$
\begin{align*}
-\int_{0}^{T} \int_{\Omega} u_{\varepsilon} \varphi_{t}-\int_{\Omega} u_{0 \varepsilon} \varphi(\cdot, 0)= & -\int_{0}^{T} \int_{\Omega}\left(A_{\varepsilon}(x, t) \cdot \nabla u_{\varepsilon}\right) \cdot \nabla \varphi-\int_{0}^{T} \int_{\Omega} u_{\varepsilon} b_{\varepsilon}(x, t) \cdot \nabla \varphi \\
& -\int_{0}^{T} \int_{\Omega} f\left(x, t, u_{\varepsilon}\right) \varphi+\int_{0}^{T} \int_{\Omega} g_{\varepsilon}(x, t) \varphi \tag{3.44}
\end{align*}
$$

for all $\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$. Here (1.22) and (1.20) directly yield

$$
\begin{equation*}
-\int_{\Omega} u_{0 \varepsilon} \varphi(\cdot, 0) \rightarrow-\int_{\Omega} u_{0} \varphi(\cdot, 0) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{3.45}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} g_{\varepsilon}(x, t) \varphi \rightarrow \int_{0}^{T} \int_{\Omega} g(x, t) \varphi \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{3.46}
\end{equation*}
$$

while relying on (1.23) and (1.24) we see that with $\left(\varepsilon_{j_{k}}\right)_{k \in \mathbb{N}}$ as provided by Theorem 1.2 we have

$$
\begin{equation*}
-\int_{0}^{T} \int_{\Omega} u_{\varepsilon} \varphi_{t} \rightarrow-\int_{0}^{T} \int_{\Omega} u \varphi_{t} \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{3.47}
\end{equation*}
$$

and

$$
\begin{equation*}
-\int_{0}^{T} \int_{\Omega} u_{\varepsilon} b_{\varepsilon}(x, t) \cdot \nabla \varphi \rightarrow-\int_{0}^{T} \int_{\Omega} u b(x, t) \cdot \nabla \varphi \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{3.48}
\end{equation*}
$$

because due to (1.24) our hypothesis $\alpha \geq 2$ in particular implies that $u_{\varepsilon} \rightharpoonup u$ in $L_{l o c}^{2}(\bar{\Omega} \times[0, T))$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$, and because $b_{\varepsilon} \rightarrow b$ in $L_{l o c}^{2}(\bar{\Omega} \times[0, T))$ as $\varepsilon=\varepsilon_{j} \searrow 0$ by (1.18).
In appropriately passing to the limit in the crucial first and third summand on the right of (3.44), we now make essential use of Lemma 3.3 by fixing the null set $N_{\star} \subset(0, T)$ as given there, and taking $t_{0} \in(0, T) \backslash N_{\star}$ sufficiently close to $T$ such that $\varphi \equiv 0$ in $\Omega \times\left(t_{0}, T\right)$, so that Lemma 3.3 guarantees that

$$
\begin{equation*}
u_{\varepsilon} \rightarrow u \quad \text { in } L^{\alpha}\left(\Omega \times\left(0, t_{0}\right)\right) \hookrightarrow L^{2}\left(\Omega \times\left(0, t_{0}\right)\right) \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 . \tag{3.49}
\end{equation*}
$$

Therefore, namely, we firstly obtain that according to the dominated convergence theorem,

$$
\begin{equation*}
\left(u_{\varepsilon}+1\right)\left(A_{\varepsilon}\right)_{i j} \rightarrow(u+1) A_{i j} \quad \text { in } L^{2}\left(\Omega \times\left(0, t_{0}\right)\right) \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{3.50}
\end{equation*}
$$

because in the majorization $\left\{\left(u_{\varepsilon}+1\right)\left(A_{\varepsilon}\right)_{i j}\right\}^{2} \leq K_{a}^{2}\left(u_{\varepsilon}+1\right)^{2}$ asserted by (1.15) the right-hand side is convergent in $L^{1}\left(\Omega \times\left(0, t_{0}\right)\right)$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$ by (3.49), and because $\left(u_{\varepsilon}+1\right)\left(A_{\varepsilon}\right)_{i j} \rightarrow(u+1) A_{i j}$ a.e. in $\Omega \times(0, T)$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$ due to (1.23) and (1.16).
Combining (3.50) with the weak convergence statement in (1.25), by means of the chain rule-type result from Lemma 3.4 we thus conclude that

$$
\begin{align*}
-\int_{0}^{T} \int_{\Omega}\left(A_{\varepsilon}(x, t) \cdot \nabla u_{\varepsilon}\right) \cdot \nabla \varphi & =-\int_{0}^{t_{0}} \int_{\Omega}\left(u_{\varepsilon}+1\right)\left\{A_{\varepsilon}(x, t) \cdot \nabla \ln \left(u_{\varepsilon}+1\right)\right\} \cdot \nabla \varphi \\
& \rightarrow-\int_{0}^{t_{0}} \int_{\Omega}(u+1)\{A(x, t) \cdot \nabla \ln (u+1)\} \cdot \nabla \varphi \\
& =-\int_{0}^{T} \int_{\Omega}(A(x, t) \cdot \nabla u) \cdot \nabla \varphi \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 . \tag{3.51}
\end{align*}
$$

We secondly make full use of the strong convergence property (3.49) in the space $L^{\alpha}\left(\Omega \times\left(0, t_{0}\right)\right)$, actually possibly smaller than $L^{2}\left(\Omega \times\left(0, t_{0}\right)\right)$, to treat the superlinear nonlinearity in (3.44): Since from (1.8) we know that

$$
\left|f\left(x, t, u_{\varepsilon}\right)\right| \leq\|f\|_{L^{\infty}\left(\Omega \times(0, T) \times\left(0, s_{0}\right)\right.}+K_{f} u_{\varepsilon}^{\alpha} \quad \text { in } \Omega \times(0, T) \quad \text { for all } \varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}
$$

and since herein $u_{\varepsilon}^{\alpha} \rightarrow u^{\alpha}$ in $L^{1}\left(\Omega \times\left(0, t_{0}\right)\right)$ as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$ by (3.49), once more employing the dominated convergence theorem we see that thanks to (1.23) and the continuity of $f$,

$$
\begin{equation*}
f\left(\cdot, \cdot, u_{\varepsilon}\right) \rightarrow f(\cdot, \cdot, u) \quad \text { in } L^{1}\left(\Omega \times\left(0, t_{0}\right)\right) \quad \text { as } \varepsilon=\varepsilon_{j_{k}} \searrow 0 \tag{3.52}
\end{equation*}
$$

and that thus

$$
-\int_{0}^{T} \int_{\Omega} f\left(x, t, u_{\varepsilon}\right) \varphi=-\int_{0}^{t_{0}} \int_{\Omega} f\left(x, t, u_{\varepsilon}\right) \varphi \rightarrow-\int_{0}^{t_{0}} \int_{\Omega} f(x, t, u) \varphi=-\int_{0}^{T} \int_{\Omega} f(x, t, u) \varphi
$$

as $\varepsilon=\varepsilon_{j_{k}} \searrow 0$. In conjunction with (3.45) and (3.48) and (3.51), this yields (3.3) as a consequence of (3.44) upon taking $\varepsilon=\varepsilon_{j_{k}} \searrow 0$, so that the proof becomes complete by noting that the regularity requirements in (3.1) and (3.2) are direct consequences of the integrability propertis implied by (1.23)(1.25) and (3.52) when combined with Lemma 3.4.

## 4 Application to logistic Keller-Segel systems

As our first concrete example, we here consider the logistic Keller-Segel system (1.28) under the permanent assumption that the reaction term $F$ therein satisfies (1.29) with some $k_{F}>0, K_{F}>0$, $s_{0}>0$ and $\alpha>1$, and that the initial data are such that

$$
\begin{equation*}
u_{0} \in L^{1}(\Omega) \quad \text { and } \quad v_{0} \in L^{2}(\Omega) \quad \text { are nonnegative. } \tag{4.1}
\end{equation*}
$$

Then adapting well-established arguments $([1,13,36,17])$ readily shows that if we fix $\left(u_{0 \varepsilon}\right)_{\varepsilon \in(0,1)} \subset$ $C^{1}(\bar{\Omega})$ and $\left(v_{0 \varepsilon}\right)_{\varepsilon \in(0,1)} \subset C^{2}(\bar{\Omega})$ such that $\frac{\partial v_{0 \varepsilon}}{\partial \nu}=0$ on $\partial \Omega$, and that

$$
\begin{equation*}
0 \leq u_{0 \varepsilon} \rightarrow u_{0} \quad \text { in } L^{1}(\Omega) \quad \text { and } \quad v_{0 \varepsilon} \rightarrow v_{0} \quad \text { in } L^{2}(\Omega) \quad \text { as } \varepsilon \searrow 0 \tag{4.2}
\end{equation*}
$$

each of the problems

$$
\begin{cases}u_{\varepsilon t}=\Delta u_{\varepsilon}-\nabla \cdot\left(u_{\varepsilon} \nabla v_{\varepsilon}\right)+\lambda u_{\varepsilon}-\mu u_{\varepsilon}^{\alpha}, & x \in \Omega, t>0  \tag{4.3}\\ v_{\varepsilon t}=\Delta v_{\varepsilon}-v_{\varepsilon}+\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}, & x \in \Omega, t>0 \\ \frac{\partial u}{\partial \nu}=\frac{\partial v}{\partial \nu}=0, & x \in \partial \Omega, t>0 \\ u(x, 0)=u_{0}(x), \quad v(x, 0)=v_{0}(x), & x \in \Omega\end{cases}
$$

admits a global classical solution $\left(u_{\varepsilon}, v_{\varepsilon}\right), \varepsilon \in(0,1)$, with $0 \leq u_{\varepsilon} \in C^{0}(\bar{\Omega} \times[0, \infty)) \cap C^{2,1}(\bar{\Omega} \times(0, \infty))$ and $0 \leq v_{\varepsilon} \in \bigcap_{q>n} C^{0}\left([0, \infty) ; W^{1, q}(\Omega)\right) \cap C^{2,1}(\bar{\Omega} \times(0, \infty))$.
In order to make our general results derived above applicable to the present particular setting, for $\varepsilon \in(0,1)$ we let $A_{i j}=\left(A_{\varepsilon}\right)_{i j}:=\delta_{i j}, i, j \in\{1, \ldots, n\}$, and $b_{\varepsilon}:=-\nabla v_{\varepsilon}$ as well as $f(x, t, s):=-F(s)$ and $g(x, t)=g_{\varepsilon}(x, t):=0$ for $x \in \bar{\Omega}, t \geq 0$ and $s \geq 0$. Then (1.5), (1.15) and (1.16) as well as (1.9), (1.19) and (1.20) are trivially satisfied, while (1.17), (1.19) (1.10), (1.21) and (1.22) are asserted by the regularity properties of $u_{\varepsilon}$ and $v_{\varepsilon}$ and the requirements on $u_{0}$ and $u_{0 \varepsilon}$ in (4.1) and (4.2); furthermore, our choice of $f$ is compatible with (1.7) and (1.8) due to (1.29).

