Single-point blow-up in the Cauchy problem for the higher-dimensional Keller-Segel system

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Abstract

The Cauchy problem in $\mathbb{R}^n$ for the Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

is considered for $n \geq 3$.

Using a basic theory of local existence and maximal extensibility of classical and spatially integrable solutions as a starting point, the study provides a result on the occurrence of finite-time blow-up within considerably large sets of radially symmetric initial data, and moreover verifies that any such explosion exclusively occurs at the spatial origin.

The detection of blow-up is accomplished by analyzing a relative of the well-known Keller-Segel energy inequality, involving a modification of the corresponding energy functional which, unlike the latter, can be seen to be favorably controlled from below by the corresponding dissipation rate through a certain functional inequality along trajectories.

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1 Introduction

The Keller-Segel system, in its most prototypical version coupling two parabolic equations according to

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.1}$$

plays an important role in the biomathematical literature, and its essential ingredients form the respective core in a growing number of increasingly complex macroscopic models for migration processes at virtually all conceivable length scales. With applications ranging from paradigmatic cell aggregation phenomena such as in populations of *Dictyostelium discoideum* or *E. coli* ([16]), over models for tumor cell invasion ([3], [20]) for virus hotspot formation ([29]) or for socially interacting animal populations ([30]), up to the description of large-scale evolution in spatial ecology ([7], [33]), its relevance seems closely connected with its ability to describe spontaneous emergence of spatial structures.

In fact, already shortly after its proposal in the 1970s the model (1.1) was conjectured to support even the formation of singular structures in the mathematically extreme sense of finite-time blow-up for some solutions ([25]; cf. also the historical remarks in [12]); however, rigorous analytical detections of such explosions were accomplished only in the 1990s, and throughout a significantly long further period remained limited to either certain parabolic-elliptic simplifications of (1.1) ([15], [22], [23], [1]), to the construction of particular and possibly non-generic initial data enforcing blow-up ([10]), or to statements on mere unboundedness without option to determine whether such phenomena indeed occur in finite or only in infinite time ([13]). This seems to rather well reflect the circumstance that in contrast to typical objects of parabolic blow-up analysis such as quite thoroughly understood scalar reaction-diffusion equations with zero-order or first-order superlinear sources ([26]), directional effects of the driving cross-diffusive nonlinearity in (1.1) follow substantially more complex mechanisms and hence require accordingly subtle analysis.

Correspondingly, only in the recent few years some additional methodological developments fostered further progress in this field. Here a first branch of novel activities concentrates on a fine analysis of the dynamics near explicit singular steady states of the two-dimensional version of (1.1) ([28]), and hence on the one hand remains somewhat local with respect to the choice of initia data, but on the other hand can be considered quite constructive by namely providing considerable qualitative information on the asymptotic behavior of the obtained solutions near their blow-up time. An independent second recent development, though more destructive in the sense of simply confirming the occurrence of explosions without significant further qualitative description, has been found capable of identifying large sets of initial data which lead to finite-time blow-up in Neumann problems for (1.1) in n-dimensional balls Ω, in both cases $n \geq 3$ ([32]) and $n = 2$ ([21]). In such situations, namely, the quantities

$$\mathcal{F}_0 := \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega v^2 - \int_\Omega uv + \int_\Omega u \ln u \tag{1.2}$$

and

$$\mathcal{D}_0 := \int_\Omega |\Delta v - v + u|^2 + \int_\Omega \Big| \frac{\nabla u}{\sqrt{u}} - \sqrt{u} \nabla v \Big|^2, \tag{1.3}$$

known to constitute the natural energy inequality

$$\frac{d}{dt}\mathcal{F}_0 \le -\mathcal{D}_0 \tag{1.4}$$

along reasonably regular solution curves of (1.1), could be shown to satisfy certain inequalities of the form

$$\mathcal{F}_0 \ge -C \cdot (\mathcal{D}_0^\theta + 1) \qquad \text{with some } C > 0 \text{ and } \theta \in (0,1) \tag{1.5}$$

throughout suitably large sets of radially symmetric functions $(u,v) = (u(r), v(r))$ over Ω which, in particular, include radial trajectories of (1.1) ([32], [21]).

As several subsequent studies have revealed, functional inequalities of the flavor in (1.5) can be derived for significantly larger classes of expressions generalizing those in (1.2) and (1.3), and can thus be applied to corresponding Neumann problems for several generalizations of (1.1) ([4], [19], [18]). In problems posed in the entire space $\mathbb{R}^n$, however, the use of inequalities in the form of (1.4) for blow-up detection so far seems limited to situations in which the corresponding energy functional is constituted by sums of integrals among which each can favorably be bounded in its negative part by the associated dissipation rate in the style of e.g. (1.5). In particular, in the Cauchy problem for (1.1) the quantities $\mathcal{F}_0$ and $\mathcal{D}_0$ from (1.2) and (1.3) in this sense apparently become inappropriate, because in domains with infinite measure the expression $\int_\Omega u \ln u$ need no longer be bounded from below along trajectories when the only a priori information available for the first component thereof seems to reduce to an L^1 bound obtained through mass conservation. Due to a corresponding lack of suitable energy-based arguments, thus somewhat contrasting with the development of a small-data solution theory in the special case $\Omega = \mathbb{R}^2$ in which (1.4) can indeed be accompanied by moment control techniques to establish results on global solvability for sucritical-mass data ([2]), already the problem of verifying the mere existence of some non-global solutions to (1.1) on $\Omega = \mathbb{R}^n$ accordingly seems open up to now.

As a further complication encountered when passing from bounded to unbounded domains, we note that beyond such a basic finding on global nonexistence, the detection of blow-up in the spirit of a genuine application-relevant aggregation moreover should most favorably be accompanied by some statement on appropriate explosion localization, at least excluding the possibility that the corresponding blow-up set be empty. As impressive caveats in this regard, we recall some classical precedents which report on the appearance of so-called "blow-up at space infinity" phenomena already in some scalar parabolic problems ([17], [9], [27]).

Main results. Henceforth concerned with (1.1) in $\Omega = \mathbb{R}^n$, the present work accordingly addresses two objectives: A first goal consists in making this problem accessible to virial-type methods of blow-up detection, such as those introduced in [32] and [4] for bounded domains, by analyzing the evolution of a relative of the functional in (1.2) which is no longer genuinely nonincreasing along trajectories, but the possible growth of which can adequately be controlled, and which moreover enjoys favorable lower bounds, thus inter alia allowing for functional inequalities of the form in (1.5). Hence set in the position to verify the occurrence of finite-time blow-up throughout considerably large sets of initial data, as a second purpose we will pursue the problem of determining the corresponding blow-up sets, and thereby not only exclude the possibility of blow-up at space infinity, but actually even make sure that blow-up exclusively occurs at the spatial origin at least in frameworks of radially symmetric solutions.

To be more precise, for $n \ge 3$ we shall subsequently consider

$$\begin{cases} u_t &= \Delta u - \nabla \cdot (u \nabla v), & x \in \mathbb{R}^n,\ t > 0, \\ v_t &= \Delta v - v + u, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

under the assumptions that with some $q > n$,

$$\begin{cases} u_0 \in BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \text{ is nonnegative with } u_0 \not\equiv 0, \quad \text{and that} \\ v_0 \in W^{1,q}(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n) \text{ is nonnegative,} \end{cases} \tag{1.7}$$

where as usual, $BUC(\mathbb{R}^n)$ denotes the Banach space of all bounded and uniformly continuous functions on $\mathbb{R}^n$. In most places we will moreover require that

$$u_0 \text{ and } v_0 \text{ are radially symmetric with respect to } x = 0. \tag{1.8}$$

Then an indispensable prerequisite not only for our analysis of the functional $\mathcal{F}$ below, but also for our basic qualitative description of blow-up given in (1.14) and (1.15), is consituted by the following result on local existence and uniqueness of smooth solutions enjoying appropriate spatial decay features. As we predominantly intend to make use of this in the context of non-global solutions, besides the uniqueness feature that will ensure radial symmetry whenever (1.8) holds, we particularly stress the practically quite convenient extensibility criterion (1.10) here. Thanks to the choice of a fixed point setting somewhat different from precedent approaches both to two- and to higher-dimensional versions of (1.6) ([1], [2], [6], [5]), this criterion will, up to an additional minor argument on regularity implied by L^∞ bounds on u (Lemma 2.7), actually result as a fairly straightforward by-product from our construction of local solutions (Lemma 2.2); in view of its accordingly significant importance for Theorem 1.2 below, for reasons of full rigorousness we include an essentially complete demonstration of Proposition 1.1 in Section 2, although neither its outcome nor its derivation bear any considerable surprise. We note that in its main part, it does not require the symmetry assumption (1.8), and we may note that it actually extends quite immediately to the case $n \le 2$ not further pursued in the sequel.

Proposition 1.1 *Let $n \ge 3$, and assume (1.7) with some $q > n$. Then there exist $T_{max} \in (0, \infty]$ and precisely one pair of functions*

$$\begin{cases} u \in C^0([0, T_{max}); BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \cap C^{2,1}(\mathbb{R}^n \times (0, T_{max})) \qquad \textit{and} \\ v \in C^0([0, T_{max}); W^{1,q}(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)) \cap C^{2,1}(\mathbb{R}^n \times (0, T_{max})) \end{cases} \tag{1.9}$$

which solve (1.6) in the classical sense in $\mathbb{R}^n \times (0, T_{max})$, and which are such that

$$\textit{if } T_{max} < \infty, \quad \textit{then} \quad \limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} = \infty. \tag{1.10}$$

Moreover, $u > 0$ and $v > 0$ in $\mathbb{R}^n \times (0, T_{max})$, and we have

$$\int_{\mathbb{R}^n} u(\cdot, t) = \int_{\mathbb{R}^n} u_0 \qquad \textit{for all } t \in (0, T_{max}). \tag{1.11}$$

Finally, if in addition (1.8) holds, then $u(\cdot, t)$ and $v(\cdot, t)$ are radially symmetric with respect to $x = 0$ for all $t \in (0, T_{max})$.

Now the core of our results restricts to the radial setting and, indeed, asserts occurrence of finite-time blow-up, localized at the spatial origin, within sets of initial data enjoying a certain density property. This will be achieved on the basis of an appropriate and rigorously verifiable relative of the identity

$$\begin{aligned} \frac{d}{dt}\mathcal{F} &= -\int_{\mathbb{R}^n} |\Delta v - v + u|^2 - \int_{\mathbb{R}^n} \Big|\frac{\nabla u}{\sqrt{u+1}} - \sqrt{u+1}\nabla v\Big|^2 \\ &\quad - \int_{\mathbb{R}^n} \Big|\frac{\nabla u}{u+1} + \frac{\nabla v}{2(u+1)}\Big|^2 + \int_{\mathbb{R}^n} \Big(1 + \frac{1}{4(u+1)^2}\Big)|\nabla v|^2 \end{aligned} \tag{1.12}$$

formally fulfilled by smooth solutions of (1.6) which decay suitably fast in space, where

$$\mathcal{F} := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v|^2 + \frac{1}{2}\int_{\mathbb{R}^n} v^2 - \int_{\mathbb{R}^n} uv + \int_{\mathbb{R}^n} u\ln(u+1)$$

(Section 3). According to a functional inequality favorably controling $\int_{\mathbb{R}^n} uv$ in terms of the dissipated quantities in (1.12) along radial trajectories (Section 4), thanks to the trivial fact that $u\ln(u+1)$ is nonnegative it can be shown that $\mathcal{F}$, along with some meaningful replacement of $\mathcal{D}_0$, satisfies a lower estimate of the form in (1.5), and hence ensures finite-time blow-up for all radial initial data with suitably large negative energy (Section 5 and Section 6). Finally, a bootstrap-like regularity reasoning will reveal boundedness of any such non-global radial solution outside arbitrary neighborhoods of the spatial origin (Section 7).

In summary, we will obtain the following statement on blow-up in which, as throughout the sequel, for $R > 0$ we abbreviate $B_R := B_R(0) \subset \mathbb{R}^n$.

Theorem 1.2 *Suppose that $n \ge 3$ and that with some $q > n$ the functions u_0 and v_0 satisfy (1.7) and (1.8) and are positive on $\mathbb{R}^n$. Then for any choice of $p \in (1, \frac{2n}{n+2})$ one can find $(u_{0j})_{j\in\mathbb{N}} \subset BUC(\mathbb{R}^n)\cap L^1(\mathbb{R}^n)$ and $(v_{0j})_{j\in\mathbb{N}} \subset W^{1,q}(\mathbb{R}^n)\cap W^{1,1}(\mathbb{R}^n)$ such that u_{0j} and v_{0j} are radially symmetric and positive for all $j \in \mathbb{N}$, that*

$$u_{0j} \to u_0 \quad \text{in } L^p(\mathbb{R}^n)\cap L^1(\mathbb{R}^n) \qquad \text{and} \qquad v_{0j} \to v_0 \quad \text{in } W^{1,2}(\mathbb{R}^n)\cap W^{1,1}(\mathbb{R}^n) \qquad \text{as } j\to\infty, \tag{1.13}$$

and that for each $j \in \mathbb{N}$ the associated classical solution (u_j, v_j) of (1.6) from Proposition 1.1 blows up in finite time at the spatial origin, in the sense that the corresponding maximal existence time $T_{max,j} > 0$ actually satisfies $T_{max,j} \le 1$, that

$$\limsup_{t\nearrow T_{max,j}} \|u_j(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = \infty, \tag{1.14}$$

but that

$$\sup_{t\in(0,T_{max,j})} \|u_j(\cdot,t)\|_{L^\infty(\mathbb{R}^n\setminus B_\delta)} < \infty \qquad \text{for all } \delta > 0. \tag{1.15}$$

2 Local existence, uniqueness and maximal extensibility. Proof of Proposition 1.1

2.1 Local existence and regularity of mild solutions

To prepare our construction of local-in-time solutions, for $\varphi \in \bigcup_{p\in[1,\infty]} L^p(\mathbb{R}^n)$ we let

$$[e^{t\Delta}\varphi](x) := \int_{\mathbb{R}^n} G(x-y,t)\varphi(y)dy, \qquad x \in \mathbb{R}^n, \ t > 0,$$

with $G(z,t) := (4\pi t)^{-\frac{n}{2}} e^{-\frac{|z|^2}{4t}}$ for $z \in \mathbb{R}^n$ and $t > 0$. Then the following lemma collects some essentially well-known facts that can readily be derived using basic integrability and regularity properties of G, and a proof of which is thus omitted here.

Lemma 2.1 *i) If $\varphi \in \bigcup_{p\in[1,\infty]} W^{1,p}(\mathbb{R}^n)$, then for each $i \in \{1,...,n\}$,*

$$\partial_{x_i} e^{t\Delta}\varphi = e^{t\Delta}\partial_{x_i}\varphi \quad \textit{in } \mathbb{R}^n \qquad \textit{for all } t > 0.$$

ii) Whenever $1 \le p \le q \le \infty$ and $\omega \in \mathbb{N}_0^n$, one can find $C(p,q,\omega) > 0$ with the property that given any $\varphi \in L^p(\mathbb{R}^n)$ we have

$$\|D_x^\omega e^{t\Delta}\varphi\|_{L^q(\mathbb{R}^n)} \le C(p,q,\omega) t^{-\frac{|\omega|}{2} - \frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \|\varphi\|_{L^p(\mathbb{R}^n)} \qquad \textit{for all } t > 0.$$

iii) If $p \in [1,\infty]$, $q \in [1,\infty]$, $T > 0$, $\lambda \in \mathbb{R}$ and $\varphi \in L^\infty((0,T); L^q(\mathbb{R}^n))$, then

$$[0,T] \ni t \mapsto \int_0^t e^{(t-s)(\Delta+\lambda)}\varphi(\cdot,s)ds \quad \textit{belongs to } C^0([0,T]; L^p(\mathbb{R}^n)) \quad \textit{if} \quad \frac{1}{p} < \frac{1}{q} + \frac{2}{n}$$

and for all $i \in \{1,...,n\}$,

$$[0,T] \ni t \mapsto \int_0^t \partial_{x_i} e^{(t-s)(\Delta+\lambda)}\varphi(\cdot,s)ds \quad \textit{lies in } C^0([0,T]; L^p(\mathbb{R}^n)) \quad \textit{if} \quad \frac{1}{p} < \frac{1}{q} + \frac{1}{n}.$$

Then the following statement on local existence of smooth solutions, along with a first though not yet very convenient extensibility criterion, can be established by application of a contraction mapping argument. In view of our eventual goal to achieve even (1.10), the function space setting chosen here will differ from those underlying apparently all precedent relatives (see [1], [5] and [6] and also [14], for instance).

Lemma 2.2 *Suppose that (1.7) holds with some $q > n$. Then there exist $T_{max} \in (0,\infty]$ and at least one classical solution (u,v) of (1.6) in $\mathbb{R}^n \times (0,T_{max})$ fulfilling (1.9) which is such that*

$$u(\cdot,t) = e^{t\Delta}u_0 - \int_0^t \nabla\cdot e^{(t-s)\Delta}[u(\cdot,s)\nabla v(\cdot,s)]ds \quad \textit{in } \mathbb{R}^n \qquad \textit{for all } t \in (0,T_{max}) \tag{2.1}$$

and

$$v(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)ds \quad \textit{in } \mathbb{R}^n \qquad \textit{for all } t \in (0,T_{max}), \tag{2.2}$$

and that

$$\text{if } T_{max} < \infty, \text{ then} \qquad \limsup_{t \nearrow T_{max}} \Big\{ \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} + \|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} + \|v(\cdot,t)\|_{L^1(\mathbb{R}^n)} + \|\nabla v(\cdot,t)\|_{L^q(\mathbb{R}^n)} + \|\nabla v(\cdot,t)\|_{L^1(\mathbb{R}^n)} \Big\} = \infty. \tag{2.3}$$

PROOF. According to Lemma 2.1, let us pick $c_1 > 0, c_2 > 0$ and $c_3 > 0$ such that

$$\|\nabla e^{t\Delta}\varphi\|_{L^\infty(\mathbb{R}^n)} \le c_1 t^{-\frac{1}{2}-\frac{n}{2q}} \|\varphi\|_{L^q(\mathbb{R}^n)} \qquad \text{for all } t > 0 \text{ and } \varphi \in L^q(\mathbb{R}^n) \tag{2.4}$$

and

$$\|\nabla e^{t\Delta}\varphi\|_{L^1(\mathbb{R}^n)} \le c_2 t^{-\frac{1}{2}} \|\varphi\|_{L^1(\mathbb{R}^n)} \qquad \text{for all } t > 0 \text{ and } \varphi \in L^1(\mathbb{R}^n), \tag{2.5}$$

as well as

$$\|\nabla e^{t\Delta}\varphi\|_{L^q(\mathbb{R}^n)} \le c_3 t^{-\frac{1}{2}} \|\varphi\|_{L^q(\mathbb{R}^n)} \qquad \text{for all } t > 0 \text{ and } \varphi \in L^q(\mathbb{R}^n), \tag{2.6}$$

and take $T \in (0,1)$ small enough fulfilling

$$\max\left\{ \frac{nc_1 R^2 T^{\frac{1}{2}-\frac{n}{2q}}}{\frac{1}{2}-\frac{n}{2q}} \, , \, 2nc_2 R^2 T^{\frac{1}{2}} \, , \, RT \, , \, 2c_3 R T^{\frac{1}{2}} \, , \, 2c_2 R T^{\frac{1}{2}} \right\} \le \frac{1}{5} \tag{2.7}$$

and

$$\max\left\{ \frac{2nc_1 R T^{\frac{1}{2}-\frac{n}{2q}}}{\frac{1}{2}-\frac{n}{2q}} \, , \, 4nc_2 R T^{\frac{1}{2}} \, , \, T \, , \, 2c_3 T^{\frac{1}{2}} \, , \, 2c_2 T^{\frac{1}{2}} \right\} \le \frac{1}{10}, \tag{2.8}$$

where

$$R := \|u_0\|_{L^\infty(\mathbb{R}^n)} + \|u_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{L^1(\mathbb{R}^n)} + \|\nabla v_0\|_{L^q(\mathbb{R}^n)} + \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + 1. \tag{2.9}$$

Then in the Banach space

$$X := C^0\bigg([0,T]; \Big(BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \Big) \times \Big(W^{1,q}(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n) \Big) \bigg),$$

equipped with the norm $\|\cdot\|_X$ defined by letting

$$\|(u,v)\|_X := \max_{t\in[0,T]} \Big\{ \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} + \|u(\cdot,t)\|_{L^1(\mathbb{R}^n)} + \|v(\cdot,t)\|_{L^1(\mathbb{R}^n)} + \|\nabla v(\cdot,t)\|_{L^q(\mathbb{R}^n)} + \|\nabla v(\cdot,t)\|_{L^1(\mathbb{R}^n)} \Big\}, \qquad (u,v) \in X,$$

we consider the closed set

$$S := \Big\{ (u,v) \in X \ \Big| \ (u,v)(\cdot,0) = (u_0,v_0) \text{ and } \|(u,v)\|_X \le R \Big\},$$

and for $(u,v) \in S$ we set $\Phi(u,v) := (\Phi_1(u,v), \Phi_2(u,v))$ with

$$\Phi_1(u,v)(\cdot,t) := e^{t\Delta}u_0 - \int_0^t \nabla \cdot e^{(t-s)\Delta}[u(\cdot,s)\nabla v(\cdot,s)]ds, \qquad t \in [0,T],$$

and

$$\Phi_2(u,v)(\cdot,t) := e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)ds, \qquad t\in[0,T].$$