### 4.1 Very weak solutions. Proof of Theorem 1.4

In light of the above observations, for an application of Theorem 1.2 it will thus be sufficient to find $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \subset(0,1)$ and $v \in L_{l o c}^{2}\left([0, \infty) ; W^{1,2}(\Omega)\right)$ such that $\varepsilon_{j} \searrow 0$ as $j \rightarrow \infty$, and that

$$
\begin{equation*}
\nabla v_{\varepsilon} \rightarrow \nabla v \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{4.4}
\end{equation*}
$$

This will be achieved through an analysis of the specific systems (1.28) and (4.3), particularly focusing on the second equation therein as the main additional ingredient in comparison to (1.4) and (1.14), but in some places as well resorting to statements derived for the latter general setting in Section 2. A fundamental property of (4.3), for instance, has been achieved in Lemma 2.2 already:

Lemma 4.1 Suppose that (1.29) holds with some $k_{F}>0, K_{F}>0$ and $\alpha>1$. Then for all $T>0$ there exists $C(T)>0$ such that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{\alpha}(x, t) d x d t \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{4.5}
\end{equation*}
$$

Proof. Noting that our above selections warrant applicability of Lemma 2.2, we immediately obtain (4.5) from (2.4).

By relying on appropriate smoothing properties of the inhomogeneous heat equation satisfied by $v_{\varepsilon}$, the previous lemma firstly entails a uniform spatial $L^{2}$ bound for $v_{\varepsilon}$ whenever $\alpha$ complies with the largeness assumption from Theorem 1.4.

Lemma 4.2 If (1.29) is valid with some $k_{F}>0, K_{F}>0$ and

$$
\begin{equation*}
\alpha>\frac{2 n+4}{n+4} \tag{4.6}
\end{equation*}
$$

then for all $T>0$ one can find $C(T)>0$ such that for any $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\left\|v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)} \leq C(T) \quad \text { for all } t \in(0, T) \tag{4.7}
\end{equation*}
$$

Proof. By a well-known $L^{p}-L^{q}$ estimate for the Neumann heat semigroup $\left(e^{t \Delta}\right)_{t \geq 0}$ on $\Omega$ ([37, Lemma 1.3]), there exists $c_{1}>0$ such that

$$
\left\|e^{t \Delta} \phi\right\|_{L^{2}(\Omega)} \leq c_{1}\left(1+t^{-\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+}}\right)\|\phi\|_{L^{\alpha}(\Omega)} \quad \text { for all } t>0 \text { and each } \phi \in L^{\alpha}(\Omega)
$$

Since $e^{t \Delta}$ acts as a contraction on $L^{2}(\Omega)$ for all $t>0$, according to a Duhamel representation associated with the second equation in (4.3) we can therefore estimate

$$
\begin{align*}
\left\|v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)} & =\left\|e^{t(\Delta-1)} v_{0 \varepsilon}+\int_{0}^{t} e^{(t-s) \Delta} \frac{u_{\varepsilon}(\cdot, s)}{1+\varepsilon u_{\varepsilon}(\cdot, s)} d s\right\|_{L^{2}(\Omega)} \\
& \leq e^{-t}\left\|v_{0 \varepsilon}\right\|_{L^{2}(\Omega)}+c_{1} \int_{0}^{t}\left(1+(t-s)^{-\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+}}\right) e^{-(t-s)} \|_{\frac{u_{\varepsilon}(\cdot, s)}{1+\varepsilon u_{\varepsilon}(\cdot, s)} \|_{L^{\alpha}(\Omega)} d s} \\
& \leq c_{2}+c_{1} \int_{0}^{t}\left(1+(t-s)^{-\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+}}\right)\left\|u_{\varepsilon}(\cdot, s)\right\|_{L^{\alpha}(\Omega)} d s \tag{4.8}
\end{align*}
$$

for all $t>0$ and $\varepsilon \in(0,1)$, with $c_{2}:=\sup _{\varepsilon \in(0,1)}\left\|v_{0 \varepsilon}\right\|_{L^{2}(\Omega)}$ being finite according to (4.2). Here using Young's inequality, given $T>0$ we see that for all $t \in(0, T)$ and $\varepsilon \in(0,1)$,

$$
\begin{align*}
\int_{0}^{t}\left(1+(t-s)^{-\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+}}\right)\left\|u_{\varepsilon}(\cdot, s)\right\|_{L^{\alpha}(\Omega)} d s & \leq \int_{0}^{t}\left(1+(t-s)^{-\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+}}\right)^{\frac{\alpha}{\alpha-1}} d s+\int_{0}^{t}\left\|u_{\varepsilon}(\cdot, s)\right\|_{L^{\alpha}(\Omega)}^{\alpha} d s \\
& \leq \int_{0}^{T}\left(1+\sigma^{-\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+}}\right)^{\frac{\alpha}{\alpha-1}} d \sigma+\int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{\alpha} \tag{4.9}
\end{align*}
$$

where by Lemma 4.1 we can find $c_{3}(T)>0$ such that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{\alpha} \leq c_{3}(T) \quad \text { for all } \varepsilon \in(0,1) \tag{4.10}
\end{equation*}
$$

and where our hypothesis (4.6) ensures that moreover

$$
c_{4}(T):=\int_{0}^{T}\left(1+\sigma^{-\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+}}\right)^{\frac{\alpha}{\alpha-1}} d \sigma<\infty:
$$

Indeed, again by means of Young's inequality we obtain that

$$
c_{4}(T) \leq 2^{\frac{1}{\alpha-1}} \int_{0}^{T}\left(1+\sigma^{-\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+} \cdot \frac{\alpha}{\alpha-1}}\right) d \sigma
$$

where if $\alpha \geq 2$ we trivially have $\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+} \cdot \frac{\alpha}{\alpha-1}=0$, and where in the case when $\alpha<2$ we may rely on (4.6) in estimating

$$
\frac{n}{2}\left(\frac{1}{\alpha}-\frac{1}{2}\right)_{+} \cdot \frac{\alpha}{\alpha-1}-1=\frac{2 n+4-(n+4) \alpha}{4(\alpha-1)}<0
$$

and in thus concluding finiteness of $c_{4}(T)$ also for such $\alpha$. As (4.8), (4.9) and (4.10) imply that

$$
\left\|v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)} \leq c_{2}+c_{1} \cdot\left(c_{4}(T)+c_{3}(T)\right) \quad \text { for all } t \in(0, T) \text { and } \varepsilon \in(0,1)
$$

we thereby arrive at (4.7).
Two straightforward testing procedures let us conclude further regularity properties of $v_{\varepsilon}$ from the latter and Lemma 4.1.

Lemma 4.3 Let (1.29) hold with positive constants $k_{F}, K_{F}$ and $\alpha$ fulfilling (4.6). Then for all $T>0$ there exists $C(T)>0$ such that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2} \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{4.11}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} v_{\varepsilon}^{\frac{2 n+4}{n}} \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{4.12}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\int_{0}^{T}\left\|v_{\varepsilon t}(\cdot, t)\right\|_{\left(W^{1,2}(\Omega)\right)^{\star}}^{\min \{\alpha, 2\}} d t \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{4.13}
\end{equation*}
$$

Proof. We abbreviate $p:=\frac{2 n+4}{n}$ and then obtain on combining the Gagliardo-Nirenberg inequality with the uniform $L^{2}$ bound for $v_{\varepsilon}$ from Lemma 4.2 to find $c_{1}>0$ and $c_{2}(T)>0$ such that

$$
\begin{align*}
\int_{0}^{T} \int_{\Omega} v_{\varepsilon}^{p} & \leq c_{1} \int_{0}^{T}\left\{\left\|\nabla v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)}^{2}\left\|v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)}^{\frac{4}{n}}+\left\|v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)}^{p}\right\} d t \\
& \leq c_{2}(T) \int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2}+c_{2}(T) \quad \text { for all } \varepsilon \in(0,1) \tag{4.14}
\end{align*}
$$

As our assumption (4.6) precisely asserts that $p>\frac{\alpha}{\alpha-1}$, through an application of Young's inequality this especially entails the existsnce of $c_{3}(T)>0$ such that

$$
\int_{0}^{T} \int_{\Omega} v_{\varepsilon}^{\frac{\alpha}{\alpha-1}} \leq \frac{1}{2 c_{2}(T)} \int_{0}^{T} \int_{\Omega} v_{\varepsilon}^{p}+c_{3}(T) \leq \frac{1}{2} \int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2}+\frac{1}{2}+c_{3}(T) \quad \text { for all } \varepsilon \in(0,1)
$$

whence testing the second equation in (4.3) by $v_{\varepsilon}$ in a standard manner we see, again by Young's inequality, that for all $\varepsilon \in(0,1)$,

$$
\begin{aligned}
\frac{1}{2} \int_{\Omega} v_{\varepsilon}^{2}(\cdot, T)-\frac{1}{2} \int_{\Omega} v_{0 \varepsilon}^{2}+\int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2}+\int_{0}^{T} \int_{\Omega} v_{\varepsilon}^{2} & =\int_{0}^{T} \int_{\Omega} \frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}} v_{\varepsilon} \\
& \leq \int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{\alpha}+\int_{0}^{T} \int_{\Omega} v_{\varepsilon}^{\frac{\alpha}{\alpha-1}} \\
& \leq \int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{\alpha}+\frac{1}{2} \int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2}+\frac{1}{2}+c_{3}(T)
\end{aligned}
$$

According to (4.2) and the outcome of Lemma 4.1, this firstly entails (4.11) and therefore, after another application of (4.14), also establishes (4.12).
The estimate in (4.13) can be achieved in a straightforward way by taking $\phi \in C^{\infty}(\bar{\Omega})$ and again using the second equation in (4.3) to find that for fixed $t>0$ and arbitrary $\varepsilon \in(0,1)$,

$$
\begin{aligned}
\left|\int_{\Omega} v_{\varepsilon t}(\cdot, t) \phi\right| & =\left|-\int_{\Omega} \nabla v_{\varepsilon} \cdot \nabla \phi-\int_{\Omega} v_{\varepsilon} \phi+\int_{\Omega} \frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}} \phi\right| \\
& \leq\left\|\nabla v_{\varepsilon}\right\|_{L^{2}(\Omega)}\|\nabla \phi\|_{L^{2}(\Omega)}+\left\|v_{\varepsilon}\right\|_{L^{2}(\Omega)}\|\phi\|_{L^{2}(\Omega)}+\left\|u_{\varepsilon}\right\|_{L^{\alpha}(\Omega)}\|\phi\|_{L^{\frac{\alpha}{\alpha-1}}(\Omega)}
\end{aligned}
$$

Since $\frac{\alpha}{\alpha-1}<\frac{2 n}{(n-2)_{+}}$and hence $W^{1,2}(\Omega) \hookrightarrow L^{\frac{\alpha}{\alpha-1}}(\Omega)$ by (4.11), we thus obtain $c_{3}>0$ such that writing $q:=\min \{\alpha, 2\}$ we have

$$
\begin{aligned}
\left\|v_{\varepsilon t}(\cdot, t)\right\|_{\left(W^{1,2}(\Omega)\right)^{\star}}^{q} & \leq c_{3} \cdot\left\{\left\|\nabla v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{q}+\left\|v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{q}+\left\|u_{\varepsilon}\right\|_{L^{\alpha}(\Omega)}^{q}\right\} \\
& \leq c_{3} \cdot\left\{\int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2}+\int_{\Omega} v_{\varepsilon}^{2}+\int_{\Omega} u_{\varepsilon}^{\alpha}+3\right\} \quad \text { for all } \varepsilon \in(0,1)
\end{aligned}
$$

due to Young's inequality. In view of (4.11), Lemma 4.2 and Lemma 4.1, an integration over $t \in(0, T)$ yields (4.13).
The following statements on convergence of both $u_{\varepsilon}$ and $v_{\varepsilon}$ are thus rather evident.
Lemma 4.4 Assume (1.29) with some $k_{F}>0, K_{F}>0$ and $\alpha$ satisfying (4.6). Then there exist $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \subset(0,1)$ such that $\varepsilon_{j} \searrow 0$ as $j \rightarrow \infty$, and nonnegative functions $u$ and $v$ which are defined on $\Omega \times(0, \infty)$ and such that for all $T>0$,

$$
\begin{equation*}
u \in L^{\alpha}(\Omega \times(0, T)) \quad \text { and } \quad v \in L^{\infty}\left((0, T) ; L^{2}(\Omega)\right) \cap L^{2}\left((0, T) ; W^{1,2}(\Omega)\right) \cap L^{\frac{2 n+4}{n}}(\Omega \times(0, T)) \tag{4.15}
\end{equation*}
$$

and that for all $T>0$ we have (1.23) and (1.24) as well as

$$
\begin{align*}
& v_{\varepsilon} \rightarrow v \quad \text { a.e. in } \Omega \times(0, T) \text { and in } L^{p}(\Omega \times(0, T)) \text { for all } p \in\left[1, \frac{2 n+4}{n}\right) \text { and }  \tag{4.16}\\
& \nabla v_{\varepsilon} \rightharpoonup \nabla v \quad \text { in } L^{2}(\Omega \times(0, T)) \tag{4.17}
\end{align*}
$$

$a s \varepsilon=\varepsilon_{j} \searrow 0$.

Proof. In view of (4.11), Lemma 2.5 applies so as to yield the statements concerning $u_{\varepsilon}$ along an appropriate sequence. Relying on the boundedness properties derived in Lemma 4.2) and Lemma 4.3 , as well as on the Vitali convergence theorem, a straightforward further subsequence extraction based on the Aubin-Lions lemma thereafter enables us to achieve also (4.16) and (4.17) with some nonnegative $v$ fulfilling (4.15).

As an application of Theorem 1.2 will require strong, rather than merely weak, $L^{2}$ convergence of $b_{\varepsilon}=-\nabla v_{\varepsilon}$, an additional consideration concerning this will be necessary:

Lemma 4.5 Let (1.29) be valid with positive parameters $k_{F}>0, K_{F}>0$ and $\alpha$ such that (4.6) holds, and let $T>0$. Then with $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ and $(u, v)$ as in Lemma 4.4, we have

$$
\begin{equation*}
\nabla v_{\varepsilon} \rightarrow \nabla v \quad \text { in } L^{2}(\Omega \times(0, T)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{4.18}
\end{equation*}
$$

Proof. We fix $\varepsilon \in(0,1)$ and $\varepsilon^{\prime} \in(0,1)$ and then obtain on taking differences in the respective second equations from (4.3) that

$$
\begin{align*}
\frac{1}{2} \int_{\Omega}\left(v_{\varepsilon}(\cdot, T)\right. & \left.-v_{\varepsilon^{\prime}}(\cdot, T)\right)^{2}+\int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}-\nabla v_{\varepsilon^{\prime}}\right|^{2}+\int_{0}^{T} \int_{\Omega}\left(v_{\varepsilon}-v_{\varepsilon^{\prime}}\right)^{2} \\
& =\frac{1}{2} \int_{\Omega}\left(v_{0 \varepsilon}-v_{0 \varepsilon^{\prime}}\right)^{2}+\int_{0}^{T} \int_{\Omega}\left(\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}-\frac{u_{\varepsilon^{\prime}}}{1+\varepsilon^{\prime} u_{\varepsilon^{\prime}}}\right)\left(v_{\varepsilon}-v_{\varepsilon^{\prime}}\right) \tag{4.19}
\end{align*}
$$

Here since $\left(\frac{u_{\varepsilon^{\prime}}}{1+\varepsilon^{\prime} u_{\varepsilon^{\prime}}}\right)_{\varepsilon^{\prime} \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}}$ is bounded in $L^{\alpha}(\Omega \times(0, T))$ by Lemma 4.1 and a.e. in $\Omega \times(0, T)$ convergent to $u$ according to (1.23), from Egorov's theorem it follows that

$$
\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}-\frac{u_{\varepsilon^{\prime}}}{1+\varepsilon^{\prime} u_{\varepsilon^{\prime}}} \rightharpoonup \frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}-u \quad \text { in } L^{\alpha}(\Omega \times(0, T)) \quad \text { as }\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \ni \varepsilon^{\prime} \searrow 0
$$

Apart from that, from (4.16) we know that

$$
\begin{equation*}
v_{\varepsilon}-v_{\varepsilon^{\prime}} \rightarrow v_{\varepsilon}-v \quad \text { in } L^{\frac{\alpha}{\alpha-1}}(\Omega \times(0, T)) \quad \text { as }\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \ni \varepsilon^{\prime} \searrow 0 \tag{4.20}
\end{equation*}
$$

once more because the hypothesis (4.6) warrants that $\frac{\alpha}{\alpha-1}<\frac{2 n+4}{n}$. In view of (4.2) and (4.17) we hence infer by employing the Hölder inequality that for all $\varepsilon \in(0,1)$,

$$
\begin{align*}
\int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}-\nabla v\right|^{2} \leq & \liminf _{\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \ni \varepsilon^{\prime} \backslash 0} \int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}-\nabla v_{\varepsilon^{\prime}}\right|^{2} \\
\leq & \liminf _{\left(\varepsilon_{j}\right)_{j \in \mathbb{N} \ni \varepsilon^{\prime}} \operatorname{ing}^{2}}\left\{\frac{1}{2} \int_{\Omega}\left(v_{0 \varepsilon}-v_{0 \varepsilon^{\prime}}\right)^{2}+\int_{0}^{T} \int_{\Omega}\left(\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}-\frac{u_{\varepsilon^{\prime}}}{1+\varepsilon^{\prime} u_{\varepsilon^{\prime}}}\right)\left(v_{\varepsilon}-v_{\varepsilon^{\prime}}\right)\right\} \\
= & \frac{1}{2} \int_{\Omega}\left(v_{0 \varepsilon}-v_{0}\right)^{2}+\int_{0}^{T} \int_{\Omega}\left(\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}-u\right)\left(v_{\varepsilon}-v\right) \\
\leq & \frac{1}{2} \int_{\Omega}\left(v_{0 \varepsilon}-v_{0}\right)^{2}+\left\|\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}-u\right\|_{L^{\alpha}(\Omega \times(0, T))}\left\|v_{\varepsilon}-v\right\|_{L^{\frac{\alpha}{\alpha-1}}(\Omega \times(0, T))} \\
\leq & \frac{1}{2} \int_{\Omega}\left(v_{0 \varepsilon}-v_{0}\right)^{2} \\
& +\left(\left\|\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}\right\|_{L^{\alpha}(\Omega \times(0, T))}+\|u\|_{\left.L^{\alpha}(\Omega \times(0, T))\right)}\left\|v_{\varepsilon}-v\right\|_{\left.L^{\frac{\alpha}{\alpha-1}}(\Omega \times(0, T))\right)}\right. \tag{4.21}
\end{align*}
$$