Since $0 \le t \mapsto e^{t\Delta}u_0$ can easily be seen to belong to $C^0([0,\infty);BUC(\mathbb{R}^n))$ due to (1.7), two applications of Lemma 2.1 iii) then readily entail that Φ maps S into X. Moreover, recalling the well-known fact that $e^{t\Delta}$ is nonexpansive on $L^p(\mathbb{R}^n)$ for all $p\in[1,\infty]$ and $t>0$, for $(u,v)\in S$ we can use the first restriction contained in (2.7) to estimate

$$\begin{aligned}
\|\Phi_1(u,v)(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} &\le \|u_0\|_{L^\infty(\mathbb{R}^n)} + nc_1\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2q}}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^q(\mathbb{R}^n)}ds \\
&\le \|u_0\|_{L^\infty(\mathbb{R}^n)} + nc_1\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2q}}\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}\|\nabla v(\cdot,s)\|_{L^q(\mathbb{R}^n)}ds \\
&\le \|u_0\|_{L^\infty(\mathbb{R}^n)} + \frac{nc_1R^2T^{\frac{1}{2}-\frac{n}{2q}}}{\frac{1}{2}-\frac{n}{2q}} \\
&\le \|u_0\|_{L^\infty(\mathbb{R}^n)} + \frac{1}{5} \qquad \text{for all } t\in[0,T],
\end{aligned}$$

because (2.4) clearly warrants that $\|\nabla\cdot e^{t\Delta}\varphi\|_{L^\infty(\mathbb{R}^n)} \le nc_1t^{-\frac{1}{2}-\frac{n}{2q}}\|\varphi\|_{L^q(\mathbb{R}^n)}$ for all $t>0$ and each $\varphi\in L^q(\mathbb{R}^n;\mathbb{R}^n)$.
Likewise, relying on (2.5) and the second requirement entailed by (2.7) we see that

$$\begin{aligned}
\|\Phi_1(u,v)(\cdot,t)\|_{L^1(\mathbb{R}^n)} &\le \|u_0\|_{L^1(\mathbb{R}^n)} + nc_2\int_0^t (t-s)^{-\frac{1}{2}}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^1(\mathbb{R}^n)}ds \\
&\le \|u_0\|_{L^1(\mathbb{R}^n)} + nc_2\int_0^t (t-s)^{-\frac{1}{2}}\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}\|\nabla v(\cdot,s)\|_{L^1(\mathbb{R}^n)}ds \\
&\le \|u_0\|_{L^1(\mathbb{R}^n)} + 2nc_2R^2T^{\frac{1}{2}} \\
&\le \|u_0\|_{L^1(\mathbb{R}^n)} + \frac{1}{5} \qquad \text{for all } t\in[0,T],
\end{aligned}$$

while the third implication of (2.7) guarantees that

$$\begin{aligned}
\|\Phi_2(u,v)(\cdot,t)\|_{L^1(\mathbb{R}^n)} &\le \|v_0\|_{L^1(\mathbb{R}^n)} + \int_0^t \|u(\cdot,s)\|_{L^1(\mathbb{R}^n)}ds \\
&\le \|v_0\|_{L^1(\mathbb{R}^n)} + RT \\
&\le \|v_0\|_{L^1(\mathbb{R}^n)} + \frac{1}{5} \qquad \text{for all } t\in[0,T].
\end{aligned}$$

Since furthermore from (2.6) we know that due to the Hölder inequality and (2.7) we have

$$\begin{aligned}
\|\nabla\Phi_2(u,v)(\cdot,t)\|_{L^q(\mathbb{R}^n)} &\le \|\nabla v_0\|_{L^q(\mathbb{R}^n)} + c_3\int_0^t (t-s)^{-\frac{1}{2}}\|u(\cdot,s)\|_{L^q(\mathbb{R}^n)}ds \\
&\le \|\nabla v_0\|_{L^q(\mathbb{R}^n)} + c_3\int_0^t (t-s)^{-\frac{1}{2}}\|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}^{\frac{q-1}{q}}\|u(\cdot,s)\|_{L^1(\mathbb{R}^n)}^{\frac{1}{q}}ds
\end{aligned}$$

$$\begin{aligned}
&\le \|\nabla v_0\|_{L^q(\mathbb{R}^n)} + 2c_3RT^{\frac{1}{2}} \\
&\le \|\nabla v_0\|_{L^q(\mathbb{R}^n)} + \frac{1}{5} \qquad \text{for all } t \in [0,T],
\end{aligned}$$

and since, again by (2.5), the condition (2.7) furthermore warrants that

$$\begin{aligned}
\|\nabla\Phi_2(u,v)(\cdot,t)\|_{L^1(\mathbb{R}^n)} &\le \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + c_2\int_0^t (t-s)^{-\frac{1}{2}}\|u(\cdot,s)\|_{L^1(\mathbb{R}^n)}ds \\
&\le \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + 2c_2RT^{\frac{1}{2}} \\
&\le \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + \frac{1}{5} \qquad \text{for all } t \in [0,T],
\end{aligned}$$

from the definition of $\|\cdot\|_X$ it follows that

$$\begin{aligned}
\|\Phi(u,v)\|_X &\le \|u_0\|_{L^\infty(\mathbb{R}^n)} + \|u_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{L^1(\mathbb{R}^n)} + \|\nabla v_0\|_{L^q(\mathbb{R}^n)} + \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + 1 \\
&= R \qquad \text{for all } (u,v) \in S.
\end{aligned} \tag{2.10}$$

In quite a similar manner, given $(u,v) \in S$ and $(\overline{u},\overline{v}) \in S$ we can use (2.4) and the first requirement in (2.8) to estimate

$$\begin{aligned}
&\Big\|\Phi_1(u,v)(\cdot,t) - \Phi_1(\overline{u},\overline{v})(\cdot,t)\Big\|_{L^\infty(\mathbb{R}^n)} \\
&\quad= \left\|\int_0^t \nabla\cdot e^{(t-s)\Delta}\Big\{[u(\cdot,s)-\overline{u}(\cdot,s)]\nabla v(\cdot,s) + \overline{u}(\cdot,s)[\nabla v(\cdot,s) - \nabla\overline{v}(\cdot,s)]\Big\}ds\right\|_{L^\infty(\mathbb{R}^n)} \\
&\quad\le nc_1\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2q}}\Big\{\|u(\cdot,s)-\overline{u}(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}\|\nabla v(\cdot,s)\|_{L^q(\mathbb{R}^n)} \\
&\qquad\qquad + \|\overline{u}(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}\|\nabla v(\cdot,s) - \nabla\overline{v}(\cdot,s)\|_{L^q(\mathbb{R}^n)}\Big\}ds \\
&\quad\le \frac{2nc_1RT^{\frac{1}{2}-\frac{n}{2q}}}{\frac{1}{2}-\frac{n}{2q}}\|(u,v)-(\overline{u},\overline{v})\|_X \\
&\quad\le \frac{1}{10}\|(u,v)-(\overline{u},\overline{v})\|_X \qquad \text{for all } t \in [0,T],
\end{aligned}$$

whereas (2.5) together with the second restriction in (2.8) ensures that

$$\begin{aligned}
&\Big\|\Phi_1(u,v)(\cdot,t) - \Phi_1(\overline{u},\overline{v})(\cdot,t)\Big\|_{L^1(\mathbb{R}^n)} \\
&\quad\le nc_2\int_0^t (t-s)^{-\frac{1}{2}}\Big\{\|u(\cdot,s)-\overline{u}(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}\|\nabla v(\cdot,s)\|_{L^1(\mathbb{R}^n)} \\
&\qquad\qquad + \|\overline{u}(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}\|\nabla v(\cdot,s) - \nabla\overline{v}(\cdot,s)\|_{L^1(\mathbb{R}^n)}\Big\}ds \\
&\quad\le 4nc_2RT^{\frac{1}{2}}\|(u,v)-(\overline{u},\overline{v})\|_X \\
&\quad\le \frac{1}{10}\|(u,v)-(\overline{u},\overline{v})\|_X \qquad \text{for all } t \in [0,T].
\end{aligned}$$

Apart from that, the three rightmost conditions contained in (2.8) imply that for $(u,v) \in S$ and $(\overline{u},\overline{v}) \in S$ we have

$$\begin{aligned}
\|\Phi_2(u,v)(\cdot,t) - \Phi_2(\overline{u},\overline{v})(\cdot,t)\|_{L^1(\mathbb{R}^n)} &= \left\| \int_0^t e^{(t-s)(\Delta-1)}[u(\cdot,s) - \overline{u}(\cdot,s)]ds \right\|_{L^1(\mathbb{R}^n)} \\
&\le \int_0^t \|u(\cdot,s) - \overline{u}(\cdot,s)\|_{L^1(\mathbb{R}^n)} ds \\
&\le T\|(u,v) - (\overline{u},\overline{v})\|_X \\
&\le \frac{1}{10}\|(u,v) - (\overline{u},\overline{v})\|_X \qquad \text{for all } t \in [0,T]
\end{aligned}$$

and, by (2.6),

$$\begin{aligned}
\|\nabla\Phi_2(u,v)(\cdot,t) - \nabla\Phi_2(\overline{u},\overline{v})(\cdot,t)\|_{L^q(\mathbb{R}^n)} &\le c_3 \int_0^t (t-s)^{-\frac{1}{2}} \|u(\cdot,s) - \overline{u}(\cdot,s)\|_{L^q(\mathbb{R}^n)} ds \\
&\le 2c_3 T^{\frac{1}{2}} \|(u,v) - (\overline{u},\overline{v})\|_X \\
&\le \frac{1}{10}\|(u,v) - (\overline{u},\overline{v})\|_X \qquad \text{for all } t \in [0,T]
\end{aligned}$$

as well as

$$\begin{aligned}
\|\nabla\Phi_2(u,v)(\cdot,t) - \nabla\Phi_2(\overline{u},\overline{v})(\cdot,t)\|_{L^1(\mathbb{R}^n)} &\le c_2 \int_0^t (t-s)^{-\frac{1}{2}} \|u(\cdot,s) - \overline{u}(\cdot,s)\|_{L^1(\mathbb{R}^n)} ds \\
&\le 2c_2 T^{\frac{1}{2}} \|(u,v) - (\overline{u},\overline{v})\|_X \\
&\le \frac{1}{10}\|(u,v) - (\overline{u},\overline{v})\|_X \qquad \text{for all } t \in [0,T]
\end{aligned}$$

according to (2.5). In summary,

$$\|\Phi(u,v) - \Phi(\overline{u},\overline{v})\|_X \le 5 \cdot \frac{1}{10}\|(u,v) - (\overline{u},\overline{v})\|_X \qquad \text{for all } (u,v) \in X \text{ and } (\overline{u},\overline{v}) \in X,$$

which combined with (2.10) enables us to invoke the Banach fixed point theorem to find an element (u,v) of S such that $\Phi(u,v) = (u,v)$.

A standard argument (cf. e.g.[11, Lemma 3.3] for a detailed demonstration in a closely related setting) thereafter shows that actually (u,v) belongs to $(C^{2,1}(\mathbb{R}^n \times (0,T))^2$ and, since $\nabla\cdot$ and $e^{t\Delta}$ commute on $C^1(\mathbb{R}^n;\mathbb{R}^n) \cap L^1(\mathbb{R}^n;\mathbb{R}^n)$, solves (1.6) classically in $\mathbb{R}^n \times (0,T)$, and from our definition of T it becomes clear through another standard reasoning that (u,v) can be extended up to a maximal $T_{max} \in (0,\infty]$ fulfilling (2.3), and that (2.1) and (2.2) actually hold throughout the entire interval $(0,T_{max})$. $\square$

As it directly refers to the integral identity (2.2) and to Lemma 2.1, let us include the following basic integrability property of ∇v already here, although it will only be used in the course of our blow-up argument in Lemma 4.1, and in the part identifying $x=0$ as blow-up point (Lemma 7.1).

Lemma 2.3 *There exists $C > 0$ such that if u_0 and v_0 satisfy (1.7), then*

$$\|\nabla v(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + C\|u_0\|_{L^1(\mathbb{R}^n)} \qquad \textit{for all } t \in (0,T_{max}). \tag{2.11}$$

PROOF. To the integral identity (2.2), we only need to once again apply Lemma 2.1 along with fact that $e^{t\Delta}$ has Lipschitz constant 1 in $L^1(\mathbb{R}^n)$. In view of (1.11), namely, this shows that with some $c_1 > 0$ we have

$$\begin{aligned} \|\nabla v(\cdot,t)\|_{L^1(\mathbb{R}^n)} &\le e^{-t}\|\nabla v_0\|_{L^1(\mathbb{R}^n)} + c_1 \int_0^t (t-s)^{-\frac{1}{2}} e^{-(t-s)} \|u(\cdot,s)\|_{L^1(\mathbb{R}^n)} ds \\ &= e^{-t}\|\nabla v_0\|_{L^1(\mathbb{R}^n)} + c_1 \|u_0\|_{L^1(\mathbb{R}^n)} \int_0^t \sigma^{-\frac{1}{2}} e^{-\sigma} d\sigma \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

from which (2.11) follows by finiteness of $\int_0^\infty \sigma^{-\frac{1}{2}} e^{-\sigma} d\sigma$. □

2.2 Uniqueness

To prepare our subsequent localization arguments not only in this but also during the next sections, we fix a nonincreasing cut-off function $\xi \in C^\infty(\mathbb{R})$ fulfilling $\xi \equiv 1$ in $(-\infty, 0]$ and $\xi \equiv 0$ in $[1,\infty)$, and for $R > 0$ we let

$$\zeta_R(x) := \xi(|x| - R), \qquad x \in \mathbb{R}^n. \tag{2.12}$$

Then ζ_R is radially symmetric about the origin, with $\zeta_R \equiv 1$ in B_R and $\operatorname{supp} \zeta_R \subset \overline{B}_{R+1}$ as well as $\operatorname{supp} \nabla\zeta_R \subset \overline{B}_{R+1} \setminus B_R$, and moreover we have $0 \le \zeta_R \le 1$ in $\mathbb{R}^n$.

A first use of the family $(\zeta_R)_{R>0}$ enables us to conclude uniqueness of classical solutions within spaces of functions satisfying spatial decay conditions in the flavor of those from Lemma 2.2.

Lemma 2.4 *Assume (1.7) with some $q > n$. Then for each $T > 0$, the problem (1.6) admits at most one classical solution (u, v) in $\mathbb{R}^n \times (0, T)$ which is such that*

$$\begin{cases} u \in C^0([0,T]; BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)) \cap C^{2,1}(\mathbb{R}^n \times (0,T)) \qquad \textit{and} \\ v \in C^0([0,T]; W^{1,q}(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)) \cap C^{2,1}(\mathbb{R}^n \times (0,T)). \end{cases} \tag{2.13}$$

PROOF. If (u, v) and $(\overline{u}, \overline{v})$ are two classical solutions in $\mathbb{R}^n \times (0, T)$ fulfilling the above regularity assumptions, then $w := u - \overline{u}$ and $z := v - \overline{v}$ satisfy

$$w_t = \Delta w - \nabla\cdot(w\nabla v) - \nabla\cdot(\overline{u}\nabla z) \quad \text{and} \quad z_t = \Delta z - z + w \qquad \text{in } \mathbb{R}^n \times (0,T),$$

so that with $(\zeta_R)_{R>0}$ as defined in (2.12), for $R > 0$ we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \zeta_R^2 w^2 &= \int_\Omega \zeta_R^2 w \nabla\cdot\Big\{\nabla w - w\nabla v - \overline{u}\nabla z\Big\} \\ &= -\int_\Omega \zeta_R^2 |\nabla w|^2 - 2\int_{\mathbb{R}^n} \zeta_R w \nabla\zeta_R\cdot\nabla w \\ &\quad + \int_{\mathbb{R}^n} \zeta_R^2 w \nabla w\cdot\nabla v + 2\int_{\mathbb{R}^n} \zeta_R w^2 \nabla\zeta_R\cdot\nabla v \\ &\quad + \int_{\mathbb{R}^n} \zeta_R^2 \overline{u}\nabla w\cdot\nabla z + 2\int_{\mathbb{R}^n} \zeta_R \overline{u} w \nabla\zeta_R\cdot\nabla z \\ &\le -\int_{\mathbb{R}^n} \zeta_R^2 |\nabla w|^2 \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\int_{\mathbb{R}^n}\zeta_R^2|\nabla w|^2+4\int_{\mathbb{R}^n}|\nabla\zeta_R|^2w^2\\
&+\frac{1}{4}\int_{\mathbb{R}^n}\zeta_R^2|\nabla w|^2+\int_{\mathbb{R}^n}\zeta_R^2w^2|\nabla v|^2\\
&+2\int_{\mathbb{R}^n}\zeta_R w^2\nabla\zeta_R\cdot\nabla v\\
&+\frac{1}{4}\int_{\mathbb{R}^n}\zeta_R^2|\nabla w|^2+\int_{\mathbb{R}^n}\zeta_R^2\overline{u}^2|\nabla z|^2\\
&+2\int_{\mathbb{R}^n}\zeta_R\overline{u}w\nabla\zeta_R\cdot\nabla z\\
=\;&-\frac{1}{4}\int_{\mathbb{R}^n}\zeta_R^2|\nabla w|^2\\
&+4\int_{\mathbb{R}^n}|\nabla\zeta_R|^2w^2+\int_{\mathbb{R}^n}\zeta_R^2w^2|\nabla v|^2+\int_{\mathbb{R}^n}\zeta_R^2\overline{u}^2|\nabla z|^2\\
&+2\int_{\mathbb{R}^n}\zeta_R w^2\nabla\zeta_R\cdot\nabla v+2\int_{\mathbb{R}^n}\zeta_R\overline{u}w\nabla\zeta_R\cdot\nabla z \qquad \text{for all } t\in(0,T) \qquad (2.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n}\zeta_R^2|\nabla z|^2 &= \int_{\mathbb{R}^n}\zeta_R^2\nabla z\cdot(\nabla\Delta z-\nabla z+\nabla w)\\
&= \frac{1}{2}\int_{\mathbb{R}^n}\zeta_R^2\Delta|\nabla z|^2-\int_{\mathbb{R}^n}\zeta_R^2|D^2z|^2\\
&\quad -\int_{\mathbb{R}^n}\zeta_R^2|\nabla z|^2+\int_{\mathbb{R}^n}\zeta_R^2\nabla z\cdot\nabla w\\
&= -2\int_{\mathbb{R}^n}\zeta_R\nabla\zeta_R\cdot(D^2z\cdot\nabla z)-\int_{\mathbb{R}^n}\zeta_R^2|D^2z|^2\\
&\quad -\int_{\mathbb{R}^n}\zeta_R^2|\nabla z|^2+\int_{\mathbb{R}^n}\zeta_R^2\nabla z\cdot\nabla w\\
&\le \int_{\mathbb{R}^n}|\nabla\zeta_R|^2|\nabla z|^2+\frac{1}{4}\int_{\mathbb{R}^n}\zeta_R^2|\nabla w|^2 \qquad \text{for all } t\in(0,T) \qquad (2.15)
\end{aligned}
$$

according to the pointwise identity $\nabla z\cdot\nabla\Delta z=\frac{1}{2}\Delta|\nabla z|^2-|D^2z|^2$ and Young's inequality.