so that again relying on (4.2) and (4.20), and on the boundedness of $\left(\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}\right)_{\varepsilon \in(0,1)}$ in $L^{\alpha}(\Omega \times(0, T))$, as resulting from Lemma 4.1, we see that (4.18) is a consequence of (4.21).
Thus having at hand all ingredients necessary for an application of Theorem 1.2 , we can utilize the latter to obtain our main results on global very weak solvability in (1.28).
Proof of Theorem 1.4. Taking $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}, u$ and $v$ as provided by Lemma 4.4, on the basis of the strong convergence result from Lemma 4.5 we may employ Theorem 1.2 to obtain a subsequence, again denoted by $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ for notational convenience, along which for the solutions of (4.3) we have $u_{\varepsilon} \rightarrow \tilde{u}$ a.e. in $\Omega \times(0, \infty)$ as $\varepsilon=\varepsilon_{j} \searrow 0$, so that clearly $\tilde{u}$ must coincide with $u$ and hence $u$ must have the claimed solution properties with regard to (1.4). In view of (1.12), the regularity features in (1.33) are therefore immediate by-products of Theorem 1.2 and Lemma 4.4, whereas the derivation of (1.34) can be chieved in a straightforward manner by taking $\varepsilon=\varepsilon_{j} \searrow 0$ in an accordingly tested version of the second sub-problem from (4.3).

### 4.2 Weak solutions. Proof of Theorem 1.5

Next, in order to derive the stronger integrability property (1.27) required for an application of Theorem 1.3, beyond (4.1) and (4.2) we will assume that

$$
\begin{equation*}
v_{0} \in D\left(A^{\beta}\right) \quad \text { and that } \quad \sup _{\varepsilon \in(0,1)}\left\|A^{\beta} v_{0 \varepsilon}\right\|_{L^{2}(\Omega)}<\infty \tag{4.22}
\end{equation*}
$$

for some $\beta \in(0,1)$, with $A=-\Delta+1$ as introduced before the formulation of Theorem 1.5. Here we note that for any nonnegative $v_{0} \in D\left(A^{\beta}\right)$, the requirements in (4.2) and (4.22) can simultaneously be fulfilled with some $\left(v_{0 \varepsilon}\right)_{\varepsilon \in(0,1)} \subset C^{1}(\bar{\Omega} ;[0, \infty))$ by e.g. fixing $m \in \mathbb{N}$ such that $m>\frac{n+2}{4}$ and letting $v_{0 \varepsilon}:=(1+\varepsilon A)^{-m} v_{0}$ for $\varepsilon \in(0,1)$, for instance: In fact, $v_{0 \varepsilon}$ then is nonnegative by order preservation of $(1+\varepsilon A)^{-1}$, and the inclusion $v_{0 \varepsilon} \in C^{1}(\bar{\Omega})$ is ensured by the fact that $m>\frac{n+2}{4}$ warrants continuity of the embeddings $D\left(A^{m}\right) \hookrightarrow W^{2 m, 2}(\Omega) \hookrightarrow C^{1}(\bar{\Omega})$ ([11]); apart from that, the $L^{2}$ convergence property in (4.2) can be seen by standard arguments ([26]), whereas the boundedness feature in (4.22) readily results from the inclusion $v_{0} \in D\left(A^{\beta}\right)$ in view of the contractivity of $(1+\varepsilon A)^{-m}$ on $L^{2}(\Omega)$ and the fact that $A^{\beta}$ and $(1+\varepsilon A)^{-m}$ commute on $D\left(A^{\beta}\right)([9])$.
Through quite straightforward smoothing properties of the parabolic operator in the second equation from (4.3), the assumption (4.22) indeed has further consequences on the regularity of $v_{\varepsilon}$.

Lemma 4.6 Assume (1.29) with some $k_{F}>0, K_{F}>0, s_{0}>0$ and $\alpha \geq 2$, and suppose that there exists $\beta \in\left(0, \frac{1}{2}\right]$ such that (4.22) holds. Then for all $T>0$ there exists $C(T)>0$ such that the solutions of (4.3) have the properties that

$$
\begin{equation*}
\int_{\Omega}\left|A^{\beta} v_{\varepsilon}(x, t)\right|^{2} d x \leq C(T) \quad \text { for all } t \in(0, T) \text { and } \varepsilon \in(0,1) \tag{4.23}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}\left|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}(x, t)\right|^{2} d x d t \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{4.24}
\end{equation*}
$$

Proof. Noting that the assumption $\beta \leq \frac{1}{2}$ along with the regularity features of $v_{\varepsilon}$ warrant appropriate smoothness of all subsequently appearing quantities, we may use the second equation in (4.3),
rewritten in the form $v_{\varepsilon t}+A v_{\varepsilon}=\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}$, to see that thanks to the self-adjointness of $A$ and all its fractional powers, with some $c_{1}>0$ we have

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} \int_{\Omega}\left|A^{\beta} v_{\varepsilon}\right|^{2} & =\int_{\Omega} A^{2 \beta} v_{\varepsilon} \cdot v_{\varepsilon t} \\
& =-\int_{\Omega}\left|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}\right|^{2}+\int_{\Omega} \frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}} A^{2 \beta v_{\varepsilon}} \\
& \leq-\int_{\Omega}\left|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}\right|^{2}+\left\|u_{\varepsilon}\right\|_{L^{\alpha}(\Omega)}\left\|A^{2 \beta} v_{\varepsilon}\right\|_{L^{\frac{\alpha}{\alpha-1}(\Omega)}} \\
& \leq-\int_{\Omega}\left|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}\right|^{2}+c_{1}\left\|u_{\varepsilon}\right\|_{L^{\alpha}(\Omega)}\left\|A^{2 \beta} v_{\varepsilon}\right\|_{L^{2}(\Omega)} \quad \text { for all } t>0 \tag{4.25}
\end{align*}
$$

according to the Hölder inequality and the fact that $\frac{\alpha}{\alpha-1} \leq 2$. Here we may rely on a standard interpolation result ([9]) to infer from the inequalities $\beta<2 \beta \leq \frac{2 \beta+1}{2}$, as ensured by the restrictions $\beta>0$ and $\beta \leq \frac{1}{2}$, that there exists $c_{2}>0$ fulfilling

$$
\left\|A^{2 \beta} v_{\varepsilon}\right\|_{L^{2}(\Omega)} \leq c_{2}\left\|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2 \beta}\left\|A^{\beta} v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{1-2 \beta} \quad \text { for all } t>0
$$

When inserted into (4.25) and combined with Young's inequality, this shows that we can find $c_{3}>0$ such that

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} \int_{\Omega}\left|A^{\beta} v_{\varepsilon}\right|^{2} & \leq-\int_{\Omega}\left|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}\right|^{2}+c_{1} c_{2}\left\|u_{\varepsilon}\right\|_{L^{\alpha}(\Omega)}\left\|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2 \beta}\left\|A^{\beta} v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{1-2 \beta} \\
& \leq-\frac{1}{2} \int_{\Omega}\left|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}\right|^{2}+c_{3}\left\|u_{\varepsilon}\right\|_{L^{\alpha}(\Omega)}^{\frac{1}{1-\beta}}\left\|A^{\beta} v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{\frac{1-2 \beta}{1-\beta}} \quad \text { for all } t>0 \tag{4.26}
\end{align*}
$$

where in the case $\beta<\frac{1}{2}$ we may two more times use Young's inequality to see that since

$$
\frac{\alpha(1-2 \beta)}{\alpha(1-\beta)-1}=\frac{1-2 \beta}{1-\beta-\frac{1}{\alpha}} \leq \frac{1-2 \beta}{1-\beta-\frac{1}{2}}=2
$$

by assumption on $\alpha$, we have

$$
\begin{aligned}
\left\|u_{\varepsilon}\right\|_{L^{\alpha}(\Omega)}^{\frac{1}{1-\beta}}\left\|A^{\beta} v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{\frac{1-2 \beta}{1-\beta}} & \leq\left\|u_{\varepsilon}\right\|_{L^{\alpha}(\Omega)}^{\alpha}+\left\|A^{\beta} v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{\frac{\alpha(1-2 \beta)}{\alpha(1-\beta) 1}} \\
& \leq\left\|u_{\varepsilon}\right\|_{L^{\alpha}(\Omega)}^{\alpha}+\left\|A^{\beta} v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2}+1 \quad \text { for all } t>0
\end{aligned}
$$

As the resulting inequality evidently extends so as to remain valid also in the borderline case $\beta=\frac{1}{2}$, from (4.26) we altogether obtain that

$$
\frac{d}{d t} \int_{\Omega}\left|A \beta v_{\varepsilon}\right|^{2}+\int_{\Omega}\left|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}\right|^{2} \leq 2 c_{3} \int_{\Omega}\left|A^{\beta} v_{\varepsilon}\right|^{2}+2 c_{3} \int_{\Omega} u_{\varepsilon}^{\alpha}+2 c_{3} \quad \text { for all } t>0
$$

and that hence both (4.23) and (4.24) follow upon integrating in time and recalling Lemma 4.1.
By suitable interpolation, the latter indeed entails further integrability properties of the crucial quantity $\nabla v_{\varepsilon}$.