Now in view of (2.13), the numbers $c_1:=\sup_{t\in(0,T)}\|\nabla v(\cdot,t)\|_{L^q(\mathbb{R}^n)}$ and $c_2:=\sup_{t\in(0,T)}\|\overline{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ are finite, so that on the right-hand side of (2.14), using the Hölder inequality and the fact that $0\le\zeta_R\le 1$ we can further estimate

$$
\int_{\mathbb{R}^n}\zeta_R^2w^2|\nabla v|^2\le c_1^2\|\zeta_R w\|^2_{L^{\frac{2q}{q-2}}(\mathbb{R}^n)} \qquad \text{for all } t\in(0,T) \qquad (2.16)
$$

and

$$
\int_{\mathbb{R}^n}\zeta_R^2\overline{u}^2|\nabla z|^2\le c_2^2\int_{\mathbb{R}^n}\zeta_R^2|\nabla z|^2 \qquad \text{for all } t\in(0,T), \qquad (2.17)
$$

where thanks to our assumption that $q > n$ and hence $\frac{2q}{q-2} < \frac{2n}{n-2}$, we may employ the Gagliardo-Nirenberg inequality and Young's inequality to find $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned}
c_1^2 \|\zeta_R w\|^2_{L^{\frac{2q}{q-2}}(\mathbb{R}^n)} &\le c_3 \|\nabla(\zeta_R w)\|^{\frac{2n}{q}}_{L^2(\mathbb{R}^n)} \|\zeta_R w\|^{\frac{2(q-n)}{q}}_{L^2(\mathbb{R}^n)} \\
&\le \frac{1}{16} \|\nabla(\zeta_R w)\|^2_{L^2(\mathbb{R}^n)} + c_4 \|\zeta_R w\|^2_{L^2(\mathbb{R}^n)} \\
&= \frac{1}{16} \int_{\mathbb{R}^n} |\zeta_R \nabla w + w \nabla \zeta_R|^2 + c_4 \int_{\mathbb{R}^n} \zeta_R^2 w^2 \\
&\le \frac{1}{8} \int_{\mathbb{R}^n} \zeta_R^2 |\nabla w|^2 + \frac{1}{8} \int_{\mathbb{R}^n} |\nabla \zeta_R|^2 w^2 + c_4 \int_{\mathbb{R}^n} \zeta_R^2 w^2 \qquad \text{for all } t \in (0,T). \quad (2.18)
\end{aligned}$$

Next, further applications of the Hölder inequality show that abbreviating $c_5 := \|\nabla \zeta_1\|_{L^\infty(\mathbb{R}^n)}$, $c_6 := \sup_{t\in(0,T)} \|w(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$ and $c_7 := \sup_{t\in(0,T)} \|\nabla z(\cdot,t)\|_{L^q(\mathbb{R}^n)}$ we can estimate

$$\begin{aligned}
2\int_{\mathbb{R}^n} \zeta_R w^2 \nabla\zeta_R \cdot \nabla v &\le 2c_5 \int_{B_{R+1}\setminus B_R} w^2 |\nabla v| \\
&\le 2c_5 \|w\|^{\frac{q+1}{q}}_{L^\infty(\mathbb{R}^n)} \|w\|^{\frac{q-1}{q}}_{L^1(B_{R+1}\setminus B_R)} \|\nabla v\|_{L^q(\mathbb{R}^n)} \\
&\le 2c_1 c_5 c_6^{\frac{q+1}{q}} \|w\|^{\frac{q-1}{q}}_{L^1(B_{R+1}\setminus B_R)} \qquad \text{for all } t \in (0,T)
\end{aligned}$$

and

$$\begin{aligned}
2\int_{\mathbb{R}^n} \zeta_R \overline{u} w \nabla\zeta_R \cdot \nabla z &\le 2c_5 \int_{B_{R+1}\setminus B_R} \overline{u} w |\nabla z| \\
&\le 2c_5 \|\overline{u}\|_{L^\infty(\mathbb{R}^n)} \|w\|^{\frac{1}{q}}_{L^\infty(\mathbb{R}^n)} \|w\|^{\frac{q-1}{q}}_{L^1(B_{R+1}\setminus B_R)} \|\nabla z\|_{L^q(\mathbb{R}^n)} \\
&\le 2c_2 c_5 c_6^{\frac{1}{q}} c_7 \|w\|^{\frac{q-1}{q}}_{L^1(B_{R+1}\setminus B_R)} \qquad \text{for all } t \in (0,T)
\end{aligned}$$

as well as

$$\int_{\mathbb{R}^n} |\nabla\zeta_R|^2 w^2 \le c_5^2 \int_{B_{R+1}\setminus B_R} w^2 \le c_5^2 c_6 \|w\|_{L^1(B_{R+1}\setminus B_R)} \qquad \text{for all } t \in (0,T)$$

and

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla\zeta_R|^2 |\nabla z|^2 &\le c_5^2 \int_{B_{R+1}\setminus B_R} |\nabla z|^2 \\
&\le c_5^2 \|\nabla z\|^{\frac{q}{q-1}}_{L^q(\mathbb{R}^n)} \|\nabla z\|^{\frac{q-2}{q-1}}_{L^1(B_{R+1}\setminus B_R)} \\
&\le c_5^2 c_7^{\frac{q}{q-1}} \|\nabla z\|^{\frac{q-2}{q-1}}_{L^1(B_{R+1}\setminus B_R)} \qquad \text{for all } t \in (0,T).
\end{aligned}$$

On combining (2.14) with (2.15) and with (2.16)-(2.18), we hence infer that $y_R(t) := \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 w^2(\cdot,t) + \frac{1}{4}\int_\Omega \zeta_R^2 |\nabla z(\cdot,t)|^2$, $t \in [0,T]$, satisfies

$$\begin{aligned}
y_R'(t) &\le \frac{33}{8} \int_{\mathbb{R}^n} |\nabla\zeta_R|^2 w^2 + c_4 \int_{\mathbb{R}^n} \zeta_R^2 w^2 + c_2^2 \int_{\mathbb{R}^n} \zeta_R^2 |\nabla z|^2 + c_8 \|w\|^{\frac{q-1}{q}}_{L^1(B_{R+1}\setminus B_R)} + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla\zeta_R|^2 |\nabla z|^2 \\
&\le c_9 y_R(t) + c_8 \|w\|^{\frac{q-1}{q}}_{L^1(B_{R+1}\setminus B_R)} + c_{10} \|w\|_{L^1(B_{R+1}\setminus B_R)} + c_{11} \|\nabla z\|^{\frac{q-2}{q-1}}_{L^1(B_{R+1}\setminus B_R)} \quad \text{for all } t \in (0,T)
\end{aligned}$$

with $c_8 := 2c_1c_5c_6^{\frac{q-1}{q}} + 2c_2c_5c_6^{\frac{1}{q}}c_7$, $c_9 := \max\{2c_4, 4c_2^2\}$, $c_{10} := \frac{33}{8}c_5^2c_6$ and $c_{11} := \frac{1}{2}c_5^2c_7^{\frac{q}{q-1}}$. Since $y_R(0) = 0$, an integration thereof yields

$$y_R(t) \le \int_0^t e^{c_9(t-s)} \cdot \Big\{c_8\|w(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)}^{\frac{q-1}{q}} + c_{10}\|w(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)} + c_{11}\|\nabla z(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)}^{\frac{q-2}{q-1}}\Big\}ds$$

for all $t \in (0,T)$, and that thus

$$y_R(t) \to 0 \quad \text{as } R \to \infty \qquad \text{for all } t \in (0,T), \tag{2.19}$$

because due to (2.13) we have

$$\sup_{s\in(0,T)} \Big\{\|w(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)} + \|\nabla z(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)}\Big\} \to 0 \qquad \text{as } R \to \infty.$$

But by definition of $(y_R)_{R>0}$, (2.19) means that $w(\cdot,t) \equiv 0$ and $\nabla z(\cdot,t) \equiv 0$ in $\mathbb{R}^n$ for all $t \in (0,T)$, and that hence $(u,v) \equiv (\overline{u},\overline{v})$ in $\mathbb{R}^n \times (0,T)$. $\Box$

2.3 Positivity, mass conservation and the refined extensibility criterion (1.10)

As straightforward applications of well-known maximum and comparison principles ([26, Appendix F]) seem unavailable in the present setting of unbounded domains and possibly unbounded system ingredients, such as e.g. the coefficients $b(x,t) := -\nabla v$ and $c(x,t) := -\Delta v$ in $u_t = \Delta u + b(x,t)\cdot\nabla u + c(x,t)u$, we once more utilize a localization argument involving the functions from (2.12) to derive the following statement on positivity.

Lemma 2.5 *Assume (1.7) with some $q > n$. Then the solution (u,v) of (1.6) from Lemma 2.2 satisfies $u > 0$ and $v > 0$ in $\mathbb{R}^n \times (0,T_{max})$.*

Proof. In view of the strong maximum principle and (1.7), it is sufficient to make sure that both u and v are nonnegative in $\mathbb{R}^n \times (0,T)$ for each $T \in (0,T_{max})$. To verify this, for $R > 0$ we take ζ_R from (2.12) and use the continuous differentiability of $\mathbb{R} \ni s \mapsto s_-^2$, with $s_- := \max\{-s,0\}$ for $s \in \mathbb{R}$, to see that thanks to Young's inequality,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \zeta_R^2 u_-^2 &= -\int_{\mathbb{R}^n} \zeta_R^2|\nabla u_-|^2 + \int_{\mathbb{R}^n} \zeta_R^2 u_-\nabla u_-\cdot\nabla v \\
&\quad -2\int_{\mathbb{R}^n} \zeta_R u_-\nabla\zeta_R\cdot\nabla u_- + 2\int_{\mathbb{R}^n} \zeta_R u_-^2\nabla\zeta_R\cdot\nabla v \\
&\le -\frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2|\nabla u_-|^2 + \int_{\mathbb{R}^n} \zeta_R^2 u_-^2|\nabla v|^2 \\
&\quad +4\int_{\mathbb{R}^n} |\nabla\zeta_R|^2 u_-^2 + 2\int_{\mathbb{R}^n} \zeta_R u_-^2\nabla\zeta_R\cdot\nabla v \qquad \text{for all } t \in (0,T_{max}).
\end{aligned} \tag{2.20}$$

Here we may proceed similarly to the proof of Lemma 2.4 in employing the Hölder inequality and the Gagliardo-Nirenberg inequality to see that since ∇v belongs to $L^\infty((0,T);L^q(\mathbb{R}^n))$, with some

$c_1 = c_1(T) > 0$ and $c_2 = c_2(T) > 0$ and for all $R > 0$ we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \zeta_R^2 u_-^2 |\nabla v|^2 &\le \|\nabla v\|_{L^q(\mathbb{R}^n)}^2 \|\zeta_R u_-\|_{L^{\frac{2q}{q-2}}(\mathbb{R}^n)}^2 \\
&\le c_1 \|\nabla(\zeta_R u_-)\|_{L^2(\mathbb{R}^n)}^{\frac{2n}{q}} \|\zeta_R u_-\|_{L^2(\mathbb{R}^n)}^{\frac{2(q-n)}{q}} \\
&\le \frac{1}{4}\|\nabla(\zeta_R u_-)\|_{L^2(\mathbb{R}^n)}^2 + c_2 \|\zeta_R u_-\|_{L^2(\mathbb{R}^n)}^2 \\
&\le \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R |\nabla u_-|^2 + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla \zeta_R|^2 u_-^2 + c_2 \int_{\mathbb{R}^n} \zeta_R^2 u_-^2 \qquad \text{for all } t \in (0,T).
\end{aligned}$$

As, furthermore, using (2.12) we can find $c_3 > 0$ and $c_4 = c_4(T) > 0$ such that for all $R > 0$,

$$\begin{aligned}
2\int_{\mathbb{R}^n} \zeta_R u_-^2 \nabla\zeta_R \cdot \nabla v &\le c_3 \int_{B_{R+1}\setminus B_R} u_-^2 |\nabla v| \\
&\le c_3 \|u_-\|_{L^{\frac{2q}{q-1}}(B_{R+1}\setminus B_R)}^2 \|\nabla v\|_{L^q(\mathbb{R}^n)} \\
&\le c_4 \|u_-\|_{L^{\frac{2q}{q-1}}(B_{R+1}\setminus B_R)}^2 \qquad \text{for all } t \in (0,T),
\end{aligned}$$

the inequality (2.20) therefore implies that $y_R(t) := \int_{\mathbb{R}^n} \zeta_R^2 u_-^2(\cdot,t)$, $t \in [0,T]$, $R > 0$, satisfies

$$y_R'(t) \le 2c_2 y_R(t) + c_5 \|u_-\|_{L^2(B_{R+1}\setminus B_R)}^2 + 2c_4 \|u_-\|_{L^{\frac{2q}{q-1}}(B_{R+1}\setminus B_R)}^2 \qquad \text{for all } t \in (0,T)$$

with $c_5 := 9\|\nabla\zeta_1\|_{L^\infty(\mathbb{R}^n)}^2$. When integrated over time, this entails that since $y_R(0) = 0$,

$$y_R(t) \le \int_0^t e^{2c_2(t-s)} \cdot \Big\{ c_5 \|u_-(\cdot,s)\|_{L^2(B_{R+1}\setminus B_R)}^2 + 2c_4 \|u_-\|_{L^{\frac{2q}{q-1}}(B_{R+1}\setminus B_R)}^2 \Big\} ds$$

for all $t \in (0,T)$ and each $R > 0$, whence observing that for all $p \in (1,\infty)$ we have

$$\begin{aligned}
\sup_{s\in(0,T)} \|u_-(\cdot,s)\|_{L^p(B_{R+1}\setminus B_R)} &\le \sup_{s\in(0,T)} \|u(\cdot,s)\|_{L^p(B_{R+1}\setminus B_R)} \\
&\le \sup_{s\in(0,T)} \Big\{ \|u(\cdot,s)\|_{L^\infty(\mathbb{R}^n)}^{\frac{p-1}{p}} \|u(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)}^{\frac{1}{p}} \Big\} \\
&\to 0 \qquad \text{as } R \to \infty,
\end{aligned}$$

we conclude that $y_R(t) \to 0$ as $R \to \infty$ for all $t \in (0,T)$.

Having thereby asserted nonnegativity of u in $\mathbb{R}^n \times (0,T)$, we can make use this to see that once more due to Young's inequality,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \zeta_R^2 v_-^2 &= -\int_{\mathbb{R}^n} \zeta_R^2 |\nabla v_-|^2 - 2\int_{\mathbb{R}^n} \zeta_R v_- \nabla\zeta_R \cdot \nabla v_- \\
&\quad - \int_{\mathbb{R}^n} \zeta_R^2 v_-^2 - \int_{\mathbb{R}^n} \zeta_R^2 u v_- \\
&\le \int_{\mathbb{R}^n} |\nabla\zeta_R|^2 v_-^2 \\
&\le c_6 \|v_-\|_{L^2(B_{R+1}\setminus B_R)}^2 \qquad \text{for all } t \in (0,T) \text{ and } R > 0
\end{aligned}$$

with $c_6 := \|\nabla\zeta_1\|^2_{L^\infty(\mathbb{R}^n)}$. Since

$$\sup_{s\in(0,T)} \|v_-(\cdot,s)\|^2_{L^2(B_{R+1}\setminus B_R)} \le \sup_{s\in(0,T)} \Big\{ \|v(\cdot,s)\|^{\frac{q}{q-1}}_{L^q(\mathbb{R}^n)} \|v(\cdot,s)\|^{\frac{q-2}{q-1}}_{L^1(B_{R+1}\setminus B_R)} \Big\} \to 0 \qquad \text{as } R\to\infty,$$

this implies that

$$\begin{aligned}
\int_{\mathbb{R}^n} v_-^2(\cdot,t) &= \lim_{R\to\infty} \int_{\mathbb{R}^n} \zeta_R^2 v_-^2(\cdot,t) \\
&\le \limsup_{R\to\infty} \Big\{ 2c_6 \int_0^t \|v_-(\cdot,s)\|^2_{L^2(B_{R+1}\setminus B_R)} ds \Big\} \\
&= 0 \qquad \text{for all } t\in(0,T)
\end{aligned}$$

and hence completes the proof. □

Two further but now quite simple testing procedures involving (2.12) next allow for controling the mass functionals of both components in quite an expected manner.

Lemma 2.6 *Assume (1.7) with some $q>n$. Then the solution (u,v) of (1.6) from Lemma 2.2 enjoys the mass conservation property (1.11), and moreover we have*

$$\|v(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le \max\Big\{ \|v_0\|_{L^1(\mathbb{R}^n)}\,,\, \|u_0\|_{L^1(\mathbb{R}^n)} \Big\} \qquad \textit{for all } t\in(0,T_{max}). \tag{2.21}$$

Proof. Again taking $(\zeta_R)_{R>0}$ from (2.12), we use (1.6) to see that for any $T\in(0,T_{max})$,

$$\begin{aligned}
\Big| \frac{d}{dt} \int_{\mathbb{R}^n} \zeta_R u \Big| &= \Big| \int_{\mathbb{R}^n} \zeta_R \Delta u - \int_{\mathbb{R}^n} \zeta_R \nabla\cdot(u\nabla v) \Big| \\
&= \Big| \int_{\mathbb{R}^n} u\Delta\zeta_R + \int_\Omega u\nabla v\cdot\nabla\zeta_R \Big| \\
&\le c_1 \|u\|_{L^1(B_{R+1}\setminus B_R)} + c_2 \|\nabla v\|_{L^1(B_{R+1}\setminus B_R)} \qquad \text{for all } t\in(0,T)
\end{aligned}$$

with $c_1 := \|\Delta\zeta_1\|_{L^\infty(\mathbb{R}^n)}$ and $c_2 := \|\nabla\zeta_1\|_{L^\infty(\mathbb{R}^n)} \cdot \sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)}$. Since

$$\sup_{s\in(0,T)} \Big\{ \|u(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)} + \|\nabla v(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)} \Big\} \to 0 \qquad \text{as } R\to\infty$$

by Lemma 2.2, this entails that

$$\begin{aligned}
\Big| \int_{\mathbb{R}^n} u(\cdot,t) - \int_{\mathbb{R}^n} u_0 \Big| &= \lim_{R\to\infty} \Big| \int_{\mathbb{R}^n} \zeta_R u(\cdot,t) - \int_{\mathbb{R}^n} \zeta_R u_0 \Big| \\
&\le \limsup_{R\to\infty} \int_0^T \Big\{ c_1 \|u(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)} + c_2 \|\nabla v(\cdot,s)\|_{L^1(B_{R+1}\setminus B_R)} \Big\} ds \\
&= 0 \qquad \text{for all } t\in(0,T)
\end{aligned}$$

and that thus (1.11) holds, for $T\in(0,T_{max})$ was arbitrary.

Likewise, for fixed $T \in (0, T_{max})$ the second equation in (1.6) implies that due to (1.11),

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} \zeta_R v + \int_{\mathbb{R}^n} \zeta_R v &= \int_{\mathbb{R}^n} \zeta_R \Delta v + \int_{\mathbb{R}^n} \zeta_R u \\ &= \int_{\mathbb{R}^n} v \Delta \zeta_R + \int_{\mathbb{R}^n} \zeta_R u \\ &\le c_1 \|v\|_{L^1(B_{R+1} \setminus B_R)} + \int_{\mathbb{R}^n} u_0 \qquad \text{for all } t \in (0, T) \end{aligned}$$

and hence

$$\int_{\mathbb{R}^n} \zeta_R v(\cdot, t) \le \max \left\{ \int_{\mathbb{R}^n} \zeta_R v_0 \, , \, c_1 \sup_{s \in (0,T)} \|v(\cdot, s)\|_{L^1(B_{R+1} \setminus B_R)} + \int_{\mathbb{R}^n} u_0 \right\} \qquad \text{for all } t \in (0, T)$$

by an ODE comparison argument. Noting that Lemma 2.2 ensures that

$$\sup_{s \in (0,T)} \|v(\cdot, s)\|_{L^1(B_{R+1} \setminus B_R)} \to 0 \qquad \text{as } R \to \infty,$$

on taking $R \to \infty$ we readily obtain (2.21) from this. □

With these preparations at hand, we can return to the mild formulation (2.2) to conclude that the extensibility criterion (2.3) can be refined so as to actually reduce to (1.10).

Lemma 2.7 *Under the assumption that (1.7) is satisfied with some $q > n$, the solution of (1.6) from Lemma 2.2 has the property that (1.10) holds.*

Proof. Let us assume on the contrary that $T_{max} < \infty$, but that there exists $c_1 > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \le c_1 \qquad \text{for all } t \in (0, T_{max}). \tag{2.22}$$

Then since from Lemma 2.6 and the nonnegativity of u we know that

$$\|u(\cdot, t)\|_{L^1(\mathbb{R}^n)} = c_2 := \|u_0\|_{L^1(\mathbb{R}^n)} \qquad \text{for all } t \in (0, T_{max}), \tag{2.23}$$

by using the Hölder inequality we see that

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \le \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)}^{\frac{q-1}{q}} \|u(\cdot, t)\|_{L^1(\mathbb{R}^n)}^{\frac{1}{q}} \le c_3 := c_1^{\frac{q-1}{q}} c_2^{\frac{1}{q}} \qquad \text{for all } t \in (0, T_{max}).$$

As Lemma 2.2 ensures validity of (2.2), on applying Lemma 2.1 ii) to the latter identity we thus infer the existence of $c_4 > 0$ such that

$$\begin{aligned} \|\nabla v(\cdot, t)\|_{L^q(\mathbb{R}^n)} &\le \|\nabla v_0\|_{L^q(\mathbb{R}^n)} + c_4 \int_0^t (t-s)^{-\frac{1}{2}} \|u(\cdot, s)\|_{L^q(\mathbb{R}^n)} ds \\ &\le \|\nabla v_0\|_{L^q(\mathbb{R}^n)} + 2 c_3 c_4 T_{max}^{\frac{1}{2}} \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{2.24}$$

while Lemma 2.1 ii) in conjunction with (2.23) shows that with some $c_5 > 0$ we have

$$\begin{aligned} \|\nabla v(\cdot, t)\|_{L^1(\mathbb{R}^n)} &\le \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + c_5 \int_0^t (t-s)^{-\frac{1}{2}} \|u(\cdot, s)\|_{L^1(\mathbb{R}^n)} ds \\ &\le \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + 2 c_2 c_5 T_{max}^{\frac{1}{2}} \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{2.25}$$

As furthermore

$$\|v(\cdot,t)\|_{L^1(\mathbb{R}^n)} \le \max\Big\{\|v_0\|_{L^1(\mathbb{R}^n)}\,,\,c_2\Big\} \qquad \text{for all } t\in(0,T_{max}) \tag{2.26}$$

by Lemma 2.6, combining (2.23)-(2.26) with (2.3) reveals that in fact T_{max} cannot be finite under the assumption (2.22). □

Now our basic theory of local existence, uniqueness, extensibility and preservation of mass and of radial symmetry is complete:

Proof of Proposition 1.1. Existence, uniqueness and validity of (2.1) and (2.2) have been found Lemma 2.2 and Lemma 2.4, whereas positivity of u and v have been asserted by Lemma 2.5. Due to Lemma 2.7, this solution satisfies the refined extensibility criterion (1.10), and the mass conservation identity (1.11) is part of the statement from Lemma 2.6. Based on the uniqueness property, a standard argument thereupon reveals the claimed radial symmetry feature under the additional hypothesis (1.8). □

3 A quasi-energy functional bounded from below by $-\int_{\mathbb{R}^n} uv$

Next addressing the problem of detecting blow-up, motivated by (1.12) we shall perform one more localized testing procedure using $(\zeta_R)_{R>0}$ from (2.12) to achieve the following counterpart of (1.12) in which time differentiation is avoided due to possibly lacking regularity features, and in which unfavorable contributions have already been estimated in a convenient manner.

Lemma 3.1 *Assume (1.7) for some $q>n$, and let*

$$\mathcal{F}(t) := \frac{1}{2}\int_{\mathbb{R}^n}|\nabla v(\cdot,t)|^2 + \frac{1}{2}\int_{\mathbb{R}^n} v^2(\cdot,t) - \int_{\mathbb{R}^n} u(\cdot,t)v(\cdot,t) + \int_{\mathbb{R}^n} u(\cdot,t)\ln\Big(u(\cdot,t)+1\Big), \qquad t\in[0,T_{max}), \tag{3.1}$$

and

$$\mathcal{D}(t) := \frac{1}{2}\int_{\mathbb{R}^n}\Big|\Delta v(\cdot,t)-v(\cdot,t)+u(\cdot,t)\Big|^2 + \int_{\mathbb{R}^n}\Big|\frac{\nabla u(\cdot,t)}{\sqrt{u(\cdot,t)+1}} + \sqrt{u(\cdot,t)+1}\nabla v(\cdot,t)\Big|^2, \qquad t\in(0,T_{max}), \tag{3.2}$$

where (u,v) is the corresponding solution of (1.6) from Lemma 2.2. Then $\mathcal{F}\in C^0([0,T_{max}))$ and $\mathcal{D}\in L^1_{loc}([0,T_{max}))$ with $\mathcal{F}(0)=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla v_0|^2+\frac{1}{2}\int_{\mathbb{R}^n}v_0^2-\int_{\mathbb{R}^n}u_0v_0+\int_{\mathbb{R}^n}u_0\ln(u_0+1)$, and we have

$$\mathcal{F}(t)+\int_0^t\mathcal{D}(s)ds \le \mathcal{F}(0)+4\int_0^t\mathcal{F}(s)ds+4\int_0^t\int_{\mathbb{R}^n}uv \qquad \textit{for all } t\in(0,T_{max}). \tag{3.3}$$

Proof. Since Proposition 1.1 clearly entails that $v\in C^0([0,T_{max});W^{1,p}(\mathbb{R}^n))$ for all $p\in[1,q]$, from the inequality $q>2$ it follows that $\mathcal{F}^{(1)}(t):=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla v(\cdot,t)|^2$ and $\mathcal{F}^{(2)}(t):=\frac{1}{2}\int_{\mathbb{R}^n}v^2(\cdot,t)$, $t\in[0,T_{max})$, define continuous functions fulfilling $\mathcal{F}^{(1)}(0)=\frac{1}{2}\int_{\mathbb{R}^n}|\nabla v_0|^2$ and $\mathcal{F}^{(2)}(0)=\frac{1}{2}\int_{\mathbb{R}^n}v_0^2$. As Proposition 1.1 apart from that ensures that u and hence also $\ln(u+1)$ belong to $C^0([0,T_{max});L^\infty(\mathbb{R}^n))$, from the inclusion $\{u,v\}\subset C^0([0,T_{max});L^1(\mathbb{R}^n))$ we moreover obtain that also $\mathcal{F}^{(3)}(t):=-\int_{\mathbb{R}^n}u(\cdot,t)v(\cdot,t)$ and $\mathcal{F}^{(4)}(t):=\int_{\mathbb{R}^n}u(\cdot,t)\ln(u(\cdot,t)+1)$, $t\in[0,T_{max})$, are continuous with $\mathcal{F}^{(3)}(0)=-\int_{\mathbb{R}^n}u_0v_0$ and $\mathcal{F}^{(4)}(0)=\int_{\mathbb{R}^n}u_0\ln(u_0+1)$.

Having thus asserted continuity of $\mathcal{F} = \mathcal{F}^{(1)} + \mathcal{F}^{(2)} + \mathcal{F}^{(3)} + \mathcal{F}^{(4)}$ as well as its claimed initial behavior, we are left with the verification of (3.3). To accomplish this, for $R > 0$ we once more take ζ_R as defined in (2.12) and note that

$$\begin{aligned} \mathcal{F}_R(t) &:= \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 |\nabla v(\cdot,t)|^2 + \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 v^2(\cdot,t) \\ &\quad - \int_{\mathbb{R}^n} \zeta_R^2 u(\cdot,t) v(\cdot,t) + \int_{\mathbb{R}^n} \zeta_R^2 u(\cdot,t) \ln\Big(u(\cdot,t)+1\Big), \qquad t \in [0,T_{max}), \end{aligned}$$

is evidently continuous on $[0,T_{max})$ and continuously differentiable on $(0,T_{max})$, and satisfies

$$\begin{aligned} \mathcal{F}_R'(t) &= \int_{\mathbb{R}^n} \zeta_R^2 \nabla v \cdot \nabla v_t + \int_{\mathbb{R}^n} \zeta_R^2 v v_t - \int_{\mathbb{R}^n} \zeta_R^2 u v_t - \int_{\mathbb{R}^n} \zeta_R u_t v \\ &\quad + \int_{\mathbb{R}^n} \zeta_R^2 \Big(\ln(u+1) + \frac{u}{u+1}\Big) u_t \qquad \text{for all } t \in (0,T_{max}). \end{aligned} \tag{3.4}$$

Here integrating by parts and using Young's inequality shows that due to the second equation in (1.6),

$$\begin{aligned} &\int_{\mathbb{R}^n} \zeta_R^2 \nabla v \cdot \nabla v_t + \int_{\mathbb{R}^n} \zeta_R^2 v v_t - \int_{\mathbb{R}^n} \zeta_R^2 u v_t \\ &\quad = -\int_{\mathbb{R}^n} \zeta_R^2 \Delta v v_t - 2\int_{\mathbb{R}^n} \zeta_R v_t \nabla\zeta_R \cdot \nabla v + \int_{\mathbb{R}^n} \zeta_R^2 v v_t - \int_{\mathbb{R}^n} \zeta_R^2 u v_t \\ &\quad = -\int_{\mathbb{R}^n} \zeta_R^2 v_t^2 - 2\int_{\mathbb{R}^n} \zeta_R v_t \nabla\zeta_R \cdot \nabla v \\ &\quad \le -\frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 v_t^2 + 2\int_{\mathbb{R}^n} |\nabla\zeta_R|^2 |\nabla v|^2 \qquad \text{for all } t \in (0,T_{max}), \end{aligned} \tag{3.5}$$

whereas from the first equation in (1.6) we obtain that

$$\begin{aligned} -\int_{\mathbb{R}^n} \zeta_R^2 u_t v &= \int_{\mathbb{R}^n} \zeta_R^2 \nabla u \cdot \nabla v - \int_{\mathbb{R}^n} \zeta_R^2 u |\nabla v|^2 \\ &\quad + 2\int_{\mathbb{R}^n} \zeta_R v \nabla\zeta_R \cdot \nabla u - 2\int_{\mathbb{R}^n} \zeta_R u v \nabla\zeta_R \cdot \nabla v \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} &\int_{\mathbb{R}^n} \zeta_R^2 \Big(\ln(u+1) + \frac{u}{u+1}\Big) u_t \\ &\quad = -\int_{\mathbb{R}^n} \zeta_R^2 \Big(\frac{1}{u+1} + \frac{1}{(u+1)^2}\Big) |\nabla u|^2 + \int_{\mathbb{R}^n} \zeta_R^2 \Big(\frac{u}{u+1} + \frac{u}{(u+1)^2}\Big) \nabla u \cdot \nabla v \\ &\quad\quad -2\int_{\mathbb{R}^n} \zeta_R \Big(\ln(u+1) + \frac{u}{u+1}\Big) \nabla\zeta_R \cdot \nabla u + 2\int_{\mathbb{R}^n} \Big(u\ln(u+1) + \frac{u^2}{u+1}\Big) \nabla\zeta_R \cdot \nabla v \end{aligned} \tag{3.7}$$

for all $t \in (0,T_{max})$. Using the identity

$$\begin{aligned} &\nabla u \cdot \nabla v - u|\nabla v|^2 - \Big(\frac{1}{u+1} + \frac{1}{(u+1)^2}\Big) |\nabla u|^2 + \Big(\frac{u}{u+1} + \frac{u}{(u+1)^2}\Big) \nabla u \cdot \nabla v \\ &\quad = -\Big| \frac{\nabla u}{\sqrt{u+1}} - \sqrt{u+1}\nabla v\Big|^2 - \frac{1}{(u+1)^2}|\nabla u|^2 + |\nabla v|^2 - \frac{1}{(u+1)^2} \nabla u \cdot \nabla v \qquad \text{in } \mathbb{R}^n \times (0,T_{max}), \end{aligned}$$

from (3.6) and (3.7) we infer that

$$
\begin{aligned}
-\int_{\mathbb{R}^n} \zeta_R^2 u_t v + \int_{\mathbb{R}^n} \zeta_R^2 \Big(\ln(u+1) + \frac{u}{u+1} \Big) u_t \\
= \; & -\int_{\mathbb{R}^n} \zeta_R^2 \Big| \frac{\nabla u}{\sqrt{u+1}} - \sqrt{u+1}\nabla v \Big|^2 - \int_{\mathbb{R}^n} \zeta_R^2 \frac{|\nabla u|^2}{(u+1)^2} + \int_{\mathbb{R}^n} \zeta_R^2 |\nabla v|^2 \\
& - \int_{\mathbb{R}^n} \zeta_R^2 \cdot \frac{1}{(u+1)^2} \nabla u \cdot \nabla v \\
& + 2\int_{\mathbb{R}^n} \zeta_R \Big(v - \ln(u+1) - \frac{u}{u+1} \Big) \nabla \zeta_R \cdot \nabla u \\
& - 2\int_{\mathbb{R}^n} \zeta_R \Big(uv - u\ln(u+1) - \frac{u^2}{u+1} \Big) \nabla \zeta_R \cdot \nabla v \qquad \text{for all } t \in (0, T_{max}). \qquad (3.8)
\end{aligned}
$$

Here by Young's inequality and the elementary estimates $\frac{1}{u+1} \le 1$ and $\ln(u+1) \le u$, we see that

$$
\begin{aligned}
-\int_{\mathbb{R}^n} \zeta_R^2 \cdot \frac{1}{(u+1)^2} \nabla u \cdot \nabla v \; & \le \; \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 \frac{|\nabla u|^2}{(u+1)^2} + \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 \frac{|\nabla v|^2}{(u+1)^2} \\
& \le \; \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 \frac{|\nabla u|^2}{(u+1)^2} + \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 |\nabla v|^2 \qquad \text{for all } t \in (0, T_{max})
\end{aligned}
$$

and

$$
\begin{aligned}
2 & \int_{\mathbb{R}^n} \zeta_R \Big(v - \ln(u+1) - \frac{u}{u+1} \Big) \nabla \zeta_R \cdot \nabla u \\
& \le \; \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 \frac{|\nabla u|^2}{(u+1)^2} + 2\int_{\mathbb{R}^n} |\nabla \zeta_R|^2 (u+1)^2 \Big(v - \ln(u+1) - \frac{u}{u+1} \Big)^2 \\
& \le \; \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 \frac{|\nabla u|^2}{(u+1)^2} + \int_{\mathbb{R}^n} |\nabla \zeta_R|^2 \Big(6(u+1)^2 v^2 + 6(u+1)^2 \ln^2(u+1) + 6u^2 \Big) \\
& \le \; \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 \frac{|\nabla u|^2}{(u+1)^2} + \int_{\mathbb{R}^n} |\nabla \zeta_R|^2 \Big(6(u+1)^2 v^2 + 6u^2(u+1)^2 + 6u^2 \Big) \qquad \text{for all } t \in (0, T_{max})
\end{aligned}
$$

as well as

$$
\begin{aligned}
-2 & \int_{\mathbb{R}^n} \zeta_R \Big(uv - u\ln(u+1) - \frac{u^2}{u+1} \Big) \nabla \zeta_R \cdot \nabla v \\
& \le \; \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 |\nabla v|^2 + 2\int_{\mathbb{R}^n} |\nabla \zeta_R|^2 \Big(uv - u\ln(u+1) - \frac{u^2}{u+1} \Big)^2 \\
& \le \; \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 |\nabla v|^2 + \int_{\mathbb{R}^n} |\nabla \zeta_R|^2 \Big(6u^2 v^2 + 6u^2 \ln^2(u+1) + 6\frac{u^4}{(u+1)^2} \Big) \\
& \le \; \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 |\nabla v|^2 + \int_{\mathbb{R}^n} |\nabla \zeta_R|^2 (6u^2 v^2 + 12u^4) \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}
$$

Therefore, inserting (3.5) and (3.8) into (3.4) shows that abbreviating $h = h(x,t) := 2|\nabla v|^2 + 6(u+1)^2 v^2 + 6u^2(u+1)^2 + 6u^2 + 6u^2 v^2 + 12u^4$, for all $R > 0$ we have

$$
\mathcal{F}_R'(t) + \frac{1}{2}\int_{\mathbb{R}^n} \zeta_R^2 v_t^2 + \int_{\mathbb{R}^n} \zeta_R^2 \Big| \frac{\nabla u}{\sqrt{u+1}} - \sqrt{u+1}\nabla v \Big|^2
$$

$$\begin{aligned}
&\le 2\int_{\mathbb{R}^n}\zeta_R^2|\nabla v|^2+\int_{\mathbb{R}^n}|\nabla\zeta_R|^2h\\
&= 4\cdot\Big\{\mathcal{F}_R(t)-\frac12\int_{\mathbb{R}^n}\zeta_R^2v^2+\int_{\mathbb{R}^n}\zeta_R^2uv-\int_{\mathbb{R}^n}\zeta_R^2u\ln(u+1)\Big\}+\int_{\mathbb{R}^n}|\nabla\zeta_R|^2h\\
&\le 4\mathcal{F}_R(t)+4\int_{\mathbb{R}^n}\zeta_R^2uv+\int_{\mathbb{R}^n}|\nabla\zeta_R|^2h \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

because $u\ln(u+1)$ is nonnegative. Upon integration, this implies that for all $t\in(0,T_{max})$ and each $R>0$,

$$\begin{aligned}
\mathcal{F}_R(t)&+\frac12\int_0^t\int_{\mathbb{R}^n}\zeta_R^2v_t^2+\int_0^t\int_{\mathbb{R}^n}\zeta_R^2\Big|\frac{\nabla u}{\sqrt{u+1}}-\sqrt{u+1}\nabla v\Big|^2\\
&\le \mathcal{F}_R(0)+4\int_0^t\mathcal{F}_R(s)ds+4\int_0^t\int_{\mathbb{R}^n}\zeta_R^2uv+\int_0^t\int_{\mathbb{R}^n}|\nabla\zeta_R|^2h,
\end{aligned}\tag{3.9}$$

where from Beppo Levi's theorem and the continuity of $\mathcal{F}^{(i)}$ for $i\in\{1,2,3,4\}$ it follows that as $R\to\infty$,

$$\begin{aligned}
&\frac12\int_0^t\int_{\mathbb{R}^n}\zeta_R^2v_t^2+\int_0^t\int_{\mathbb{R}^n}\zeta_R^2\Big|\frac{\nabla u}{\sqrt{u+1}}-\sqrt{u+1}\nabla v\Big|^2\\
&\qquad\to\ \frac12\int_0^t\int_{\mathbb{R}^n}v_t^2+\int_0^t\int_{\mathbb{R}^n}\Big|\frac{\nabla u}{\sqrt{u+1}}-\sqrt{u+1}\nabla v\Big|^2=\int_0^t\mathcal{D}(s)ds \qquad \text{for all } t\in(0,T_{max})
\end{aligned}$$

and

$$4\int_0^t\int_{\mathbb{R}^n}\zeta_R^2uv\to 4\int_0^t\int_{\mathbb{R}^n}uv \qquad \text{for all } t\in(0,T_{max})$$

as well as

$$\mathcal{F}_R(t)\to\mathcal{F}(t) \qquad \text{for all } t\in[0,T_{max})$$

and

$$4\int_0^t\mathcal{F}_R(s)ds\to 4\int_0^t\mathcal{F}(s)ds \qquad \text{for all } t\in(0,T_{max}).$$

Since furthermore another application of Proposition 1.1 readily reveals that h belongs to $L^1(\mathbb{R}^n\times(0,t))$ for all $t\in(0,T_{max})$, and since thus

$$\int_0^t\int_{\mathbb{R}^n}|\nabla\zeta_R|^2h\to 0 \qquad \text{as } R\to\infty$$

by (2.12) and the dominated conergence theorem, the validity of (3.3) results from (3.9). $\Box$

4 Dissipation controls superlinear powers of $\int_{\mathbb{R}^n} uv$: A functional inequality along radial trajectories

Now inspired by the strategy from [32], as a key step toward revealing blow-up we shall establish a link between the negative contribution $-\int_{\mathbb{R}^n} uv$ to $\mathcal{F}$ and the dissipation rate functional $\mathcal{D}$ from (3.2). This will be achieved in Lemma 4.5 below, and is to be prepared by four lemmata, each of which has quite a close relative in [32], but in the derivation of each of which we need to adequately account for the unboundedness of the domain on the one hand, and for the differences between $(\mathcal{F}, \mathcal{D})$ and $(\mathcal{F}_0, \mathcal{D}_0)$ from (1.2) and (1.3) on the other. Here and below, whenever convenient we shall without further explicit mentioning switch to the standard notation for functions radially symmetric about the origin, thus writing e.g. $u = u(r,t)$ for $r = |x| \geq 0$.

We begin with a pointwise estimate for v gained upon combining Lemma 2.3 with (2.21) and making essential use of radial symmetry.

Lemma 4.1 *There exists $C > 0$ such that if with some $q > n$, u_0 and v_0 are such that besides (1.7) the condition (1.8) holds, then the solution of (1.6) from Lemma 2.2 satisfies*

$$v(x,t) \leq CK \cdot (1 + |x|^{1-n}) \qquad \text{for all } x \in \mathbb{R}^n \setminus \{0\} \text{ and each } t \in (0, T_{max}), \tag{4.1}$$

where

$$K := \|u_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{L^1(\mathbb{R}^n)} + \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + 1. \tag{4.2}$$

Proof. We first recall that due to Lemma 2.3 and Lemma 2.6 there exists $c_1 > 0$ such that whenever (1.7) and (1.8) hold, with $K \geq 1$ as accordingly defined by (4.2) we have

$$\int_0^\infty r^{n-1} |v_r(r,t)| dr \leq c_1 K \qquad \text{for all } t \in (0, T_{max}) \tag{4.3}$$

and

$$\int_0^\infty r^{n-1} v(r,t) dr \leq K \qquad \text{for all } t \in (0, T_{max}),$$

where the latter especially implies that for fixed (u_0, v_0) and each $t \in (0, T_{max})$ we can pick $r_0(t) \in [1,2]$ fulfilling

$$v(r_0(t),t) \leq r_0^{n-1}(t) v(r_0(t),t) = \int_1^2 r^{n-1} v(r,t) dr \leq K. \tag{4.4}$$

Now fixing any such (u_0, v_0), for $t \in (0, T_{max})$ and $r \in (0, r_0(t)]$ we can use (4.4) together with (4.3) to see that

$$\begin{aligned} v(r,t) &= v(r_0(t),t) - \int_r^{r_0(t)} v_r(\rho,t) d\rho \\ &\leq K + \int_r^{r_0(t)} |v_r(\rho,t)| d\rho \\ &\leq K + r^{1-n} \int_r^{r_0(t)} \rho^{n-1} |v_r(\rho,t)| d\rho \\ &\leq K + c_1 K r^{1-n}, \end{aligned} \tag{4.5}$$

whereas if $t \in (0, T_{max})$ and $r \geq r_0(t)$, then similarly

$$
\begin{aligned}
v(r,t) &\leq K + \int_{r_0(t)}^{r} |v_r(\rho,t)| d\rho \\
&\leq K + r_0^{1-n}(t) \int_{r_0(t)}^{r} \rho^{n-1} |v_r(\rho,t)| d\rho \\
&\leq K + c_1 K r_0^{1-n}(t) \\
&\leq K + c_1 K, \qquad (4.6)
\end{aligned}
$$

because $r_0(t) \geq 1$. In combination, (4.5) and (4.6) yield (4.1). □

By means of the previous lemma, we can next apply a standard testing procedure to the second equation in (1.6), involving one specific among the cut-off functions from (2.12), to relate the integral under consideration to a Dirichlet integral of v, up to a sublinear, and hence favorably small, power of $\mathcal{D}$.