Lemma 4.7 Let (1.29) be satisfied with some $k_{F}>0, K_{F}>0, s_{0}>0$ and $\alpha \geq 2$, and suppose that there exists $\beta \in\left(0, \frac{1}{2}\right]$ such that (4.22) is valid. Then for any $T>0$ and each $q>2$ fulfilling

$$
\begin{cases}q<\frac{2(n+2)}{n+2-4 \beta} & \text { if } \beta<\frac{1}{2}  \tag{4.27}\\ q \leq \frac{2(n+2)}{n} & \text { if } \beta=\frac{1}{2}\end{cases}
$$

one can find $C(T, q)>0$ such that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{q} \leq C(T, q) \quad \text { for all } \varepsilon \in(0,1) \tag{4.28}
\end{equation*}
$$

Proof. We first consider the case $\beta=\frac{1}{2}$, in which due to elliptic regularity theory $([9,10])$ it is well-known that $\left\|A^{\frac{2 \beta+1}{2}}(\cdot)\right\|_{L^{2}(\Omega)}=\|(-\Delta+1)(\cdot)\|_{L^{2}(\Omega)}$ and $\left\|A^{\beta}(\cdot)\right\|_{L^{2}(\Omega)}=\left(\|\nabla(\cdot)\|_{L^{2}(\Omega)}^{2}+\|\cdot\|_{L^{2}(\Omega)}^{2}\right)^{\frac{1}{2}}$ define norms equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ and $\|\cdot\|_{W^{1,2}(\Omega)}$, respectively, so that by a Gagliardo-Nirenberg interpolation we find $c_{1}>0$ and $c_{2}>0$ such that writing $q_{0}:=\frac{2(n+2)}{n}$ we have

$$
\|\nabla \phi\|_{L^{q_{0}(\Omega)}}^{q_{0}} \leq c_{1}\|\phi\|_{W^{2,2}(\Omega)}^{2}\|\phi\|_{W^{1,2}(\Omega)}^{q_{0}-2} \leq c_{2}\left\|A^{\frac{2 \beta+1}{2}} \phi\right\|_{L^{2}(\Omega)}^{2}\left\|A^{\beta} \phi\right\|_{L^{2}(\Omega)}^{q_{0}-2} \quad \text { for all } \phi \in D(A)
$$

According to (4.27) and Young's inequality, we thus obtain that in this case,

$$
\begin{aligned}
\int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{q} & \leq \int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{q_{0}}+|\Omega| T \\
& \leq c_{2} \int_{0}^{T}\left\|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)}^{2}\left\|A^{\beta} v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)}^{q_{0}-2} d t+|\Omega| T \quad \text { for all } \varepsilon \in(0,1)
\end{aligned}
$$

and that hence (4.28) results from Lemma 4.6.
If $\beta<\frac{1}{2}$, however, we first make use of the strivt inequality in (4.27) to fix $\gamma>\beta$ such that

$$
\begin{equation*}
\frac{n}{4}+\frac{1}{2}-\frac{n}{2 q}<\gamma \leq \beta+\frac{1}{q} \tag{4.29}
\end{equation*}
$$

noting that then a known embedding result ([11]) warrants that $D\left(A^{\gamma}\right) \hookrightarrow W^{1, q}(\Omega)$. As furthermore our assumption $q>2$ ensures that $\gamma \leq \beta+\frac{1}{q} \leq \frac{2 \beta+1}{2}$, once more according to an appropriate interpolation property of fractional powers ([9, Part 2, Theorem 14.1]) we can fix $c_{3}>0$ and $c_{4}>0$ such that

$$
\begin{aligned}
\int_{0}^{T}\left\|\nabla v_{\varepsilon}(\cdot, t)\right\|_{L^{q}(\Omega)}^{q} d t & \leq c_{3} \int_{0}^{T}\left\|A^{\gamma} v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)}^{q} d t \\
& \leq c_{4} \int_{0}^{T}\left\|A^{\frac{2 \beta+1}{2}} v_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)}^{2 q(\gamma-\beta)}\left\|A^{\beta} v_{\varepsilon}\right\|_{L^{2}(\Omega)}^{q-2 q(\gamma-\beta)} d t \quad \text { for all } \varepsilon \in(0,1)
\end{aligned}
$$

Observing that herein $2 q(\gamma-\beta) \leq 2$ thanks to the right inequality in (4.29), again invoking Lemma 4.6 we infer (4.28) from this.

We can thereby proceed to make sure that our very weak solutions are in fact weak solutions whenever the hypotheses from Theorem 1.5 are met.

Proof of Theorem 1.5. We take $u, v$ and $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ as given by Theorem 1.4, and then infer from Lemma 4.6 and Lemma 4.7 that $u$ and $v$ have the regularity properties stated in (1.37). In particular, in the case $\beta=\frac{1}{2}$ this entails the inclusion $\nabla v \in L_{l o c}^{q}\left(\bar{\Omega} \times[0, \infty) ; \mathbb{R}^{n}\right)$ with $q=\frac{2(n+2)}{n}$ complying with the requirement in (1.27) due to the fact that then

$$
q-\frac{2 \alpha}{\alpha-1}=\frac{2(n+2)}{n}-\frac{2}{1-\frac{1}{\alpha}} \geq \frac{2(n+2)}{n}-\frac{2}{1-\frac{2}{n+2}}=0
$$

by (1.35). The claim therefore results by combining Theorem 1.3 with Theorem 1.4.
If $\beta \in\left(\frac{n+2}{4 \alpha}, \frac{1}{2}\right)$, and hence $\alpha>\frac{n+2}{2}$, observing that then

$$
\frac{2(n+2)}{n+2-4 \beta}>\frac{2(n+2)}{n+2-4 \cdot \frac{n+2}{4 \alpha}}=\frac{2 \alpha}{\alpha-1}
$$

we may pick any $q \in\left[\frac{2 \alpha}{\alpha-1}, \frac{2(n+2)}{n+2-4 \beta}\right)$ to similarly conclude on the basis of (1.37) that Theorem 1.3 and Theorem 1.4 imply the stated solution properties of $(u, v)$.

Proof of Corollary 1.6. We only need to apply Theorem 1.5 to $\alpha:=2, \beta:=\frac{1}{2}$ and $q:=4$, and once more make use of the well-known fact that then $D\left(A^{\beta}\right)=D\left(A^{\frac{1}{2}}\right)=W^{1,2}(\Omega)$.

## 5 Application to a Shigesada-Kawasaki-Teramoto system

We will next focus on the Shigesada-Kawasaki-Teramoto system (1.39) under the standing assumptions that $d_{1}, d_{2}$ and $\mu_{1}$ are positive, that $a_{12}, a_{22}, \mu_{2}, a_{1}$ and $a_{2}$ are nonnegative, and that

$$
\begin{equation*}
u_{0} \in L^{1}(\Omega) \quad \text { and } \quad v_{0} \in L^{\infty}(\Omega) \quad \text { are nonnegative. } \tag{5.1}
\end{equation*}
$$

As approximations of (1.39) convenient for our purposes, for $\varepsilon \in(0,1)$ we shall consider

$$
\begin{cases}u_{\varepsilon t}=d_{1} \Delta u_{\varepsilon}+a_{12} \Delta\left(u_{\varepsilon} v_{\varepsilon}\right)+\mu_{1} u_{\varepsilon}\left(1-u_{\varepsilon}-a_{1} v_{\varepsilon}\right), & x \in \Omega, t>0  \tag{5.2}\\ v_{\varepsilon t}=d_{2} \Delta v_{\varepsilon}+a_{22} \Delta v_{\varepsilon}^{2}+\mu_{2} v_{\varepsilon}\left(1-v_{\varepsilon}-\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}\right), & x \in \Omega, t>0 \\ \frac{\partial u_{\varepsilon}}{\partial \nu}=\frac{\partial v_{\varepsilon}}{\partial \nu}=0, & x \in \partial \Omega, t>0 \\ u_{\varepsilon}(x, 0)=u_{0 \varepsilon}(x), \quad v_{\varepsilon}(x, 0)=v_{0 \varepsilon}(x), & x \in \Omega,\end{cases}
$$

where we take any $\left(u_{0 \varepsilon}\right)_{\varepsilon \in(0,1)} \subset C^{1}(\bar{\Omega})$ and $\left(v_{0 \varepsilon}\right)_{\varepsilon \in(0,1)} \subset C^{3}(\bar{\Omega})$ such that $\frac{\partial v_{0 \varepsilon}}{\partial \nu}=0$ on $\partial \Omega$, that

$$
\begin{array}{r}
0 \leq u_{0 \varepsilon} \rightarrow u_{0} \quad \text { in } L^{1}(\Omega) \quad \text { and } \quad 0 \leq v_{0 \varepsilon} \rightarrow v_{0} \quad \text { a.e. in } \Omega \quad \text { as } \varepsilon \searrow 0, \quad \text { and that } \\
\sup _{\varepsilon \in(0,1)}\left\|v_{0 \varepsilon}\right\|_{L^{\infty}(\Omega)}<\infty . \tag{5.3}
\end{array}
$$

We shall see that (1.39) and (5.2) fall among the class of problems covered by our general theory if we let $A(x, t):=\left(d_{1}+a_{12} v(x, t)\right)\left(\delta_{i j}\right)_{i, j=1, \ldots, n},\left(A_{\varepsilon}\right)(x, t):=\left(d_{1}+a_{12} v_{\varepsilon}(x, t)\right)\left(\delta_{i j}\right)_{i, j=1, \ldots, n}, b(x, t):=$ $a_{12} \nabla v(x, t), b_{\varepsilon}(x, t):=a_{12} \nabla v_{\varepsilon}(x, t), f(x, t, s):=-\mu_{1} s+\mu_{1} s^{2}, g(x, t):=-\mu_{1} a_{1} u(x, t) v(x, t)$ and $g_{\varepsilon}(x, t):=-\mu_{1} a_{1} u_{\varepsilon}(x, t) v_{\varepsilon}(x, t)$ for $x \in \bar{\Omega}, t>0, s \geq 0$ and $\varepsilon \in(0,1)$.
Several of our overall requirements, and in particular (1.15), are already asserted by the following basic statement on global classical solvability of (5.2) that can be derived by straightforward adaptation of standard arguments:

Lemma 5.1 For any $\varepsilon \in(0,1)$, the problem (5.2) admits a global classical solution $\left(u_{\varepsilon}, v_{\varepsilon}\right)$ with

$$
\left\{\begin{array}{l}
u_{\varepsilon} \in C^{0}(\bar{\Omega} \times[0, \infty)) \cap C^{2,1}(\bar{\Omega} \times(0, \infty)) \quad \text { and }  \tag{5.4}\\
v_{\varepsilon} \in \bigcap_{q>n} C^{0}\left([0, \infty) ; W^{1, q}(\Omega)\right) \cap C^{2,1}(\bar{\Omega} \times(0, \infty)) .
\end{array}\right.
$$

Moreover, $u_{\varepsilon}$ and $v_{\varepsilon}$ are nonnegative with

$$
\begin{equation*}
v_{\varepsilon}(x, t) \leq \max \left\{1,\left\|v_{0 \varepsilon}\right\|_{L^{\infty}(\Omega)}\right\} \quad \text { for all } x \in \Omega \text { and } t>0 . \tag{5.5}
\end{equation*}
$$

Proof. Standard theory ([1]) asserts local existence of a solution with the indicated regularity properties, extensible up to a maximal existence time $T_{\text {max }, \varepsilon} \in(0, \infty]$ such that either $T_{\text {max }, \varepsilon}=$ $\infty$, or $\lim \sup _{t} \not T_{\text {max }, \varepsilon}\left\|u_{\varepsilon}(\cdot, t)\right\|_{L^{\infty}(\Omega)}=\infty$. As three applications of the comparison principle assert nonnegativity of $u_{\varepsilon}$ and $v_{\varepsilon}$ as well as the inequality in (5.5), by relying on the boundedness of the reaction term in the second equation from (5.2) we may invoke known results on gradient regularity in scalar parabolic problems ([20]) to see that if $T_{\max , \varepsilon}<\infty$ then $\nabla v_{\varepsilon}$ is bounded in $\Omega \times\left(0, T_{\max , \varepsilon}\right)$. By means of a straightforward reasoning based on $L^{p}-L^{q}$ estimates for the Neumann heat semigroup, this in turn warrants boundedness of $u_{\varepsilon}$ throughout $\Omega \times\left(0, T_{\max , \varepsilon}\right)$ in this case, by contradiction to the above thus showing that actually $T_{\max , \varepsilon}=\infty$.

### 5.1 Very weak solutions. Proof of Theorem 1.7

As before starting with the construction of very weak solutions, we first collect some basic properties of solutions to (5.2), and especially of the second solution component $v_{\varepsilon}$.
Lemma 5.2 Let $T>0$. Then there exists $C(T)>0$ such that

$$
\begin{equation*}
\int_{\Omega} u_{\varepsilon}(x, t) d x \leq C(T) \quad \text { for all } t \in(0, T) \text { and } \varepsilon \in(0,1) \tag{5.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{2}(x, t) d x d t \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{5.7}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}(x, t)\right|^{2} d x d t \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{5.8}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{T}\left\|v_{\varepsilon t}(\cdot, t)\right\|_{\left(W^{1,2}(\Omega)\right)^{*}}^{2} d t \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{5.9}
\end{equation*}
$$

Proof. The estimates in (5.6) and (5.8) directly result from Lemma 2.2. To verify (5.8), we test the second equation in (5.2) by $v_{\varepsilon}$ and thereby obtain that for all $t>0$,
$\frac{1}{2} \frac{d}{d t} \int_{\Omega} v_{\varepsilon}^{2}+d_{2} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2}=-2 a_{22} \int_{\Omega} v_{\varepsilon}\left|\nabla v_{\varepsilon}\right|^{2}+\mu_{2} \int_{\Omega} v_{\varepsilon}^{2}-\mu_{2} \int_{\Omega} v_{\varepsilon}^{3}-\mu_{2} a_{2} \int_{\Omega} \frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}} v_{\varepsilon}^{2} \leq \mu_{2} \int_{\Omega} v_{\varepsilon}^{2}$,
so that

$$
d_{2} \int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2} \leq \frac{1}{2} \int_{\Omega} v_{0 \varepsilon}^{2}+\mu_{2} \int_{0}^{T} \int_{\Omega} v_{\varepsilon}^{2}
$$

from which (5.8) follows due to (5.5).
Finally, (5.9) can be derived from this in a standard manner by using (5.2) to see that for all $\phi \in C^{1}(\bar{\Omega})$ with $\|\nabla \phi\|_{L^{2}(\Omega)}^{2}+\|\phi\|_{L^{2}(\Omega)}^{2} \leq 1$,

$$
\begin{aligned}
\left|\int_{\Omega} v_{\varepsilon t}(\cdot, t) \phi\right|= & \mid-d_{2} \int_{\Omega} \nabla v_{\varepsilon} \cdot \nabla \phi-2 a_{22} \int_{\Omega} v_{\varepsilon} \nabla v_{\varepsilon} \cdot \nabla \phi \\
& \left.+\mu_{2} \int_{\Omega} v_{\varepsilon} \phi-\mu_{2} \int_{\Omega} v_{\varepsilon}^{2} \phi-\mu_{2} a_{2} \int_{\Omega} \frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}} v_{\varepsilon} \phi \right\rvert\, \\
\leq & d_{2}\left\|\nabla v_{\varepsilon}\right\|_{L^{2}(\Omega)}+2 a_{22}\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega)}\left\|\nabla v_{\varepsilon}\right\|_{L^{2}(\Omega)} \\
& +\mu_{2}\left\|v_{\varepsilon}\right\|_{L^{2}(\Omega)}+\mu_{2}\left\|v_{\varepsilon}\right\|_{L^{4}(\Omega)}^{2}+\mu_{2} a_{2}\left\|u_{\varepsilon}\right\|_{L^{2}(\Omega)}\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega)}
\end{aligned}
$$

for all $t>0$ and $\varepsilon \in(0,1)$, whence by Young's inequality,

$$
\begin{aligned}
\left\|v_{\varepsilon t}(\cdot, t)\right\|_{\left(W^{1,2}(\Omega)\right)^{\star} \leq}^{2} \leq & 5\left(d_{2}^{2}+a_{22}^{2}\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{2}\right) \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2} \\
& +5 \mu_{2}^{2} \int_{\Omega} v_{\varepsilon}^{2}+5 \mu_{2}^{2} \int_{\Omega} v_{\varepsilon}^{4}+5 \mu_{2}^{2} a_{2}^{2}\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{2} \int_{\Omega} u_{\varepsilon}^{2} \quad \text { for all } t>0 \text { and } \varepsilon \in(0,1)
\end{aligned}
$$

implying (5.9) upon integrating and using (5.5), (5.8) and (5.7).
Again, some approximation properties of (5.2) thereby become quite obvious.
Lemma 5.3 There exists $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \subset(0,1)$ such that $\varepsilon_{j} \searrow 0$ as $j \rightarrow \infty$, and nonnegative functions $u$ and $v$ on $\Omega \times(0, \infty)$ which for each $T>0$ satisfy

$$
\left\{\begin{array}{l}
u \in L^{\infty}\left((0, T) ; L^{1}(\Omega) \cap L^{2}(\Omega \times(0, T)) \quad\right. \text { and }  \tag{5.10}\\
v \in L^{\infty}(\Omega \times(0, T)) \cap L^{2}\left((0, T) ; W^{1,2}(\Omega)\right),
\end{array}\right.
$$

and which are such that for all $T>0$,

$$
\begin{align*}
& u_{\varepsilon} \rightarrow u \quad \text { in } L^{1}(\Omega \times(0, T)) \text { and a.e. in } \Omega \times(0, T),  \tag{5.11}\\
& u_{\varepsilon} \rightharpoonup u \quad \text { in } L^{2}(\Omega \times(0, T)),  \tag{5.12}\\
& v_{\varepsilon} \rightarrow v \quad \text { in } L^{2}(\Omega \times(0, T)) \text { and a.e. in } \Omega \times(0, T), \quad \text { and that }  \tag{5.13}\\
& \nabla v_{\varepsilon} \rightharpoonup \nabla v \quad \text { in } L^{2}(\Omega \times(0, T)) \tag{5.14}
\end{align*}
$$

as $\varepsilon=\varepsilon_{j} \searrow 0$. Moreover, the second equation in (1.39) is satisfied in the sense that (1.41) holds for all $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty))$.

Proof. The existence of a sequence $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ and limit functions $u$ and $v$ with the properties in (5.11)-(5.14) immediately results from a straighforward extraction process based on Lemma 2.5, Lemma 5.2 , (5.5) and the Aubin-Lions lemma. The verification of (1.41) can thereupon be achieved on testing the second equation in (5.2) by $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty))$ and observing that (5.12)-(5.14) are especially sufficient for passing to the limit in each of the respective nonlinear contributions in the sense that

$$
\int_{0}^{\infty} \int_{\Omega} v_{\varepsilon} \nabla v_{\varepsilon} \cdot \nabla \varphi \rightarrow \int_{0}^{\infty} \int_{\Omega} v \nabla v \cdot \nabla \varphi \quad \text { and } \quad \int_{0}^{\infty} \int_{\Omega} v_{\varepsilon}^{2} \varphi \rightarrow \int_{0}^{\infty} \int_{\Omega} v^{2} \varphi
$$

as well as

$$
\int_{0}^{\infty} \int_{\Omega} \frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}} v_{\varepsilon} \varphi \rightarrow \int_{0}^{\infty} \int_{\Omega} u v \varphi
$$

as $\varepsilon=\varepsilon_{j} \searrow 0$, the latter because clearly also $\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}} \rightharpoonup u$ in $L_{l o c}^{2}(\bar{\Omega} \times[0, \infty))$ according to (5.11), (5.12) and Egorov's theorem.

In comparison with the corresponding statement from the previous section in Lemma 4.5, due to the presence of nonlinear diffusion in the second equation from (5.2) the derivation of a strong convergence feature of $\nabla v_{\varepsilon}$ here requires an additional argument.