Lemma 4.2 *There exists $C > 0$ such that whenever (1.7) and (1.8) are valid with some $q > n$, for the solution of (1.6) from Lemma 2.2 we have*

$$
\int_{\mathbb{R}^n} uv \leq 3 \int_{B_2} |\nabla v|^2 + CK^2 + CK^{\frac{4}{n+4}} \Big\| \Delta v - v + u \Big\|_{L^2(\mathbb{R}^n)}^{\frac{2n+4}{n+4}} \qquad \textit{for all } t \in (0, T_{max}), \qquad (4.7)
$$

where K is as in (4.2).

Proof. We take $\zeta := \zeta_1$ with ζ_1 as defined in (2.12), and then observe that since Lemma 4.1 provides $c_1 > 0$ fulfilling

$$
v(r,t) \leq c_1 K \qquad \text{for all } r \geq 1 \text{ and } t \in (0, T_{max}), \qquad (4.8)
$$

in the decomposition

$$
\int_{\mathbb{R}^n} uv = \int_{\mathbb{R}^n} \zeta^2 uv + \int_{\mathbb{R}^n} (1 - \zeta^2) uv, \qquad t \in (0, T_{max}), \qquad (4.9)
$$

we may estimate

$$
\int_{\mathbb{R}^n} (1 - \zeta^2) uv \leq \int_{\mathbb{R}^n \setminus B_1} uv \leq c_1 K \int_{\mathbb{R}^n} u = c_1 K \int_{\mathbb{R}^n} u_0 \leq c_1 K^2 \qquad \text{for all } t \in (0, T_{max}) \qquad (4.10)
$$

according to (1.11). To appropriately handle the first integral on the right of (4.9), we write $f := -\Delta v + v - u$ and test this defining identity by $\zeta^2 v$ to see that due to Young's inequality and the Hölder inequality,

$$
\begin{aligned}
\int_{\mathbb{R}^n} \zeta^2 uv &= \int_{\mathbb{R}^n} \zeta^2 |\nabla v|^2 + 2 \int_{\mathbb{R}^n} \zeta v \nabla \zeta \cdot \nabla v + \int_{\mathbb{R}^n} \zeta^2 v^2 - \int_{\mathbb{R}^n} \zeta^2 f v \\
&\leq 2 \int_{\mathbb{R}^n} \zeta^2 |\nabla v|^2 + \int_{\mathbb{R}^n} |\nabla \zeta|^2 v^2 + \int_{\mathbb{R}^n} \zeta^2 v^2 + \|f\|_{L^2(\mathbb{R}^n)} \|\zeta v\|_{L^2(\mathbb{R}^n)} \qquad \text{for all } t \in (0, T_{max}) \qquad (4.11)
\end{aligned}
$$

because $0 \le \zeta \le 1$. Since $\operatorname{supp} \nabla\zeta \subset \overline{B}_2 \setminus B_1$, we may once again rely on (4.8) in estimating

$$\int_{\mathbb{R}^n} |\nabla\zeta|^2 v^2 \le c_2 K^2 \qquad \text{for all } t \in (0, T_{max}) \tag{4.12}$$

with $c_2 := c_1^2 \int_{\mathbb{R}^n} |\nabla\zeta|^2$, while invoking the Gagliardo-Nirenberg inequality and again Young's inequality as well as (4.11) we can find positive constents c_3, c_4, c_5 and c_6 such that

$$\begin{aligned}
\|f\|_{L^2(\mathbb{R}^n)} \|\zeta v\|_{L^2(\mathbb{R}^n)} &\le c_3 \|f\|_{L^2(\mathbb{R}^n)} \|\nabla(\zeta v)\|_{L^2(\mathbb{R}^n)}^{\frac{n}{n+2}} \|\zeta v\|_{L^1(\mathbb{R}^n)}^{\frac{2}{n+2}} \\
&\le \frac{1}{4} \|\nabla(\zeta v)\|_{L^2(\mathbb{R}^n)}^2 + c_4 \|f\|_{L^2(\mathbb{R}^n)}^{\frac{2n+4}{n+4}} \|\zeta v\|_{L^1(\mathbb{R}^n)}^{\frac{4}{n+4}} \\
&= \frac{1}{4} \int_{\mathbb{R}^n} |\zeta\nabla v + v\nabla\zeta|^2 + c_4 \|f\|_{L^2(\mathbb{R}^n)}^{\frac{2n+4}{n+4}} \Big\{ \int_{\mathbb{R}^n} \zeta v \Big\}^{\frac{4}{n+4}} \\
&\le \frac{1}{2} \int_{\mathbb{R}^n} \zeta^2 |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla\zeta|^2 v^2 + c_4 \|f\|_{L^2(\mathbb{R}^n)}^{\frac{2n+4}{n+4}} \Big\{ \int_{\mathbb{R}^n} \zeta v \Big\}^{\frac{4}{n+4}} \\
&\le \frac{1}{2} \int_{\mathbb{R}^n} \zeta^2 |\nabla v|^2 + \frac{1}{2} c_2 K^2 + c_4 K^{\frac{4}{n+2}} \|f\|_{L^2(\mathbb{R}^n)}^{\frac{2n+4}{n+4}} \qquad \text{for all } t \in (0, T_{max}) \qquad (4.13)
\end{aligned}$$

and that, similarly,

$$\begin{aligned}
\int_{\mathbb{R}^n} \zeta^2 v^2 = \|\zeta v\|_{L^2(\mathbb{R}^n)}^2 &\le c_5 \|\nabla(\zeta v)\|_{L^2(\mathbb{R}^n)}^{\frac{2n}{n+2}} \|\zeta v\|_{L^1(\mathbb{R}^n)}^{\frac{4}{n+2}} \\
&\le \frac{1}{4} \|\nabla(\zeta v)\|_{L^2(\mathbb{R}^n)}^2 + c_6 \|\zeta v\|_{L^1(\mathbb{R}^n)}^2 \\
&\le \frac{1}{2} \int_{\mathbb{R}^n} \zeta^2 |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^n} |\nabla\zeta|^2 v^2 + c_6 \Big\{ \int_{\mathbb{R}^n} \zeta v \Big\}^2 \\
&\le \frac{1}{2} \int_{\mathbb{R}^n} \zeta^2 |\nabla v|^2 + \frac{1}{2} c_2 K^2 + c_6 K^2 \qquad \text{for all } t \in (0, T_{max}), \qquad (4.14)
\end{aligned}$$

as due to Lemma 2.6 we have $\int_{\mathbb{R}^n} \zeta v \le \int_{\mathbb{R}^n} v \le K$ for all $t \in (0, T_{max})$.
It thus only remains to insert (4.12) and (4.13) into (4.11) and combine the latter with (4.10) to infer (4.7) from (4.9). $\Box$

To appropriately estimate the crucial contribution $\int_{B_2} |\nabla v|^2$ to the right-hand side of (4.7), we subdivide the ball appearing therein and first concentrate on certain annuli with yet flexible radii. On multiplying the second equation in (1.6) by the positive but sublinear power $v^{\frac{1}{2}}$ of v, by means of another localization using (2.12) we can achieve the following estimate of the corresponding integral against small portions of our original target object, as well as two summands explicitly containing certain negative powers of the respective cutting radius.

Lemma 4.3 *For each $\varepsilon > 0$ one can find $C(\varepsilon) > 0$ with the property that if (1.7) and (1.8) hold with some $q > n$, then for any choice of $r_0 \in (0, 2)$, the solution of (1.6) from Lemma 2.2 satisfies*

$$\int_{B_2 \setminus B_{r_0}} |\nabla v|^2 \le \varepsilon \int_{\mathbb{R}^n} uv + C(\varepsilon) K^2 r_0^{-(n-1)} + C(\varepsilon) K r_0^{-\frac{n-1}{2}} \Big\| \Delta v - v + u \Big\|_{L^2(\mathbb{R}^n)} \qquad \textit{for all } t \in (0, T_{max}), \tag{4.15}$$

where again K is taken from (4.2).

PROOF. We once more abbreviate $f := -\Delta v + v - u$, and we take $\zeta := \zeta_2$ with ζ_2 introduced in (2.12). Then multiplying the equation $-\Delta v = u - v + f$ by $\zeta^2 v^{\frac{1}{2}}$, upon an integration by parts we obtain that thanks to Young's inequality and the nonnegativity of $\zeta^2 v^{\frac{3}{2}}$,

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^n} \zeta^2 v^{-\frac{1}{2}}|\nabla v|^2 &= -2\int_{\mathbb{R}^n} \zeta v^{\frac{1}{2}} \nabla\zeta\cdot\nabla v + \int_{\mathbb{R}^n} \zeta u v^{\frac{1}{2}} - \int_{\mathbb{R}^n} \zeta^2 v^{\frac{3}{2}} + \int_{\mathbb{R}^n} \zeta^2 f v^{\frac{1}{2}} \\ &\le \frac{1}{4}\int_{\mathbb{R}^n} \zeta^2 v^{-\frac{1}{2}}|\nabla v|^2 + 4\int_{\mathbb{R}^n} |\nabla\zeta|^2 v^{\frac{3}{2}} + \int_{\mathbb{R}^n} u v^{\frac{1}{2}} + \int_{\mathbb{R}^n} |f| v^{\frac{1}{2}} \qquad (4.16) \end{aligned}$$

for all $t \in (0, T_{max})$, where since $\operatorname{supp}\nabla\zeta \subset \overline{B}_3\backslash B_2$, Lemma 4.1 says that with some $c_1 > 0$ independent of (u_0, v_0) we have

$$4\int_{\mathbb{R}^n} |\nabla\zeta|^2 v^{\frac{3}{2}} \le c_1 K^{\frac{3}{2}} \qquad \text{for all } t \in (0, T_{max}). \qquad (4.17)$$

Since (4.1) moreover states that with some $c_2 > 0$, for any (u_0, v_0) fulfilling (1.7) and (1.8) we have

$$v(x,t) \le c_2 K r_0^{1-n} \qquad \text{for all } x \in B_2 \setminus B_{r_0} \text{ and } t \in (0, T_{max}),$$

and since thus for any choice of $r_0 \in (0,2)$ the inequality

$$\frac{1}{4}\int_{\mathbb{R}^n} \zeta^2 v^{-\frac{1}{2}}|\nabla v|^2 \ge \frac{1}{4} c_2^{-\frac{1}{2}} K^{-\frac{1}{2}} r_0^{\frac{n-1}{2}} \int_{B_2\backslash B_{r_0}} |\nabla v|^2$$

holds for all $t \in (0, T_{max})$, from (4.16) and (4.17) we infer that for all $t \in (0, T_{max})$ and each $r_0 \in (0,2)$,

$$\int_{B_2\backslash B_{r_0}} |\nabla v|^2 \le 4c_1 c_2^{\frac{1}{2}} K^2 r_0^{\frac{1-n}{2}} + 4c_2^{\frac{1}{2}} K^{\frac{1}{2}} r_0^{\frac{1-n}{2}} \int_{\mathbb{R}^n} u v^{\frac{1}{2}} + 4c_2^{\frac{1}{2}} K^{\frac{1}{2}} r_0^{\frac{1-n}{2}} \int_{\mathbb{R}^n} |f| v^{\frac{1}{2}}. \qquad (4.18)$$

Here given $\varepsilon > 0$ we may again use Young's inequality to see that

$$\begin{aligned} 4c_2^{\frac{1}{2}} K^{\frac{1}{2}} r_0^{\frac{1-n}{2}} \int_{\mathbb{R}^n} u v^{\frac{1}{2}} &\le \varepsilon \int_{\mathbb{R}^n} uv + \frac{4c_2 K r_0^{1-n}}{\varepsilon} \int_{\mathbb{R}^n} u \\ &\le \varepsilon \int_{\mathbb{R}^n} uv + \frac{4c_2}{\varepsilon} K^2 r_0^{1-n} \qquad \text{for all } t \in (0, T_{max}) \text{ and } r_0 \in (0,2), \ (4.19) \end{aligned}$$

whereas employing the Cauchy-Schwarz inequality along with Lemma 2.6 we find that

$$\begin{aligned} 4c_2^{\frac{1}{2}} K^{\frac{1}{2}} r_0^{\frac{1-n}{2}} \int_{\mathbb{R}^n} |f| v^{\frac{1}{2}} &\le 4c_2^{\frac{1}{2}} K^{\frac{1}{2}} r_0^{\frac{1-n}{2}} \|f\|_{L^2(\mathbb{R}^n)} \|v\|_{L^1(\mathbb{R}^n)}^{\frac{1}{2}} \\ &\le 4c_2^{\frac{1}{2}} K r_0^{\frac{1-n}{2}} \|f\|_{L^2(\mathbb{R}^n)} \qquad \text{for all } t \in (0, T_{max}) \text{ and } r_0 \in (0,2). \end{aligned}$$

Together with (4.19), this shows that (4.18) entails the claimed inequality, because $r_0^{\frac{1-n}{2}} \le 2^{\frac{n-1}{2}} r_0^{-(n-1)}$ whenever $r_0 \in (0,2)$. □

We finally follow an idea from [32, Lemma 4.4] in deriving an inequality for the associated inner Dirichlet integral in terms of, essentially, the product of $\mathcal{D}$ with a factor that contains a positive power of the dividing radius, and hence can be enforced to become conveniently small.

Lemma 4.4 *There exists $C>0$ such that if (1.7) and (1.8) are satisfied with some $q>n$, and if $r_0\in(0,2)$, then taking (u,v) from Lemma 2.2 and K from (4.2) we have*

$$\int_{B_{r_0}} |\nabla v|^2 \le CK + Cr_0 \Big\|\Delta v - v + u\Big\|_{L^2(\mathbb{R}^n)}^2 + C\sqrt{K}\Big\|\frac{\nabla u}{\sqrt{u+1}} - \sqrt{u+1}\nabla v\Big\|_{L^2(\mathbb{R}^n)} + C\int_{B_2} v^2 \quad (4.20)$$

for all $t\in(0,T_{max})$.

PROOF. Letting $f\equiv f(r,t) := -\Delta v + v - u$ and $g\equiv g(r,t) := \frac{u_r}{\sqrt{u+1}} - \sqrt{u+1}v_r$ for $r\ge 0$ and $t\in(0,T_{max})$, following [32] we multiply the identity $r^{1-n}(r^{n-1}v_r)_r = -u+v-f$ by $2r^{2n-2}v_r$ and apply Young's inequality to see that since $uv_r = u_r - v_r - \sqrt{u+1}g$,

$$\begin{aligned}
\partial_r(r^{2n-2}v_r^2) &= -2r^{2n-2}uv_r + 2r^{2n-2}vv_r - 2r^{2n-2}fv_r \\
&= -2r^{2n-2}u_r + 2r^{2n-2}v_r + 2r^{2n-2}\sqrt{u+1}g + 2r^{2n-2}vv_r - 2r^{2n-2}fv_r \\
&\le -2r^{2n-2}u_r + 2r^{2n-2}v_r + 2r^{2n-2}\sqrt{u+1}g + 2r^{2n-2}vv_r + (n-1)r^{2n-2}v_r^2 + \frac{1}{n-1}r^{2n-2}f^2
\end{aligned}$$

for all $r>0$ and $t\in(0,T_{max})$, Thus, by integration,

$$\begin{aligned}
r^{2n-2}v_r^2(r,t) \le \int_0^r e^{(n-1)(r-\rho)}\cdot\Big\{ &-2\rho^{2n-2}u_r(\rho,t) + 2\rho^{2n-2}v_r(\rho,t) + 2\rho^{2n-2}\sqrt{u(\rho,t)+1}g(\rho,t) \\
&+2\rho^{2n-2}v(\rho,t)v_r(\rho,t) + \frac{1}{n-1}\rho^{2n-2}f^2(\rho,t)\Big\}d\rho \qquad (4.21)
\end{aligned}$$

for all $r>0$ and $t\in(0,T_{max})$, where three integrations by parts show that if we restrict our considerations to the range $0<r<2$, then for any such r and $t\in(0,T_{max})$,

$$\begin{aligned}
&\int_0^r e^{(n-1)(r-\rho)}\cdot\Big\{-2\rho^{2n-2}u_r(\rho,t) + 2\rho^{2n-2}v_r(\rho,t) + 2\rho^{2n-2}v(\rho,t)v_r(\rho,t)\Big\}d\rho \\
&= e^{(n-1)(r-\rho)}\cdot\Big\{-2\rho^{2n-2}u(\rho,t) + 2\rho^{2n-2}v(\rho,t) + \rho^{2n-2}v^2(\rho,t)\Big\}\Big|_{\rho=0}^{\rho=r} \\
&\quad + \int_0^r \partial_\rho\Big\{e^{(n-1)(r-\rho)}\rho^{2n-2}\Big\}\cdot\Big\{2u(\rho,t) - 2v(\rho,t) - v^2(\rho,t)\Big\}d\rho \\
&= -2r^{2n-2}u(r,t) + 2r^{2n-2}v(r,t) + r^{2n-2}v^2(r,t) \\
&\quad +(n-1)\int_0^r e^{(n-1)(r-\rho)}\rho^{2n-3}(2-\rho)\cdot\Big\{2u(\rho,t) - 2v(\rho,t) - v^2(\rho,t)\Big\}d\rho \\
&\le 2r^{2n-2}v(r,t) + r^{2n-2}v^2(r,t) + 2(n-1)\int_0^r e^{(n-1)(r-\rho)}\rho^{2n-3}(2-\rho)u(\rho,t)d\rho \\
&\le 2r^{2n-2}v(r,t) + r^{2n-2}v^2(r,t) + 4(n-1)e^{2(n-1)}\int_0^r \rho^{2n-3}u(\rho,t)d\rho. \qquad (4.22)
\end{aligned}$$

Since within this range of r we may furthermore use the Cauchy-Schwarz inequality along with (1.11) to find $c_1>0$ such that

$$2\int_0^r e^{(n-1)(r-\rho)}\rho^{2n-2}\sqrt{u(\rho,t)+1}g(\rho,t)d\rho$$

$$
\begin{aligned}
&\le 2e^{2(n-1)} r^{n-1} \int_0^r \rho^{n-1} \sqrt{u(\rho,t)+1}\,|g(\rho,t)|d\rho \\
&\le 2e^{2(n-1)} r^{n-1} \Big\{ \int_0^r \rho^{n-1} \Big(u(\rho,t)+1\Big) d\rho \Big\}^{\frac{1}{2}} \Big\{ \int_0^r \rho^{n-1} g^2(\rho,t) d\rho \Big\}^{\frac{1}{2}} \\
&\le c_1\sqrt{K} r^{n-1} \Big\{ \int_0^\infty \rho^{n-1} g^2(\rho,t) d\rho \Big\}^{\frac{1}{2}} \qquad \text{for all } r\in(0,2) \text{ and } t\in(0,T_{max}),
\end{aligned}
$$

and since clearly

$$
\frac{1}{n-1}\int_0^r e^{(n-1)(r-\rho)} \rho^{2n-2} f^2(\rho,t)d\rho \le \frac{e^{2(n-1)}}{n-1} r^{n-1} \int_0^\infty \rho^{n-1} f^2(\rho,t)d\rho
$$

for all $r\in(0,2)$ and $t\in(0,T_{max})$, from (4.21) and (4.22) we conclude upon another integration that whenever $r_0\in(0,2)$,

$$
\begin{aligned}
\int_0^{r_0} r^{n-1} v_r^2(r,t)dr \le\ & 2\int_0^{r_0} r^{n-1} v(r,t)dr + \int_0^{r_0} r^{n-1} v^2(r,t)dr \\
&+4(n-1)e^{2(n-1)} \int_0^{r_0} r^{1-n} \int_0^r \rho^{2n-3} u(\rho,t)d\rho dr \\
&+c_1\sqrt{K} r_0 \Big\{ \int_0^\infty \rho^{n-1} g^2(\rho,t)d\rho \Big\}^{\frac{1}{2}} \\
&+\frac{e^{2(n-1)}}{n-1} r_0 \int_0^\infty \rho^{n-1} f^2(\rho,t)d\rho \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}
\tag{4.23}
$$

Here by the Fubini theorem and our overall assumption that $n\ge 3$,

$$
\begin{aligned}
&4(n-1)e^{2(n-1)} \int_0^{r_0} r^{1-n} \int_0^r \rho^{2n-3} u(\rho,t)d\rho dr \\
&\quad = 4(n-1)e^{2(n-1)} \int_0^{r_0} \Big\{ \int_\rho^{r_0} r^{1-n} dr \Big\} \cdot \rho^{2n-3} u(\rho,t)d\rho \\
&\quad = \frac{4(n-1)e^{2(n-1)}}{n-2} \int_0^{r_0} (\rho^{2-n} - r_0^{2-n}) \rho^{2n-3} u(\rho,t)d\rho \\
&\quad \le \frac{4(n-1)e^{2(n-1)}}{n-2} \int_0^{r_0} \rho^{n-1} u(\rho,t)d\rho \qquad \text{for all } t\in(0,T_{max}) \text{ and any } r_0\in(0,2),
\end{aligned}
$$

so that recalling (2.21) and (1.11) we obtain that with some $c_2>0$,

$$
\begin{aligned}
&2\int_0^{r_0} r^{n-1} v(r,t)dr + 4(n-1)e^{2(n-1)} \int_0^{r_0} r^{1-n} \int_0^r \rho^{2n-3} u(\rho,t)d\rho dr \\
&\qquad \le c_2 K \qquad \text{for all } t\in(0,T_{max}) \text{ and each } r_0\in(0,2).
\end{aligned}
$$

In view of the definitions of f and g, (4.20) therefore immediately results from (4.23) upon trivially estimating $r_0\le 2$ in the second last summand therein. $\square$

Indeed, appropriate choices of ε and r_0 in the above preparations enable us to bound $\int_{\mathbb{R}^n} uv$ by a sublinear power of D and a lower order expression:

Lemma 4.5 *There exist $\theta \in (\frac{1}{2}, 1)$ and $C > 0$ such that if (1.7) and (1.8) hold with some $q > n$, then the solution of (1.6) from Lemma 2.2 has the property that*

$$\int_{\mathbb{R}^n} uv \le CK^2 \cdot \left\{ \left\| \Delta v - v + u \right\|_{L^2(\mathbb{R}^n)}^{2\theta} + \left\| \frac{\nabla u}{\sqrt{u+1}} - \sqrt{u+1}\nabla v \right\|_{L^2(\mathbb{R}^n)} + 1 \right\} \qquad \text{for all } t \in (0, T_{max}), \tag{4.24}$$

where again K is as in (4.2).

Proof. Fixing any $\alpha \in (0, \frac{2}{n-1})$, given (u_0, v_0) such that (1.7) and (1.8) hold we define $f := -\Delta v + v - u$ and $g := \frac{\nabla u}{\sqrt{u+1}} - \sqrt{u+1}\nabla v$ and let

$$r_0 \equiv r_0(t) := \min\left\{1\,,\, \|f(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{-\alpha}\right\}, \qquad t \in (0, T_{max}). \tag{4.25}$$

Then application of Lemma 4.2, Lemma 4.3 and Lemma 4.4 to these values of r_0 and to $\varepsilon := \frac{1}{12}$ provide positive constants c_1, c_2 and c_3 independent of (u_0, v_0) such that

$$\int_{\mathbb{R}^n} uv \le 3\int_{B_2} |\nabla v|^2 + c_1 K^2 + c_1 K^{\frac{4}{n+4}} \|f\|_{L^2(\mathbb{R}^n)}^{\frac{2n+4}{n+4}} \qquad \text{for all } t \in (0, T_{max}) \tag{4.26}$$

and

$$\int_{B_2 \setminus B_{r_0}} |\nabla v|^2 \le \frac{1}{12}\int_{\mathbb{R}^n} uv + c_2 K^2 r_0^{-(n-1)} + c_2 K r_0^{-\frac{n-1}{2}} \|f\|_{L^2(\mathbb{R}^n)} \qquad \text{for all } t \in (0, T_{max}) \tag{4.27}$$

as well as

$$\int_{B_{r_0}} |\nabla v|^2 \le c_3 K + c_3 r_0 \|f\|_{L^2(\mathbb{R}^n)}^2 + c_3 K^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^n)} + c_3 \int_{B_2} v^2 \qquad \text{for all } t \in (0, T_{max}). \tag{4.28}$$

Here by compactness of the embedding $W^{1,2}(B_2) \hookrightarrow L^2(B_2)$, an associated Ehrling-type lemma in conjunction with Lemma 2.6 shows that with some $c_4 > 0$, again independent of u_0 and v_0, we have

$$c_3 \int_{B_2} v^2 \le \frac{1}{2}\int_{B_2} |\nabla v|^2 + c_4 \left\{ \int_{B_2} v \right\}^2 \le \frac{1}{2}\int_{B_2} |\nabla v|^2 + c_4 K^2 \qquad \text{for all } t \in (0, T_{max}),$$

whence combining (4.27) with (4.28) firstly shows that

$$\begin{aligned} \frac{1}{2}\int_{B_2} |\nabla v|^2 \;\; &\le \;\; \frac{1}{12}\int_{\mathbb{R}^n} uv + c_2 K^2 r_0^{-(n-1)} + c_2 K r_0^{-\frac{n-1}{2}} \|f\|_{L^2(\mathbb{R}^n)} \\ &\quad + c_3 K + c_3 r_0 \|f\|_{L^2(\mathbb{R}^n)}^2 + c_3 K^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^n)} + c_4 K^2 \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

and therefore, secondly, implies that due to (4.26),

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^n} uv \;\; &\le \;\; 6c_2 K^2 r_0^{-(n-1)} + 6c_2 K r_0^{-\frac{n-1}{2}} \|f\|_{L^2(\mathbb{R}^n)} + 6c_3 K + 6c_3 r_0 \|f\|_{L^2(\mathbb{R}^n)}^2 + 6c_3 K^{\frac{1}{2}} \|g\|_{L^2(\mathbb{R}^n)} \\ &\quad + 6c_4 K^2 + c_1 K^2 + c_1 K^{\frac{4}{n+4}} \|f\|_{L^2(\mathbb{R}^n)}^{\frac{2n+4}{n+4}} \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{4.29}$$

Now if $t \in (0, T_{max})$ is such that $\|f(\cdot,t)\|_{L^2(\mathbb{R}^n)} > 1$, then (4.25) means that $r_0(t) = \|f(\cdot,t)\|_{L^2(\mathbb{R}^n)}^{-\alpha}$ and that thus, by (4.29), writing $\theta := \max\{\frac{(n-1)\alpha}{2}, \frac{1}{2} + \frac{(n-1)\alpha}{4}, \frac{2-\alpha}{2}, \frac{n+2}{n+4}\}$ we have

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^n} uv &\le 6c_2K^2\|f\|_{L^2(\mathbb{R}^n)}^{\frac{(n-1)\alpha}{2}} + 6c_2K\|f\|_{L^2(\mathbb{R}^n)}^{1+\frac{(n-1)\alpha}{2}} + 6c_3K + 6c_3\|f\|_{L^2(\mathbb{R}^n)}^{2-\alpha} + 6c_3K^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^n)} \\
&\quad + 6c_4K^2 + c_1K^2 + c_1K^{\frac{4}{n+4}}\|f\|_{L^2(\mathbb{R}^n)}^{\frac{2n+4}{n+4}} \\
&\le (6c_2K^2 + 6c_2K + 6c_3 + c_1K^{\frac{4}{n+4}}) \cdot (\|f\|_{L^2(\mathbb{R}^n)}^{2\theta} + 1) + 6c_3K^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^n)} \\
&\quad + 6c_3K + 6c_4K^2 + c_1K^2 \\
&\le (12c_2 + 6c_3 + c_1)K^2 \cdot (\|f\|_{L^2(\mathbb{R}^n)}^{2\theta} + 1) + 6c_3K^2\|g\|_{L^2(\mathbb{R}^n)} + (6c_3 + 6c_4 + c_1)K^2 \qquad (4.30)
\end{aligned}$$

according to Young's inequality and the fact that $K \ge 1$. If, conversely, $t \in (0, T_{max})$ is such that $\|f\|_{L^2(\mathbb{R}^n)} \le 1$, then (4.29) directly entails that again since $K \ge 1$,

$$\begin{aligned}
\frac{1}{2}\int_{\mathbb{R}^n} uv &\le 6c_2K^2 + 6c_2K\|f\|_{L^2(\mathbb{R}^n)} + 6c_3K + 6c_3\|f\|_{L^2(\mathbb{R}^n)} + 6c_3K^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^n)} \\
&\quad + 6c_4K^2 + c_1K^2 + c_1K^{\frac{4}{n+4}}\|f\|_{L^2(\mathbb{R}^n)}^{\frac{2n+4}{n+4}} \\
&\le 6c_2K^2 + 6c_2K + 6c_3K + 6c_3 + 6c_3K^{\frac{1}{2}}\|g\|_{L^2(\mathbb{R}^n)} + 6c_4K^2 + c_1K^2 + c_1K^{\frac{4}{n+4}} \\
&\le 6c_3K^2\|g\|_{L^2(\mathbb{R}^n)} + (12c_2 + 12c_3 + 6c_4 + 2c_1)K^2. \qquad (4.31)
\end{aligned}$$

Upon an evident defintion of C, a combination of (4.30) and (4.31) yields (4.24), with the exponent θ indeed fulfilling $\theta \in (\frac{1}{2}, 1)$ due to the inequalities $\frac{(n-1)\alpha}{2} < 1$ and $\frac{1}{2} + \frac{(n-1)\alpha}{4} < 1$ ensured by our initial restriction on α. □

5 Blow-up of low-energy radial solutions

As a last ingredient for our analysis of the inequaliy (3.3), let us add a Gronwall-type statement on blow-up in an integral inequality that can be viewed as a counterpart of a superlinearly forced differential inequality.

Lemma 5.1 *Suppose that $a > 0, b > 0$ and $\beta > 1$, and that for some $T > 0$, a nonnegative function $y \in C^0([0,T))$ satisfies*

$$y(t) \ge a + b\int_0^t y^\beta(s)ds \qquad \text{for all } t \in (0,T). \qquad (5.1)$$

Then

$$T \le \frac{1}{(\beta-1)a^{\beta-1}b}. \qquad (5.2)$$

Proof. For $\varepsilon \in (0,a)$, the function $y_\varepsilon \in C^1([0,T_\varepsilon))$ defined by

$$y_\varepsilon(t) := \Big\{(a-\varepsilon)^{1-\beta} - (\beta-1)bt\Big\}^{-\frac{1}{\beta-1}}, \qquad 0 \le t < T_\varepsilon := \frac{1}{(\beta-1)(a-\varepsilon)^{\beta-1}b}, \qquad (5.3)$$

satisfies $y_\varepsilon' = by_\varepsilon^\beta$ on $(0,T_\varepsilon)$ and $y_\varepsilon(0)=a-\varepsilon$, so that

$$y_\varepsilon(t) = a-\varepsilon + b\int_0^t y_\varepsilon^\beta(s)ds \qquad \text{for all } t\in[0,T_\varepsilon). \tag{5.4}$$

Now if (5.2) was false, then since $T_\varepsilon \searrow T_0 := \frac{1}{(\beta-1)a^{\beta-1}b}$ as $\varepsilon\searrow 0$, it would be possible to find $\varepsilon_0\in(0,a)$ such that for any $\varepsilon\in(0,\varepsilon_0)$, y would belong to $C^0([0,T_\varepsilon])$ and the number $t_\varepsilon := \sup\{\tilde t\in (0,T_\varepsilon) \mid y>y_\varepsilon \text{ on } [0,\tilde t]\}$ would be well-defined, because $y(0)\ge a>a-\varepsilon=y_\varepsilon(0)$. To see that actually $t_\varepsilon = T_\varepsilon$ for any such ε, assuming on the contrary that $t_\varepsilon\in(0,T_\varepsilon)$ we could use the continuity of y and y_ε to infer that $y>y_\varepsilon$ on $(0,t_\varepsilon)$ but $y(t_\varepsilon)=y_\varepsilon(t_\varepsilon)$, by monotonicity of $0<\sigma\mapsto\sigma^\beta$ implying that

$$y_\varepsilon(t_\varepsilon) = y(t_\varepsilon) \ge a + b\int_0^{t_\varepsilon} y^\beta(s)ds \ge a + b\int_0^{t_\varepsilon} y_\varepsilon^\beta(s)ds > a-\varepsilon+b\int_0^{t_\varepsilon} y_\varepsilon^\beta(s)ds = y_\varepsilon(t_\varepsilon)$$

according to (5.1) and (5.4). As thus indeed $y>y_\varepsilon$ throughout $[0,T_\varepsilon)$ for all $\varepsilon\in(0,\varepsilon_0)$, from the observation that $y_\varepsilon(T_0) = \{(a-\varepsilon)^{1-\beta} - a^{1-\beta}\}^{-\frac{1}{\beta-1}} \to +\infty$ as $\varepsilon\searrow 0$ it follows that y could not be bounded on $[0,T_0]$, in contradiction to our hypothesis on T. $\square$

We are thereby prepared to combine Lemma 3.1 with Lemma 4.5 in order to reveal a criterion on radial initial data as sufficient for finite-time blow-up:

Lemma 5.2 *There exist $M>0$ and $\gamma>0$ with the property that if for some $q>n$, u_0 and v_0 comply with (1.7) and (1.8) and are such that the corresponding solution of (1.6) satisfies*

$$\mathcal{F}(0) \le -M\cdot\Big\{\|u_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{L^1(\mathbb{R}^n)} + \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + 1\Big\}^\gamma, \tag{5.5}$$

with $\mathcal{F}$ taken from (3.1), then

$$T_{max} \le 1. \tag{5.6}$$

Proof. According to Lemma 4.5, we can find $\theta\in(\frac{1}{2},1)$ and $c_1>0$ such that whenever (1.7) and (1.8) hold, taking $\mathcal{D}$ and K as defined in (3.2) and (4.2) we have

$$\int_{\mathbb{R}^n} uv \le c_1K^2(\mathcal{D}^\theta(t)+1) \qquad \text{for all } t\in(0,T_{max}). \tag{5.7}$$

Using that $\theta<1$, we may employ Young's inequality here to find $c_2>0$ such that for any such solution we moreover have

$$\int_{\mathbb{R}^n} uv \le \frac{1}{8}\mathcal{D}(t) + c_2K^{\frac{2}{1-\theta}} \qquad \text{for all } t\in(0,T_{max}), \tag{5.8}$$

whereupon we abbreviate

$$c_3 := 2^{-\frac{\theta+1}{\theta}}c_1^{-\frac{1}{\theta}} \tag{5.9}$$

and fix $c_4>0$ large enough fulfilling both

$$c_4 > 2c_1+4c_2 \tag{5.10}$$

and

$$c_4 \ge 4c_2 + \Big(\frac{2\theta}{(1-\theta)c_3}\Big)^{\frac{\theta}{1-\theta}}. \tag{5.11}$$

Then assuming (1.7) and (1.8) to hold for some (u_0, v_0) which is such that for the corresponding solution we have

$$\mathcal{F}(0) \le -c_4 K^{\frac{2}{1-\theta}} \tag{5.12}$$

with K as accordingly defined through (4.2), we claim that necessarily (5.6) must be valid. To verify this, we suppose for contradiction that $T_{max} > 1$, and then first observe that by the continuity property of $\mathcal{F}$ asserted by Lemma 3.1,

$$t_0 := \sup \Big\{ \widetilde{t} \in (0,1) \ \Big| \ \mathcal{F}(t) < -2c_1 K^2 \text{ for all } t \in (0, \widetilde{t}) \Big\}$$

would be a well-defined element of $(0,1]$, because since $\frac{2}{1-\theta} > 2$ and $K \ge 1$, from (5.12) and (5.10) we especially know that $\mathcal{F}(0) < -(2c_1 + 4c_2) K^{\frac{2}{1-\theta}} < -2c_1 K^2$. To see that actually

$$t_0 = 1, \tag{5.13}$$

we note that the converse assumption $t_0 < 1$ implies that, again by continuity of $\mathcal{F}$,

$$\mathcal{F}(t) < -2c_1 K^2 \quad \text{for all } t \in (0, t_0) \qquad \text{and} \qquad \mathcal{F}(t_0) = -2c_1 K^2. \tag{5.14}$$

In view of Lemma 3.1 and (5.8), this particularly entails that

$$\begin{aligned}
\mathcal{F}(t) &\le -\int_0^t \mathcal{D}(s) ds + \mathcal{F}(0) + 4 \int_0^t \mathcal{F}(s) ds + 4 \int_0^t \int_{\mathbb{R}^n} uv \\
&\le -\int_0^t \mathcal{D}(s) ds + \mathcal{F}(0) + 4 \int_0^t \Big\{ \frac{1}{8} \mathcal{D}(s) + c_2 K^{\frac{2}{1-\theta}} \Big\} ds \\
&\le -\frac{1}{2} \int_0^t \mathcal{D}(s) ds + \mathcal{F}(0) + 4c_2 K^{\frac{2}{1-\theta}} \qquad \text{for all } t \in (0, t_0),
\end{aligned} \tag{5.15}$$

because $t_0 \le 1$. Therefore, by (5.14), the nonnegativity of $\mathcal{D}$, (5.12) and (5.10),

$$-2c_1 K^2 = \mathcal{F}(t_0) \le \mathcal{F}(0) + 4c_2 K^{\frac{2}{1-\theta}} \le (-c_4 + 4c_2) K^{\frac{2}{1-\theta}} < -2c_1 K^{\frac{2}{1-\theta}} \le -2c_1 K^2,$$

which is absurd and hence confirms that indeed $t_0 = 1$. But since, on the other hand, by definition (3.1) of $\mathcal{F}$ and (5.7) we have

$$\mathcal{F}(t) \ge -\int_{\mathbb{R}^n} uv \ge -c_1 K^2 (\mathcal{D}^\theta(t) + 1) \qquad \text{for all } t \in (0, T_{max})$$

and hence

$$\mathcal{D}^\theta(t) \ge \frac{-\mathcal{F}(t)}{c_1 K^2} - 1 \ge \frac{-\mathcal{F}(t)}{2c_1 K^2} \qquad \text{for all } t \in (0, t_0)$$

according to the fact that $1 \le \frac{-\mathcal{F}(t)}{2c_1 K^2}$ for all $t \in (0, t_0)$, from (5.15) we infer that

$$\begin{aligned}
-\mathcal{F}(t) &\ge \frac{1}{2} \int_0^t \Big(\frac{-\mathcal{F}(t)}{2c_1 K^2} \Big)^{\frac{1}{\theta}} ds - \mathcal{F}(0) - 4c_2 K^{\frac{2}{1-\theta}} \\
&= c_3 K^{-\frac{2}{\theta}} \int_0^t (-\mathcal{F}(s))^{\frac{1}{\theta}} ds - \mathcal{F}(0) - 4c_2 K^{\frac{2}{1-\theta}} \qquad \text{for all } t \in (0, t_0).
\end{aligned} \tag{5.16}$$

As, by (5.11),

$$-\mathcal{F}(0) - 4c_2K^{\frac{2}{1-\theta}} \ge (c_4 - 4c_2)K^{\frac{2}{1-\theta}} \ge \Big(\frac{2\theta}{(1-\theta)c_3}\Big)^{\frac{\theta}{1-\theta}} K^{\frac{2}{1-\theta}},$$

however, we may invoke Lemma 5.1 to conclude from (5.16) that

$$t_0 \le \frac{1}{\Big(\frac{1}{\theta}-1\Big)\cdot \Big\{\Big(\frac{2\theta}{(1-\theta)c_3}\Big)^{\frac{\theta}{1-\theta}} K^{\frac{2}{1-\theta}}\Big\}^{\frac{1}{\theta}-1} \cdot c_3 K^{-\frac{2}{\theta}}} = \frac{1}{2},$$

which is incompatible with (5.13) and thus shows that in fact our hypothesis that $T_{max} > 1$ must have been wrong, and that hence the claimed implication holds with $M := c_4$ and $\gamma := \frac{2}{1-\theta}$. □

6 A density property of blow-up enforcing radial data

In order to complete our argument ensuring blow-up within large sets of initial data, we now only need to resort to a known and essentially explicit construction of explosion-enforcing initial data in arbitrarily small neighbourhoods of any prescribed pair of positive functions fulfilling (1.7) and (1.8).