Lemma 5.4 Let $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ and $v$ be as in Lemma 5.3. Then there exist a null set $N \subset(0, \infty)$ and a subsequence, again denoted by $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$, such that

$$
\begin{equation*}
\nabla v_{\varepsilon} \rightarrow \nabla v \quad \text { in } L^{2}\left(\Omega \times\left(0, t_{0}\right)\right) \quad \text { for all } t_{0} \in(0, \infty) \backslash N \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{5.15}
\end{equation*}
$$

Proof. As $v$ belongs to $L^{\infty}(\Omega \times(0, T)) \cap L^{2}\left((0, T) ; W^{1,2}(\Omega)\right)$ for each $T>0$, by means of a standard approximation procedure (see e.g. [40, Lemma 8.2]) it can be verified that $v$ can be used as a test function in (1.41) in the sense that if we fix a null set $N_{1} \subset(0, \infty)$ such that $(0, \infty) \backslash N_{1}$ exclusively consists of Lebesgue points of $(0, \infty) \ni t \mapsto \int_{\Omega} v^{2}(\cdot, t)$, then

$$
\begin{align*}
\frac{1}{2} \int_{\Omega} v^{2}\left(\cdot, t_{0}\right)-\frac{1}{2} \int_{\Omega} v_{0}^{2} \geq & -d_{2} \int_{0}^{t_{0}} \int_{\Omega}|\nabla v|^{2}-2 a_{22} \int_{0}^{t_{0}} \int_{\Omega} v|\nabla v|^{2} \\
& +\mu_{2} \int_{0}^{t_{0}} \int_{\Omega} v^{2}-\mu_{2} \int_{0}^{t_{0}} \int_{\Omega} v^{3}-\mu_{2} a_{2} \int_{0}^{t_{0}} \int_{\Omega} u v^{2} \tag{5.16}
\end{align*}
$$

for all $t_{0} \in(0, \infty) \backslash N_{1}$, where actually even equality can be achieved but will not be needed here. To further exploit this, according to (5.13) we fix a second null set $N_{2} \subset(0, \infty)$ such that for all $t_{0} \in(0, \infty) \backslash N_{2}$ we have $\int_{\Omega} v_{\varepsilon}^{2}\left(\cdot, t_{0}\right) \rightarrow \int_{\Omega} v^{2}\left(\cdot, t_{0}\right)$ as $\varepsilon=\varepsilon_{j} \searrow 0$. Apart from that, we note that in the most complex case when $a_{22}$ is positive, writing $\rho(s):=\frac{1}{3 a_{22}} \sqrt{d_{2}+2 a_{22} s}, s \geq 0$, we see that due to (5.5) and (5.8) the family $\left(\nabla \rho\left(v_{\varepsilon}\right)\right)_{\varepsilon \in(0,1)} \equiv\left(\sqrt{d_{2}+2 a_{22} v_{\varepsilon}} \nabla v_{\varepsilon}\right)_{\varepsilon \in(0,1)}$ is bounded in $L^{2}(\Omega \times(0, T))$ for all $T>0$, which in view of (5.13) means that for any such $T$,

$$
\begin{equation*}
\nabla \rho\left(v_{\varepsilon}\right) \rightharpoonup \nabla \rho(v) \quad \text { in } L^{2}(\Omega \times(0, T)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 . \tag{5.17}
\end{equation*}
$$

By lower semicontinuity of $L^{2}$ norms with respect to weak convergence, this implies that

$$
\begin{align*}
\int_{0}^{t_{0}} \int_{\Omega}|\nabla \rho(v)|^{2} & \leq \liminf _{\varepsilon=\varepsilon_{j} \searrow 0} \int_{0}^{t_{0}} \int_{\Omega}\left|\nabla \rho\left(v_{\varepsilon}\right)\right|^{2} \\
& =\liminf _{\varepsilon=\varepsilon_{j} \searrow 0}\left\{d_{2} \int_{0}^{t_{0}} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2}+2 a_{22} \int_{0}^{t_{0}} \int_{\Omega} v_{\varepsilon}\left|\nabla v_{\varepsilon}\right|^{2}\right\} \quad \text { for all } t_{0}>0 \tag{5.18}
\end{align*}
$$

while on the other hand, by definition of $N_{2}$ and by (5.3), (5.13), (5.12), (5.5) and the dominated convergence theorem, testing the second equation in (5.2) by $v_{\varepsilon}$ we readily find that for all $t_{0} \in$

$$
\begin{aligned}
&(0, \infty) \backslash N_{2} \\
& d_{2} \int_{0}^{t_{0}} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2}+2 a_{22} \int_{0}^{t_{0}} \int_{\Omega} v_{\varepsilon}\left|\nabla v_{\varepsilon}\right|^{2} \\
&=\frac{1}{2} \int_{\Omega} v_{0 \varepsilon}^{2}-\frac{1}{2} \int_{\Omega} v_{\varepsilon}^{2}\left(\cdot, t_{0}\right)+\mu_{2} \int_{0}^{t_{0}} \int_{\Omega} v_{\varepsilon}^{2}-\mu_{2} \int_{0}^{t_{0}} \int_{\Omega} v_{\varepsilon}^{3}-\mu_{2} a_{2} \int_{0}^{t_{0}} \int_{\Omega} \frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}} v_{\varepsilon}^{2} \\
& \rightarrow \frac{1}{2} \int_{\Omega} v_{0}^{2}-\frac{1}{2} \int_{\Omega} v^{2}\left(\cdot, t_{0}\right)+\mu_{2} \int_{0}^{t_{0}} \int_{\Omega} v^{2}-\mu_{2} \int_{0}^{t_{0}} \int_{\Omega} v^{3}-\mu_{2} a_{2} \int_{0}^{t_{0}} \int_{\Omega} u v^{2}
\end{aligned}
$$

as $\varepsilon=\varepsilon_{j} \searrow 0$. In conjunction with (5.16) and (5.18), this shows that if we let $N:=N_{1} \cup N_{2}$, then

$$
\int_{0}^{t_{0}} \int_{\Omega}\left|\nabla \rho\left(v_{\varepsilon}\right)\right|^{2} \rightarrow \int_{0}^{t_{0}} \int_{\Omega}|\nabla \rho(v)|^{2} \quad \text { for all } t_{0} \in(0, \infty) \backslash N \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0
$$

and that hence, by (5.17),

$$
\begin{equation*}
\nabla \rho\left(v_{\varepsilon}\right) \rightharpoonup \nabla \rho(v) \quad \text { in } L^{2}\left(\Omega \times\left(0, t_{0}\right)\right) \quad \text { for all } t_{0} \in(0, \infty) \backslash N \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{5.19}
\end{equation*}
$$

and therefore also

$$
\begin{equation*}
\left|\nabla \rho\left(v_{\varepsilon}\right)\right|^{2} \rightarrow|\nabla \rho(v)|^{2} \quad \text { in } L^{1}\left(\Omega \times\left(0, t_{0}\right)\right) \quad \text { for all } t_{0} \in(0, \infty) \backslash N \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{5.20}
\end{equation*}
$$

On particularly choosing $t_{0}=t_{0 k}$ here, with $\left(t_{0 k}\right)_{k \in \mathbb{N}} \subset(0, \infty) \backslash N$ fulfilling $t_{0 k} \nearrow \infty$ as $k \rightarrow \infty$, we easily infer from (5.19) that passing to a conveniently relabeled subsequence we can achieve that also $\nabla \rho\left(v_{\varepsilon}\right) \rightarrow \nabla \rho(v)$ a.e. in $\left.\Omega \times(0, \infty)\right)$ and thus, by (5.13) and positivity of $\rho^{\prime}(s)=\sqrt{d_{2}+2 a_{22} s}$ on $[0, \infty)$,

$$
\begin{equation*}
\nabla v_{\varepsilon} \rightarrow \nabla v \quad \text { a.e. in } \Omega \times(0, \infty) \tag{5.21}
\end{equation*}
$$

as $\varepsilon=\varepsilon_{j} \searrow 0$. Since furthermore

$$
\left|\nabla v_{\varepsilon}\right|^{2}=\frac{1}{d_{2}+2 a_{22} v_{\varepsilon}}\left|\nabla \rho\left(v_{\varepsilon}\right)\right|^{2} \leq \frac{1}{d_{2}}\left|\nabla \rho\left(v_{\varepsilon}\right)\right|^{2} \quad \text { in } \Omega \times(0, \infty) \quad \text { for all } \varepsilon \in(0,1)
$$

a combination of (5.21) with (5.20) and the dominated convergence theorem shows that

$$
\int_{0}^{t_{0}} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{2} \rightarrow \int_{0}^{t_{0}} \int_{\Omega}|\nabla v|^{2} \quad \text { for all } t_{0} \in(0, \infty) \backslash N \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0
$$

which together with (5.14) entails (5.15) when $a_{22}>0$. In the case $a_{22}=0$ of linear diffusion, the argument actually becomes much simpler and may thus be omitted here.

We can thus apply Theorem 1.2 in a straightforward manner to achieve the claimed results on very weak solvability in (1.39).
Proof of Theorem 1.7. Thanks to the strong convergence result from Lemma 5.4, in view of Theorem 1.2 and Lemma 5.3 we only need to make sure that $g_{\varepsilon}:=-\mu_{1} a_{1} u_{\varepsilon} v_{\varepsilon}$ satisfies $g_{\varepsilon} \rightarrow g$ in $L_{l o c}^{1}(\bar{\Omega} \times[0, T))$ as $\varepsilon=\varepsilon_{j} \searrow 0$ for each $T>0$. However, since $g_{\varepsilon} \rightarrow g$ a.e. in $\Omega \times(0, \infty)$ by (5.11) and (5.12), and since $\left|g_{\varepsilon}\right| \leq c_{1}(T) u_{\varepsilon}$ in $\Omega \times(0, T)$ with $c_{1}(T):=\mu_{1} a_{1} \sup _{\varepsilon \in(0,1)}\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega \times(0, T))}$ being finite due to (5.5), by means of the dominated convergence theorem this directly results from the $L^{1}$ convergence property of $\left(u_{\varepsilon}\right)_{\varepsilon \in\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}}$ in (5.10).