Lemma 6.1 *Assume that with some $q > n$, u_0 and v_0 satisfy (1.7) and (1.8) with $u_0 > 0$ and $v_0 > 0$ in $\mathbb{R}^n$. Then there exist radially symmetric positive functions $u_{0j} \in BUC(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $v_{0j} \in W^{1,q}(\mathbb{R}^n) \cap W^{1,1}(\mathbb{R}^n)$, $j \in \mathbb{N}$, such that (1.13) holds, and that for the corresponding solutions (u_j, v_j) of (1.6), maximally extended up to $T_{max,j} \in (0,\infty]$ according to Proposition 1.1, we have $T_{max,j} \le 1$ for all $j \in \mathbb{N}$.*

Proof. Following the construction from [32, Lemma 6.1], we take any $(r_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $r_j \searrow 0$ as $j \to \infty$, and use that $\int_0^1 \rho^{n-1}(\rho^2+\varepsilon)^{-\frac{n}{2}}d\rho \nearrow +\infty$ as $\varepsilon \searrow 0$ to fix $(\eta_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $r_j^n \int_0^1 \rho^{n-1}(\rho^2 + \frac{\eta_j}{r_j^2})^{-\frac{n}{2}}d\rho \ge j$ for all $j \in \mathbb{N}$. Next, picking any $\kappa \in (n - \frac{n}{p}, \frac{n-2}{2})$, given positive functions u_0 and v_0 fulfilling (1.7) and (1.8) we let $(u_{0j}, v_{0j}) \equiv (u_{0j}(r), v_{0j}(r))$ be defined by

$$u_{0j}(r) := \begin{cases} a_j(r^2+\eta_j)^{-\frac{n-\kappa}{2}}, & r\in[0,r_j],\\ u_0(r), & r>r_j,\end{cases} \qquad \text{and} \qquad v_{0j}(r) := \begin{cases} b_j(r^2+\eta_j)^{-\frac{\kappa}{2}}, & r\in[0,r_j],\\ v_0(r), & r>r_j,\end{cases} \tag{6.1}$$

with $a_j := (r_j^2+\eta_j)^{\frac{n-\kappa}{2}}u_0(r_j)$ and $b_j := (r_j^2+\eta_j)^{\frac{\kappa}{2}}v_0(r_j)$ for $j \in \mathbb{N}$. Then clearly u_{0j} and v_{0j} have the claimed regularity properties, and the argument in [32, Lemma 6.1] precisely shows that

$$u_{0j} \to u_0 \quad \text{in } L^p(B_1) \quad \text{and} \quad v_{0j} \to v_0 \quad \text{in } W^{1,2}(B_1) \qquad \text{as } j \to \infty, \tag{6.2}$$

that moreover

$$\sup_{j\in\mathbb{N}} \Big\{ \frac{1}{2}\int_{B_1} |\nabla v_{0j}|^2 + \frac{1}{2}\int_{B_1} v_{0j}^2 + \int_{B_1} u_{0j}\ln u_{0j} \Big\} < \infty, \tag{6.3}$$

and that

$$\int_{B_1} u_{0j}v_{0j} \to \infty \qquad \text{as } j \to \infty. \tag{6.4}$$

Now in view of the identities $u_{0j} \equiv u_0$ and $v_{0j} \equiv v_0$ in $\mathbb{R}^n \setminus B_1$, (6.2) immediately implies (1.13) and thus furthermore especially entails that

$$\begin{aligned} K_j &:= \|u_{0j}\|_{L^1(\mathbb{R}^n)} + \|v_{0j}\|_{L^1(\mathbb{R}^n)} + \|\nabla v_{0j}\|_{L^1(\mathbb{R}^n)} + 1 \\ &\to \|u_0\|_{L^1(\mathbb{R}^n)} + \|v_0\|_{L^1(\mathbb{R}^n)} + \|\nabla v_0\|_{L^1(\mathbb{R}^n)} + 1 =: K \qquad \text{as } j \to \infty. \end{aligned} \tag{6.5}$$

Apart from that, the validity of $\ln(1+\sigma) \le \sigma$ for all $\sigma \ge 0$ implies that

$$\begin{aligned} \int_{B_1} u_{0j} \ln(u_{0j}+1) &= \int_{B_1} u_{0j} \ln u_{0j} + \int_{B_1} u_{0j} \ln\Big(1 + \frac{1}{u_{0j}}\Big) \\ &\le \int_{B_1} u_{0j} \ln u_{0j} + |B_1| \qquad \text{for all } j \in \mathbb{N}, \end{aligned}$$

whence (6.3) along with (6.4) and, again, (6.1) ensures that for the corresponding solutions (u_j, v_j) of (1.6) we have

$$\mathcal{F}_j(0) \to -\infty \qquad \text{as } j \to \infty, \tag{6.6}$$

where $\mathcal{F}_j(t) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla v_j(\cdot,t)|^2 + \frac{1}{2}\int_{\mathbb{R}^n} v_j^2(\cdot,t) - \int_{\mathbb{R}^n} u_j(\cdot,t)v_j(\cdot,t) + \int_{\mathbb{R}^n} u_j(\cdot,t)\ln(u_j(\cdot,t)+1)$, $t \in [0, T_{max,j})$. Therefore, if we take $M>0$ and $\gamma>0$ as provided by Lemma 5.2, then according to the convergence statements in (6.5) and (6.6) we can pick $j_0 \in \mathbb{N}$ large enough such that $\mathcal{F}_j(0) \le -MK_j^\gamma$ for all $j \ge j_0$, whence upon replacing j with $j - j_0$ if necessary we can achieve the claimed conclusion as a consequence of (5.6). $\Box$

7 Localization of blow-up points: Bounds for u outside the origin

Our final objective consists in establishing the inequality (1.15) for a radial solution that is already known to blow up within finite time. Although our final result in this direction will essentially parallel knowledge on behavior in Neumann problems for (1.1) on planar disks ([24]), its derivation here will need to considerably deviate from that in the corresponding precedent, inter alia due to unboundedness of the physical domain. To accomplish uniform bounds outside the origin through several steps on the basis of arguments from parabolic regularity theory, for localization procedures different from those previously performed let us choose a family $(\overline{\chi}_\delta)_{\delta\in(0,1)}$ of cut-off functions $\overline{\chi}_\delta \in C_0^\infty(\mathbb{R})$ fulfilling $0 \le \overline{\chi}_\delta \le 1$ on $\mathbb{R}$ as well as $\overline{\chi}_\delta \equiv 1$ on $[0,1]$ and $\operatorname{supp}\overline{\chi}_\delta \subset (-\frac{\delta}{2}, 2)$ for all $\delta \in (0,1)$, and let

$$\chi_{\delta R}(r) := \overline{\chi}_\delta(r - R), \qquad r \in \mathbb{R}, \tag{7.1}$$

for $\delta \in (0,1)$ and $R > \delta$. Then

$$\chi_{\delta R} \equiv 1 \quad \text{on } [R, R+1] \qquad \text{and} \qquad \operatorname{supp}\chi_{\delta R} \subset \Big(R - \frac{\delta}{2}, R+2\Big) \quad \text{for all } \delta \in (0,1) \text{ and } R > \delta, \tag{7.2}$$

and furthermore

$$\sup_{R>\delta} \Big\{ \|\chi'_{\delta R}\|_{L^\infty(\mathbb{R})} + \|\chi''_{\delta R}\|_{L^\infty(\mathbb{R})} \Big\} < \infty \qquad \text{for each fixed } \delta \in (0,1). \tag{7.3}$$

Assuming (1.7) and (1.8), we next observe that with $\chi \equiv \chi_{\delta R}$ we have

$$(\chi u)_t = (\chi u)_{rr} + \Big(a_1(r,t)u\Big)_r + a_2(r,t)u, \qquad r > 0,\ t \in (0, T_{max}), \tag{7.4}$$

and

$$(\chi v)_t = (\chi v)_{rr} + b(r,t), \qquad r>0,\ t\in(0,T_{max}), \tag{7.5}$$

where

$$a_1(r,t) := \frac{n-1}{r}\chi - 2\chi_r - \chi v_r, \qquad r>0,\ t\in(0,T_{max}), \tag{7.6}$$

and

$$a_2(r,t) := -(n-1)\cdot\Big(\frac{\chi}{r}\Big)_r + \chi_{rr} + \Big(\chi_r - \frac{n-1}{r}\chi\Big)v_r, \qquad r>0,\ t\in(0,T_{max}), \tag{7.7}$$

as well as

$$b(r,t) := -2\chi_r v_r + \frac{n-1}{r}\chi v_r - \chi_{rr}v - \chi v + \chi u, \qquad r>0,\ t\in(0,T_{max}). \tag{7.8}$$

Then Lemma 2.3 and (1.11) entail some temporally uniform L^1 control over b in intervals not touching the point $r=0$, which in the context of the one-dimensional inhomogeneous heat equation is already sufficient to warrant corresponding L^p bounds for the gradient v_r with arbitrary finite p.

Lemma 7.1 *Assume that (1.7) and (1.8) hold with some $q>n$, and that $T_{max}<\infty$. Then for all $p>1$ and each $\delta\in(0,1)$ there exists $C(p,\delta)>0$ such that the solution of (1.6) from Lemma 2.2 satisfies*

$$\int_R^{R+1} |v_r(r,t)|^p dr \le C(p,\delta) \qquad \textit{for all } t\in(\tfrac{1}{2}T_{max},T_{max}) \textit{ and } R>\delta. \tag{7.9}$$

PROOF. We first recall Lemma 2.3 to fix $c_1>0$ such that

$$\int_0^\infty r^{n-1}|v_r(r,t)|dr \le c_1 \qquad \text{for all } t\in(0,T_{max}),$$

and note that (1.11) and Lemma 2.6 provide $c_2>0$ and $c_3>0$ fulfilling

$$\int_0^\infty r^{n-1}u(r,t)dr \le c_2 \quad \text{and} \quad \int_0^\infty r^{n-1}v(r,t)dr \le c_3 \qquad \text{for all } t\in[0,T_{max}).$$

Given $\delta\in(0,1)$, we thereby see that

$$\begin{aligned}\int_{R-\frac{\delta}{2}}^{R+2} |v_r(r,t)|dr &\le \Big(R-\frac{\delta}{2}\Big)^{1-n}\int_{R-\frac{\delta}{2}}^\infty r^{n-1}|v_r(r,t)|dr \\ &\le \Big(\frac{\delta}{2}\Big)^{1-n}c_1 \qquad \text{for all } t\in(0,T_{max}) \text{ and } R>\delta,\end{aligned}$$

and that, similarly,

$$\int_{R-\frac{\delta}{2}}^{R+2} u(r,t)dr \le \Big(\frac{\delta}{2}\Big)^{1-n}c_2 \quad \text{and} \quad \int_{R-\frac{\delta}{2}}^{R+2} v(r,t)dr \le \Big(\frac{\delta}{2}\Big)^{1-n}c_3 \qquad \text{for all } t\in[0,T_{max}) \text{ and } R>\delta. \tag{7.10}$$

In view of (7.3) and the second relation in (7.2), we thus readily obtain $c_4(\delta)>0$ such that the accordingly defined function b from (7.8), extended by zero to all of $\mathbb{R}\times(0,T_{max})$ if necessary, satisfies

$$\|b(\cdot,t)\|_{L^1(J_R)} \le c_4(\delta) \qquad \text{for all } \in(0,T_{max}) \text{ and } R>\delta, \tag{7.11}$$

where we have set $J_R := (R-\frac{1}{2}, R+2)$. To make appropriate use of this, we note that by translation invariance of the one-dimensional heat equation and known smoothing properties of the Neumann heat semigroup $(e^{t\Delta_J})_{t\ge 0}$ over open bounded intervals $J \subset \mathbb{R}$ ([31]), given $p>1$ we can fix $c_5(p,\delta)>0$ such that for any choice of $R\in\mathbb{R}$,

$$\|\partial_r e^{t\Delta_{J_R}}\varphi\|_{L^p(J_R)} \le c_5(p,\delta) t^{-1+\frac{1}{2p}} \|\varphi\|_{L^1(J_R)} \qquad \text{for all } t>0 \text{ and any } \varphi\in L^1(J_R). \tag{7.12}$$

Then using that with $\chi=\chi_{\delta R}$, and once more with a trivial extension if appropriate, we have $(\chi v)_r = 0$ on ∂J_R by (7.2), we may apply (7.12) to a variation-of-constants representation associated with (7.5) to see that again thanks to (7.10),

$$\begin{aligned}
\|(\chi v)_r(\cdot,t)\|_{L^p(J_R)} &= \bigg\| \partial_r e^{t\Delta_{J_R}}[\chi v(\cdot,0)] + \int_0^t \partial_r e^{(t-s)\Delta_{J_R}} b(\cdot,s)ds \bigg\|_{L^p(J_R)} \\
&\le c_5(p,\delta) t^{-1+\frac{1}{2p}} \|\chi v(\cdot,0)\|_{L^1(J_R)} + c_5(p,\delta)\int_0^t (t-s)^{-1+\frac{1}{2p}} \|b(\cdot,s)\|_{L^1(J_R)} ds \\
&\le c_5(p,\delta)\cdot\Big(\frac{1}{2}T_{max}\Big)^{-1+\frac{1}{2p}} \|v(\cdot,0)\|_{L^1((R-\frac{\delta}{2},R+2))} + c_4(\delta)c_5(p,\delta)\int_0^t (t-s)^{-1+\frac{1}{2p}} ds \\
&\le \Big(\frac{\delta}{2}\Big)^{1-n} c_3 c_5(p,\delta)\cdot \Big(\frac{1}{2}T_{max}\Big)^{-1+\frac{1}{2p}} + 2pc_4(\delta)c_5(p,\delta) T_{max}^{\frac{1}{2p}}
\end{aligned}$$

for all $t\in(\frac{1}{2}T_{max}, T_{max})$ and $R>\delta$. As $(\chi v)_r \equiv v_r$ in $(R,R+1)\times(0,T_{max}) \subset J_R\times(0,T_{max})$ by (7.2), this establishes (7.9). $\Box$

By tracking suitably localized versions of $\int u^{p_0}$ with some sublinear $p_0>0$, as a first consequence of Lemma 7.1 we can derive an integrability property of u involving arbitrary subcubic powers.

Lemma 7.2 *Assume that (1.7) and (1.8) hold with some $q>n$, and that $T_{max}<\infty$. Then for all $p\in(2,3)$ and any $\delta\in(0,1)$ one can find $C(p,\delta)>0$ such that for the solution of (1.6) from Lemma 2.2 we have*

$$\int_{\frac{1}{2}T_{max}}^{T_{max}} \int_R^{R+1} u^p(r,t)dr \le C(p,\delta) \qquad \textit{for all } R>\delta. \tag{7.13}$$

Proof. Given $p\in(2,3)$ we set $p_0\equiv p_0(p) := p-2 \in (0,1)$, and for $\delta\in(0,1)$ and $R>\delta$ we let $\chi\equiv\chi_{\delta R}$ and use (7.4) to see that with a_1 and a_2 as defined through (7.6) and (7.7) we have

$$\begin{aligned}
\frac{1}{p_0}\frac{d}{dt}\int_0^\infty \chi^2 u^{p_0}(r,t)dr &= \int_0^\infty \chi u^{p_0-1}(\chi u)_{rr}dr + \int_0^\infty \chi u^{p_0-1}(a_1u)_r dr + \int_0^\infty \chi u^{p_0-1}\cdot a_2 u dr \\
&= -\int_0^\infty (\chi u^{p_0-1})_r\cdot(\chi u)_r dr - \int_0^\infty (\chi u^{p_0-1})_r \cdot a_1 u dr + \int_0^\infty \chi a_2 u^{p_0} dr \\
&= (1-p_0)\int_0^\infty \chi^2 u^{p_0-2}u_r^2 dr + (1-p_0)\int_0^\infty \chi\chi_r u^{p_0-1}u_r dr \\
&\quad -\int_0^\infty \chi\chi_r u^{p_0-1}u_r dr - \int_0^\infty \chi_r^2 u^{p_0} dr \\
&\quad +(1-p_0)\int_0^\infty \chi a_1 u^{p_0-1}u_r dr - \int_0^\infty \chi_r a_1 u^{p_0}dr + \int_0^\infty \chi a_2 u^{p_0} dr
\end{aligned}$$

$$\begin{aligned}
&= (1-p_0)\int_0^\infty \chi^2 u^{p_0-2}u_r^2 dr \\
&\quad +\int_0^\infty \Big\{-p_0\chi_r+(1-p_0)a_1\Big\}\chi u^{p_0-1}u_r dr \\
&\quad -\int_0^\infty (\chi_r^2+\chi_r a_1-\chi a_2)u^{p_0}dr \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{7.14}$$

Since by Young's inequality,

$$\begin{aligned}
\Big|\int_0^\infty \Big\{-p_0\chi_r+(1-p_0)a_1\Big\}\chi u^{p_0-1}u_r dr\Big| &\le \frac{1-p_0}{2}\int_0^\infty \chi^2 u^{p_0-2}u_r^2 dr \\
&\quad +\frac{1}{2(1-p_0)}\int_{\operatorname{supp}\chi}\Big\{-p_0\chi_r+(1-p_0)a_1\Big\}^2 u^{p_0}dr \\
&\le \frac{1-p_0}{2}\int_0^\infty \chi^2 u^{p_0-2}u_r^2 dr \\
&\quad +\int_{\operatorname{supp}\chi}\Big\{\frac{p_0^2}{1-p_0}\chi_r^2+(1-p_0)a_1^2\Big\}u^{p_0}dr
\end{aligned}$$

for all $t\in(0,T_{max})$, and since $\operatorname{supp}\chi\subset(R-\frac{\delta}{2},R+2)$ and $\chi^2\equiv 1$ in $[R,R+1]$, from (7.14) we obtain that

$$\frac{1}{p_0}\frac{d}{dt}\int_0^\infty \chi^2 u^{p_0}(r,t)dr \ge \frac{1-p_0}{2}\int_R^{R+1} u^{p_0-2}u_r^2 dr - \int_{R-\frac{\delta}{2}}^{R+2} h(r,t)u^{p_0}dr \qquad \text{for all } t\in(0,T_{max}), \tag{7.15}$$

where due to Young's inequality, (7.6), (7.7) and (7.1),

$$h(r,t) := \frac{p_0^2}{1-p_0}\chi_r^2+(1-p_0)a_1^2+\chi_r^2+|\chi_r|\cdot|a_1|+\chi|a_2|, \qquad r>0,\ t\in(0,T_{max}),$$

has the property that for some $c_1(p,\delta)>0$ and any $R>\delta$,

$$\begin{aligned}
|h(r,t)| &\le \Big(\frac{3}{2}-p_0\Big)a_1^2+|a_2|+\Big(\frac{p_0^2}{1-p_0}+\frac{3}{2}\Big)\chi_r^2 \\
&\le 3\Big(\frac{3}{2}-p_0\Big)\cdot\Big\{\frac{(n-1)^2}{r^2}+4\chi_r^2+v_r^2\Big\} \\
&\quad +(n-1)\cdot\Big|\Big(\frac{\chi}{r}\Big)_r\Big|+|\chi_{rr}|+\Big|\chi_r-\frac{n-1}{r}\chi\Big|\cdot|v_r|+\Big(\frac{p_0^2}{1-p_0}+\frac{3}{2}\Big)\chi_r^2 \\
&\le c_1(p,\delta)(v_r^2+1) \qquad \text{for all } r>\frac{\delta}{2} \text{ and } t\in(0,T_{max}).
\end{aligned}$$

Accordingly, in view of Young's inequality and (1.11), an application of Lemma 7.1 to the summability power $\frac{2}{1-p_0}$ reveals the existence of $c_2(p,\delta)>0$ such that for all $R>\delta$, writing $\tau:=\frac{1}{2}T_{max}$ we have

$$\Big|\int_{R-\frac{\delta}{2}}^{R+2} h(r,t)u^{p_0}dr\Big| \le c_1(p,\delta)\int_{R-\frac{\delta}{2}}^{R+2}(v_r^2+1)u^{p_0}dr$$

$$
\begin{aligned}
&\le c_1(p,\delta)\int_{R-\frac{\delta}{2}}^{R+2}(v_r^2+1)^{\frac{1}{1-p_0}}dr + c_1(p,\delta)\int_{\frac{\delta}{2}}^{\infty} udr \\
&\le 2^{\frac{1}{1-p_0}}c_1(p,\delta)\int_{R-\frac{\delta}{2}}^{R+2}(|v_r|^{\frac{2}{1-p_0}}+1)dr + c_1(p,\delta)\cdot\Big(\frac{\delta}{2}\Big)^{1-n}\int_{\frac{\delta}{2}}^{\infty} r^{n-1}udr \\
&\le c_2(p,\delta) \qquad \text{for all } t\in(\tau,T_{max}),
\end{aligned}
$$

because for any such R we know that $R-\frac{\delta}{2}>\frac{\delta}{2}$.
Therefore, a further integration of (7.15) shows that again by (7.2), Young's inequality and (1.11),

$$
\begin{aligned}
\frac{1-p_0}{2}\int_\tau^T\int_R^{R+1}u^{p_0-2}u_r^2drdt &\le \frac{1}{p_0}\int_0^\infty \chi^2u^{p_0}(r,T)dr - \frac{1}{p_0}\int_0^\infty \chi^2u^{p_0}(r,\tau)dr + c_2(p,\delta)\cdot(T-\tau) \\
&\le \frac{1}{p_0}\int_{R-\frac{\delta}{2}}^{R+2}u^{p_0}(r,T)dr + c_2(p,\delta)T_{max} \\
&\le \frac{1}{p_0}\int_{R-\frac{\delta}{2}}^{R+1}\Big(u(r,T)+1\Big)dr + c_2(p,\delta)T_{max} \\
&\le \frac{1}{p_0}\cdot\Big(\frac{\delta}{2}\Big)^{1-n}\int_{\frac{\delta}{2}}^{\infty} r^{n-1}u(r,T)dr + \frac{1}{p_0}\cdot\Big\{(R+2)-\Big(R-\frac{\delta}{2}\Big)\Big\} \\
&\quad + c_2(p,\delta)T_{max} \qquad \text{for all } T\in(\tau,T_{max}) \text{ and } R>\delta,
\end{aligned}
$$

so that since $(R+2)-(R-\frac{\delta}{2})\le\frac{5}{2}$ for arbitrary $R\in\mathbb{R}$, once more thanks to (1.11) we can find $c_3(p,\delta)>0$ such that

$$
\int_\tau^{T_{max}}\|(u^{\frac{p_0}{2}})_r(\cdot,t)\|^2_{L^2((R,R+1))}dt \le c_3(p,\delta) \qquad \text{for all } R>\delta. \tag{7.16}
$$

As the one-dimensional Gagliardo-Nirenberg inequality provides $c_4(p)>0$ fulfilling

$$
\begin{aligned}
\|\varphi\|^{\frac{2p}{p-2}}_{L^{\frac{2p}{p-2}}((R,R+1))} &\le c_4(p)\|\varphi_r\|^2_{L^2((R,R+1))}\|\varphi\|^{\frac{4}{p-2}}_{L^{\frac{2}{p-2}}((R,R+1))} + c_4(p)\|\varphi\|^{\frac{2p}{p-2}}_{L^{\frac{2}{p-2}}((R,R+1))} \\
&\qquad \text{for all } R\in\mathbb{R} \text{ and each } \varphi\in W^{1,2}((R,R+1)),
\end{aligned}
$$

recalling that $p_0=p-2$ and using that, again by (1.11), with some $c_5(\delta)>0$ we have

$$
\|u^{\frac{p_0}{2}}(\cdot,t)\|^{\frac{2}{p-2}}_{L^{\frac{2}{p-2}}((R,R+1))} = \int_R^{R+1}u(r,t)dr \le \delta^{1-n}\int_\delta^\infty r^{n-1}u(r,t)dr \le c_5(\delta)
$$

for all $R>\delta$ and $t\in(0,T_{max})$, from (7.16) we infer that

$$
\begin{aligned}
\int_\tau^{T_{max}}\int_R^{R+1}u^p(r,t)drdt &= \int_\tau^{T_{max}}\|u^{\frac{p_0}{2}}(\cdot,t)\|^{\frac{2p}{p-2}}_{L^{\frac{2p}{p-2}}((R,R+1))}dt \\
&\le c_4(p)\int_\tau^{T_{max}}\|(u^{\frac{p_0}{2}})_r(\cdot,t)\|^2_{L^2((R,R+1))}\|u^{\frac{p_0}{2}}(\cdot,t)\|^{\frac{4}{p-2}}_{L^{\frac{2}{p-2}}((R,R+1))}dt
\end{aligned}
$$

$$
\begin{aligned}
&\quad + c_4(p) \int_\tau^{T_{max}} \|u^{\frac{p_0}{2}}(\cdot,t)\|_{L^{\frac{2}{p-2}}((R,R+1))}^{\frac{2p}{p-2}} dt \\
&\le c_4(p) c_5^2(p) \int_\tau^{T_{max}} \|(u^{\frac{p_0}{2}})_r(\cdot,t)\|_{L^2((R,R+1))}^2 dt + \frac{1}{2} c_4(p) c_5^p(\delta) T_{max} \\
&\le c_3(p,\delta) c_4(p) c_5^2(p) + \frac{1}{2} c_4(p) c_5^p(\delta) T_{max} \qquad \text{for all } R > \delta,
\end{aligned}
$$

and that thus indeed (7.13) holds. □

The above two integrability properties are sufficient to ensure applicability of L^p-L^q estimates for one-dimensional heat semigroups to achieve bounds for u actually in arbitrary L^p spaces.