### 5.2 Weak solutions. Proof of Theorem 1.8

As in Section 4, higher regularity of the flux term $\nabla v$ will result from suitably strengthened assumptions on the corresponding initial data. Accordingly and in line with the hypotheses from Theorem 1.8 , we now assume that beyond (5.1) and (5.3) we have

$$
\begin{equation*}
v_{0} \in W^{1,2}(\Omega) \quad \text { and } \quad \sup _{\varepsilon \in(0,1)}\left\|\nabla v_{0 \varepsilon}\right\|_{L^{2}(\Omega)}<\infty \tag{5.22}
\end{equation*}
$$

Then using a standard multiplier for the nonlinear diffusion equation for $v_{\varepsilon}$ in (5.2) yields the following.
Lemma 5.5 Assume (5.22), and let

$$
\begin{equation*}
P(s):=d_{2} s+a_{22} s^{2}, \quad s \geq 0 \tag{5.23}
\end{equation*}
$$

Then for all $T>0$ there exists $C(T)>0$ such that

$$
\begin{equation*}
\int_{\Omega}\left|\nabla P\left(v_{\varepsilon}(\cdot, t)\right)\right|^{2} \leq C(T) \quad \text { for all } t \in(0, T) \text { and } \varepsilon \in(0,1) \tag{5.24}
\end{equation*}
$$

and that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}\left|\Delta P\left(v_{\varepsilon}\right)\right|^{2} \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{5.25}
\end{equation*}
$$

Proof. Integrating by parts and using Young's inequality in the second equation from (5.2), for all $t>0$ we obtain

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} \int_{\Omega}\left|\nabla P\left(v_{\varepsilon}\right)\right|^{2} & =-\int_{\Omega} \Delta P\left(v_{\varepsilon}\right) \cdot P^{\prime}\left(v_{\varepsilon}\right) v_{\varepsilon t} \\
& =-\int_{\Omega} P^{\prime}\left(v_{\varepsilon}\right)\left|\Delta P\left(v_{\varepsilon}\right)\right|^{2}-\mu_{2} \int_{\Omega} P^{\prime}\left(v_{\varepsilon}\right) \Delta P\left(v_{\varepsilon}\right) \cdot v_{\varepsilon}\left(1-v_{\varepsilon}-\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}\right) \\
& \leq-\frac{1}{2} \int_{\Omega} P^{\prime}\left(v_{\varepsilon}\right)\left|\Delta P\left(v_{\varepsilon}\right)\right|^{2}+\frac{\mu_{2}^{2}}{2} \int_{\Omega} P^{\prime}\left(v_{\varepsilon}\right) v_{\varepsilon}^{2}\left(1-v_{\varepsilon}-\frac{u_{\varepsilon}}{1+\varepsilon u_{\varepsilon}}\right)^{2} \\
& \leq-\frac{1}{2} \int_{\Omega} P^{\prime}\left(v_{\varepsilon}\right)\left|\Delta P\left(v_{\varepsilon}\right)\right|^{2}+\frac{3 \mu_{2}^{2}}{2} \int_{\Omega} P^{\prime}\left(v_{\varepsilon}\right) v_{\varepsilon}^{2}\left(1+v_{\varepsilon}^{2}+a_{2}^{2} u_{\varepsilon}^{2}\right) \tag{5.26}
\end{align*}
$$

Here we recall that by (5.5) and (5.3) we can find $c_{1}>0$ such that $v_{\varepsilon} \leq c_{1}$ in $\Omega \times(0, T)$ for all $\varepsilon \in(0,1)$, so that in view of (5.23),

$$
\begin{equation*}
d_{2} \leq P^{\prime}\left(v_{\varepsilon}\right) \leq c_{2}:=d_{2}+2 a_{22} c_{1} \quad \text { in } \Omega \times(0, T) \quad \text { for all } \varepsilon \in(0,1) \tag{5.27}
\end{equation*}
$$

Accordingly,

$$
\int_{\Omega} P^{\prime}\left(v_{\varepsilon}\right) v_{\varepsilon}^{2}\left(1+v_{\varepsilon}^{2}+a_{2}^{2} u_{\varepsilon}^{2}\right) \leq c_{1}^{2} c_{2}|\Omega|+c_{1}^{4} c_{2}|\Omega|+a_{2}^{2} c_{2} \int_{\Omega} u_{\varepsilon}^{2} \quad \text { for all } t \in(0, T)
$$

and thus (5.26) implies that

$$
\frac{d}{d t} \int_{\Omega}\left|\nabla P\left(v_{\varepsilon}\right)\right|^{2}+\int_{\Omega} P^{\prime}\left(v_{\varepsilon}\right)\left|\Delta v_{\varepsilon}\right|^{2} \leq c_{3}+c_{4} \int_{\Omega} u_{\varepsilon}^{2} \quad \text { for all } t \in(0, T)
$$

with $c_{3}:=3 \mu_{2}^{2}\left(c_{1}^{2} c_{2}|\Omega|+c_{1}^{4} c_{2}|\Omega|\right)$ and $c_{4}:=3 \mu_{2}^{2} a_{2}^{2} c_{2}$. Upon integration, again by (5.27) this entails that

$$
\begin{aligned}
\int_{\Omega}\left|\nabla P\left(v_{\varepsilon}(\cdot, t)\right)\right|^{2}+d_{2} \int_{0}^{t} \int_{\Omega}\left|\Delta P\left(v_{\varepsilon}\right)\right|^{2} & \leq \int_{\Omega}\left|\nabla P\left(v_{\varepsilon}(\cdot, t)\right)\right|^{2}+\int_{0}^{t} \int_{\Omega} P^{\prime}\left(v_{\varepsilon}\right)\left|\Delta P\left(v_{\varepsilon}\right)\right|^{2} \\
& \leq \int_{\Omega} P^{2}\left(v_{0 \varepsilon}\right)\left|\nabla v_{0 \varepsilon}\right|^{2}+c_{3} T+c_{4} \int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{2} \\
& \leq c_{2}^{2} \int_{\Omega}\left|\nabla v_{0 \varepsilon}\right|^{2}+c_{3} T+c_{4} \int_{0}^{T} \int_{\Omega} u_{\varepsilon}^{2} \quad \text { for all } t \in(0, T)
\end{aligned}
$$

and hence establishes (5.24) and (5.25) due to (5.22) and (5.7).
Once more by interpolation, this has a favorable consequence on integrability of $\nabla v_{\varepsilon}$.
Lemma 5.6 If (5.22) holds, then for all $T>0$ one can find $C(T)>0$ fulfilling

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{4} \leq C(T) \quad \text { for all } \varepsilon \in(0,1) \tag{5.28}
\end{equation*}
$$

Proof. Again taking $P$ as defined in (5.23), from the Gagliardo-Nirenberg inequality and elliptic regularity theory $([10])$ we obtain $c_{1}>0$ such that

$$
\begin{align*}
\int_{0}^{T} \int_{\Omega}\left|\nabla P\left(v_{\varepsilon}\right)\right|^{4} & \leq c_{1} \int_{0}^{T}\left\|\Delta P\left(v_{\varepsilon}(\cdot, t)\right)\right\|_{L^{2}(\Omega)}^{2} \| P\left(v_{\varepsilon}(\cdot, t) \|_{L^{\infty}(\Omega)}^{2} d t\right. \\
& \leq c_{1} c_{2} \int_{0}^{T} \int_{\Omega}\left|\Delta P\left(v_{\varepsilon}\right)\right|^{2} \quad \text { for all } \varepsilon \in(0,1) \tag{5.29}
\end{align*}
$$

where $c_{1}:=\sup _{\varepsilon \in(0,1)}\left\|P\left(v_{\varepsilon}\right)\right\|_{L^{\infty}(\Omega \times(0, T))}=\sup _{\varepsilon \in(0,1)}\left\{d_{2}\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega \times(0, T))}+a_{22}\left\|v_{\varepsilon}\right\|_{L^{\infty}(\Omega \times(0, T))}^{2}\right\}$ is finite according to Lemma 5.1. Since $\left|\nabla P\left(v_{\varepsilon}\right)\right|=\left(d_{2}+2 a_{22} v_{\varepsilon}\right)\left|\nabla v_{\varepsilon}\right| \geq d_{2}\left|\nabla v_{\varepsilon}\right|$ in $\Omega \times(0, \infty)$, due to Lemma 5.5 we directly infer (5.28) from (5.29).

In conclusion, Theorem 1.3 can be applied so as to yield our claimed results on global existence of weak solutions in (1.39) for initial data merely belonging to $L^{1} \times\left(W^{1,2} \cap L^{\infty}\right)$.
Proof of Theorem 1.8. We let $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}, u$ and $v$ be as provided by Theorem 1.7. Then due to the fact that clearly $\int_{0}^{T} \int_{\Omega}|\nabla v|^{4} \leq \liminf _{\varepsilon=\varepsilon_{j} \searrow 0} \int_{0}^{T} \int_{\Omega}\left|\nabla v_{\varepsilon}\right|^{4}$ for all $T>0$ by Lemma 5.6 , we may apply Theorem 1.3 to $q:=4$ and $\alpha:=2$ and thereby infer the claimed additional regularity and solution properties, beyond those guaranteed by Theorem 1.7, of $u$. It thus remains to note that the inclusion $v \in L_{l o c}^{\infty}\left([0, \infty) ; W^{1,2}(\Omega)\right)$ is a by-product of $(5.24)$ when once more combined with the fact that $P$ from (5.23) satisfies $P^{\prime} \geq d_{2}>0$ throughout [0, $\infty$ ).

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