Lemma 7.3 *Assume that (1.7) and (1.8) hold with some $q > n$, and that $T_{max} < \infty$. Then for all $p > 3$ and arbitrary $\delta \in (0,1)$ there exists $C(p,\delta) > 0$ such that with (u,v) taken from Lemma 2.2 we have*

$$
\int_R^{R+1} u^p(r,t)dr \le C(p,\delta) \qquad \textit{for all } t \in (\tfrac{1}{2}T_{max}, T_{max}) \textit{ and } R > \delta. \tag{7.17}
$$

PROOF. Since $p > 3$, we can fix $p_0 = p_0(p) \in (2,3)$ such that $p_0 > \frac{3p}{p+1}$, which ensures that $\frac{p+1}{p} > \frac{3}{p_0}$ and hence $1 + \frac{1}{p} - \frac{2}{p_0} > \frac{1}{p_0}$, and thus enables us to choose some $\lambda = \lambda(p) \in (1, p_0)$ fulfilling

$$
\frac{1}{\lambda} < 1 + \frac{1}{p} - \frac{2}{p_0}. \tag{7.18}
$$

Then using that $\lambda < p_0$, we may recall Lemma 7.1 to see that in view of (7.2), for each $\delta \in (0,1)$ we can find $c_1(p,\delta) > 0$ such that for any choice of $R > \delta$, the functions a_1 and a_2, as defined in (7.6) and (7.7) and trivially extended to all of $\mathbb{R} \times (0, T_{max})$ if necessary, satisfy

$$
\|a_1(\cdot,t)\|_{L^{\frac{p_0\lambda}{p_0-\lambda}}(J_R)} \le c_1(p,\delta) \quad \text{and} \quad \|a_2(\cdot,t)\|_{L^{\frac{p_0\lambda}{p_0-\lambda}}(J_R)} \le c_1(p,\delta) \qquad \text{for all } t \in (\tau, T_{max}), \tag{7.19}
$$

where $\tau := \frac{1}{2}T_{max}$ and $J_R := (R - \frac{1}{2}, R+2)$ for $R \in \mathbb{R}$. Apart from that, thanks to the restriction that $p_0 \in (2,3)$ we may invoke Lemma 7.2 to fix $c_2(p,\delta) > 0$ such that

$$
\int_\tau^{T_{max}} \int_{R-\frac{\delta}{2}}^{R+2} u^{p_0}(r,t)drdt \le c_2(p,\delta) \qquad \text{for all } R > \delta, \tag{7.20}
$$

and to make adequate use of this in the framework of the inhomogeneous linear heat equation (7.4) for χu, we employ known smoothing properties of the Neumann heat semigroup $(e^{t\Delta_{J_R}})_{t\ge 0}$ on J_R ([31]) to see that once more due to translation invariance, we can find $c_3(p) > 0$ and $c_4(p) > 0$ such that for all $R \in \mathbb{R}$,

$$
\|e^{t\Delta_{J_R}} \varphi_r\|_{L^p(J_R)} \le c_3(p) t^{-\frac{1}{2} - \frac{1}{2}(\frac{1}{\lambda} - \frac{1}{p})} \|\varphi\|_{L^\lambda(J_R)} \quad \text{for all } t > 0 \text{ and } \varphi \in C^1(\overline{J}_R) \text{ with } \varphi|_{\partial J_R} = 0 \tag{7.21}
$$

and

$$
\|e^{t\Delta_{J_R}} \varphi\|_{L^p(J_R)} \le c_4(p) t^{-\frac{1}{2}(\frac{1}{\lambda} - \frac{1}{p})} \|\varphi\|_{L^\lambda(J_R)} \qquad \text{for all } t > 0 \text{ and } \varphi \in L^\lambda(J_R). \tag{7.22}
$$

An application of (7.21), (7.22) and the maximum principle to a Duhamel representation of χu on the basis of (7.4) thus shows that for all $\delta \in (0,1)$ and $R > \delta$,

$$
\begin{aligned}
\|\chi u(\cdot,t)\|_{L^p(J_R)} &= \Big\| e^{(t-\tau)\Delta_{J_R}}[\chi u(\cdot,\tau)] + \int_\tau^t e^{(t-s)\Delta_{J_R}} \partial_r [a_1(\cdot,s)u(\cdot,s)]ds \\
&\qquad + \int_\tau^t e^{(t-s)\Delta_{J_R}}[a_2(\cdot,s)u(\cdot,s)]ds \Big\|_{L^p(J_R)} \\
&\le \|\chi u(\cdot,\tau)\|_{L^p(J_R)} \\
&\quad + c_3(p) \int_\tau^t (t-s)^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})} \|a_1(\cdot,s)u(\cdot,s)\|_{L^\lambda(J_R)} ds \\
&\quad + c_4(p) \int_\tau^t (t-s)^{-\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})} \|a_2(\cdot,s)u(\cdot,s)\|_{L^\lambda(J_R)} ds \qquad \text{for all } t \in (\tau, T_{max}), \qquad (7.23)
\end{aligned}
$$

where by the Hölder inequality, (7.2), (7.19) and (7.20),

$$
\begin{aligned}
& c_3(p) \int_\tau^t (t-s)^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})} \|a_1(\cdot,s)u(\cdot,s)\|_{L^\lambda(J_R)} ds \\
&\le c_3(p) \int_\tau^t (t-s)^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})} \|a_1(\cdot,s)\|_{L^{\frac{p_0\lambda}{p_0-\lambda}}(J_R)} \|u(\cdot,s)\|_{L^{p_0}((R-\frac{\delta}{2},R+2))} ds \\
&\le c_1(p,\delta) c_3(p) \int_\tau^t (t-s)^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})} \|u(\cdot,s)\|_{L^{p_0}((R-\frac{\delta}{2},R+2))} ds \\
&\le c_1(p,\delta) c_3(p) \Big\{ \int_\tau^t (t-s)^{-[\frac{1}{2}+\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})]\cdot\frac{p_0}{p_0-1}} ds \Big\}^{\frac{p_0-1}{p_0}} \cdot \Big\{ \int_\tau^t \|u(\cdot,s)\|^{p_0}_{L^{p_0}((R-\frac{\delta}{2},R+2))} ds \Big\}^{\frac{1}{p_0}} \\
&\le c_1(p,\delta) c_2^{\frac{1}{p_0}}(p,\delta) c_3(p) c_5^{\frac{p_0-1}{p_0}}(p) \qquad \text{for all } t \in (\tau, T_{max}),
\end{aligned}
$$

with $c_5(p) := \int_0^{T_{max}-\tau} \sigma^{-[\frac{1}{2}+\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})]\cdot\frac{p_0}{p_0-1}} d\sigma$ being finite due to the fact that by (7.18),

$$
\Big[\frac{1}{2} + \frac{1}{2}\Big(\frac{1}{\lambda} - \frac{1}{p}\Big)\Big] \cdot \frac{p_0}{p_0-1} < \Big[\frac{1}{2} + \frac{1}{2}\Big(1 - \frac{2}{p_0}\Big)\Big] \cdot \frac{p_0}{p_0-1} = 1.
$$

Since, similarly,

$$
\begin{aligned}
& c_4(p) \int_\tau^t (t-s)^{-\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})} \|a_2(\cdot,s)u(\cdot,s)\|_{L^\lambda(J_R)} ds \\
&\le c_1(p,\delta) c_4(p) \int_\tau^t (t-s)^{-\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})} \|u(\cdot,s)\|_{L^{p_0}(R-\frac{\delta}{2},R+2))} ds \\
&\le c_1(p,\delta) c_2^{\frac{1}{p_0}}(p,\delta) c_4(p) \cdot \Big\{ \int_\tau^t (t-s)^{-\frac{1}{2}(\frac{1}{\lambda}-\frac{1}{p})\cdot\frac{p_0}{p_0-1}} ds \Big\}^{\frac{p_0-1}{p_0}} \qquad \text{for all } t \in (\tau, T_{max}),
\end{aligned}
$$

and since (7.18) clearly entails that also $\frac{1}{2}(\frac{1}{\lambda} - \frac{1}{p}) \cdot \frac{p_0}{p_0-1} < 1$, in view of the finiteness of $\sup_{r>0} u(r,\tau)$ asserted by Proposition 1.1 we infer from (7.23) that for all $\delta \in (0,1)$ there exists $c_6(p,\delta) > 0$ such that for all $R > \delta$,

$$
\|\chi u(\cdot,t)\|_{L^p(J_R)} \le c_6(p,\delta) \qquad \text{for all } t \in (\tau, T_{max}),
$$

which by (7.2) and our definition of τ establishes (7.17). $\Box$

In a last bootstrap step, this information can quite similarly be turned into a genuine pointwise bound in the intended flavor.

Lemma 7.4 *Assume that (1.7) and (1.8) are satisfied for some $q > n$, and that $T_{max} < \infty$. Then given any $\delta \in (0,1)$ one can find $C(\delta) > 0$ with the property that for the corresponding solution of (1.6) we have*

$$u(r,t) \le C(\delta) \qquad \textit{for all } r > \delta \textit{ and } t \in (0, T_{max}). \tag{7.24}$$

PROOF. We fix an arbitrary $p > 1$ and then infer from (7.2), Lemma 7.1 and Lemma 7.3 that for all $\delta \in (0,1)$ there exist $c_1(\delta) > 0$ and $c_2(\delta) > 0$ such that whenever $R > \delta$, with $\chi \equiv \chi_{\delta R}, a_1$ and a_2 from (7.1), (7.6) and (7.7) extended by zero to $(0,\infty) \times (0, T_{max})$, and again with $\tau := \frac{1}{2}T_{max}$ and $J := (R - \frac{1}{2}, R+2)$, we have

$$\|a_1(\cdot,s)\|_{L^{2p}(J_R)} \le c_1(\delta) \quad \text{and} \quad \|a_2(\cdot,s)\|_{L^{2p}(J_R)} \le c_1(\delta) \qquad \text{for all } t \in (\tau, T_{max})$$

as well as

$$\|u(\cdot,t)\|_{L^{2p}((R-\frac{\delta}{2},R+2))} \le c_2(\delta) \qquad \text{for all } t \in (\tau, T_{max}).$$

Then proceeding similarly to the argument in Lemma 7.3, by means of (7.4), of known regularization features of the Neumann heat semigroup $(e^{t\Delta_{J_R}})_{t\ge 0}$ on J_R ([8]), and of the Hölder inequality we see that with some $c_3 > 0$ and $c_4 > 0$, for any choice of $\delta \in (0,1)$ and $R > \delta$ we have

$$\begin{aligned}
\|\chi u(\cdot,t)\|_{L^\infty(J_R)} &\le \|\chi u(\cdot,\tau)\|_{L^\infty(J_R)} + c_3 \int_\tau^t (t-s)^{-\frac{1}{2}-\frac{1}{2p}} \|a_1(\cdot,s)u(\cdot,s)\|_{L^p(J_R)} ds \\
&\qquad + c_4 \int_\tau^t (t-s)^{-\frac{1}{2p}} \|a_2(\cdot,s)u(\cdot,s)\|_{L^p(J_R)} ds \\
&\le \|\chi u(\cdot,\tau)\|_{L^\infty(J_R)} + c_3 \int_\tau^t (t-s)^{-\frac{1}{2}-\frac{1}{2p}} \|a_1(\cdot,s)\|_{L^{2p}(J_R)} \|u(\cdot,s)\|_{L^{2p}((R-\frac{\delta}{2},R+2))} ds \\
&\qquad + c_4 \int_\tau^t (t-s)^{-\frac{1}{2p}} \|a_2(\cdot,s)\|_{L^{2p}(J_R)} \|u(\cdot,s)\|_{L^{2p}((R-\frac{\delta}{2},R+2))} ds \\
&\le \|\chi u(\cdot,\tau)\|_{L^\infty(J_R)} + c_1(\delta)c_2(\delta)c_3 \cdot \frac{(T_{max}-\tau)^{\frac{1}{2}-\frac{1}{2p}}}{\frac{1}{2}-\frac{1}{2p}} \\
&\qquad + c_1(\delta)c_2(\delta)c_4 \cdot \frac{(T_{max}-\tau)^{1-\frac{1}{2p}}}{1-\frac{1}{2p}}
\end{aligned}$$

for all $t \in (\tau, T_{max})$. According to (7.2) and the fact that u is bounded in $\mathbb{R}^n \times [0,\tau]$ by Proposition 1.1, this already entails (7.24). $\Box$

8 Proof of Theorem 1.2

The proof of our main result on blow-up in (1.6) now reduces to collecting actually completed pieces only:

Proof of Theorem 1.2. To construct a sequence $((u_{0j}, v_{0j}))_{j\in\mathbb{N}}$ which fulfils the claimed regularity and approximation properties and which is such that the corresponding solutions of (1.6) satisfy $T_{max,j} \le 1$ as well as (1.14), we only need to combine the outcome of Lemma 6.1 with the extensibility criterion (1.10). The boundedness feature (1.15) thereafter immediately results from Lemma 7.4. $\Box$

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References

[1] Biler, P.: *Local and global solvability of some parabolic systems modelling chemotaxis.* Adv. Math. Sci. Appl. **8**, 715-743 (1998)

[2] Calvez, V., Corrias, L.: *The parabolic-parabolic Keller-Segel model in* $\mathbb{R}^2$. Comm. Math. Sci. **6**, 417-446 (2008)

[3] Chaplain, M.A.J., Lolas, G.: *Mathematical modelling of cancer invasion of tissue: The role of the urokinase plasminogen activation system.* Math. Mod. Meth. Appl. Sci. **15**, 1685-1734 (2005)

[4] Cieślak, T., Stinner, C.: *Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions.* J. Differ. Eq. **252** (10), 5832-5851 (2012)

[5] Corrias, L., Escobedo, M., Matos, J.: *Existence, uniqueness and asymptotic behavior of the solutions to the fully parabolic Keller-Segel system in the plan.* J. Differential Eq. **257**, 1840-1878 (2014)

[6] Corrias,L., Perthame, B.: *Asymptotic decay for the solutions of the parabolic-parabolic Keller-Segel chemotaxis system in critical spaces.* Mathematical and Computer Modelling **47**, 755-764 (2008)

[7] Cosner, C.: *Reaction-diffusion-advection models for the effects and evolution of dispersal.* Discr. Cont. Dyn. Syst. **34**, 1701-1745 (2014)

[8] Fujie, K., Ito, A., Winkler, M., Yokota, T.: *Stabilization in a chemotaxis model for tumor invasion.* Discr. Cont. Dyn. Syst. **36**, 151-169 (2016)

[9] Giga, Y., Umeda, N.: *On blow-up at space infinity for semilinear heat equations.* Acta Math. Univ. Comenianae **76**, 63-76 (2007)

[10] Herrero, M. A., Velázquez, J. J. L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scuola Normale Superiore Pisa **24**, 633-683 (1997)

[11] Hillen, T., Painter, K.J., Winkler, M.: *Global solvability and explicit bounds for non-local adhesion models.* Eur. J. Appl. Math. **29**, 645-684 (2018)

[12] Horstmann, D.: *From 1970 until present: The Keller-Segel model in chemotaxis and its consequences I.* Jahresberichte DMV **105** (3), 103-165 (2003)

[13] Horstmann, D., Wang, G.: Blow-up in a chemotaxis model without symmetry assumptions. Eur. J. Appl. Math. **12**, 159-177 (2001)

[14] Horstmann, D., Winkler, M.: *Boundedness vs. blow-up in a chemotaxis system.* J. Differential Equations **215**, 52-107 (2005)

[15] Jäger, W., Luckhaus, S.: *On explosions of solutions to a system of partial differential equations modelling chemotaxis.* Trans. Am. Math. Soc. **329**, 819-824 (1992)

[16] Keller, E.F., Segel, L.A.: *Initiation of slime mold aggregation viewed as an instability.* J. Theor. Biol. **26**, 399-415 (1970)

[17] Lacey, A.A.: *The form of blow-up for nonlinear parabolic equations.* Proc. Roy. Soc. Edinburgh Sect. A **98**, 183-202 (1984)

[18] Laurençot, Ph., Mizoguchi, N.: *Finite time blowup for the parabolic-parabolic KellerSegel system with critical diffusion.* Ann. Inst. H. Poincaré Anal. Non Linéaire **34**, 197-220 (2017)

[19] Li, Y., Li, Y.: *Finite-time blow-up in higher dimensional fully-parabolic chemotaxis system for two species.* Nonlin. Anal. **109**, 72-84 (2014)

[20] Meral, G., Stinner, C., Surulescu, C.: *On a multiscale model involving cell contractivity and its effects on tumor invasion.* Discr. Cont. Dyn. Syst. B **20**, 198-213 (2015)

[21] Mizoguchi, N., Winkler, M.: *Finite-time blow-up in the two-dimensional parabolic Keller-Segel system.* Preprint

[22] Nagai, T.: *Blow-up of radially symmetric solutions to a chemotaxis system.* Adv. Math. Sci. Appl. **5**, 581-601 (1995)

[23] Nagai, T.: *Blowup of Nonradial Solutions to Parabolic-Elliptic Systems Modeling Chemotaxis in Two-Dimensional Domains.* J. Inequal. Appl. **6**, 37-55 (2001)

[24] Nagai, T., Senba, T., Suzuki, T.: *Chemotactic collapse in a parabolic system of mathematical biology.* Hiroshima Math. J. **30**, 463-497 (2000)

[25] Nanjundiah, V.: *Chemotaxis, signal relaying and aggregation morphology.* J. Theor. Biol. **42**, 63-105 (1973)

[26] Quittner, P., Souplet, Ph.: *Superlinear Parabolic Problems. Blow-up, Global Existence and Steady States.* Birkhäuser Advanced Texts, Basel/Boston/Berlin, 2007

[27] Sato, S.: *Blow-up at space infinity of a solution with a moving singularity for a semilinear parabolic equation.* Comm. Pure Appl. Anal. **10**, 1225-1237 (2011)

[28] Schweyer, R.: *Stable blow-up dynamic for the parabolic-parabolic Patlak-Keller-Segel model.* `arXiv:1403.4975`

[29] Stancevic, O., Angstmann, C.N., Murray, J.M., Henry, B.I.: *Turing patterns from dynamics of early HIV infection.* Bull. Math. Biol. **75**, 774-795 (2013)

[30] Tania, N., Vanderlei, B., Heath, J.P., Edelstein-Keshet, L.: *Role of social interactions in dynamic patterns of resource patches and forager aggregation.* Proc. Nat. Acad. Sci. USA **109**, 11228-11233 (2012)

[31] Winkler, M.: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Equations **248**, 2889-2905 (2010)

[32] Winkler, M: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013), `arXiv:1112.4156v1`

[33] Wu, S., Shi, J., Wu, B.: *Global existence of solutions and uniform persistence of a diffusive predator-prey model with prey-taxis.* J. Differential Eq. **260**, 5847-5874 (2016)