# Small-mass solutions in the two-dimensional Keller-Segel system coupled to the Navier-Stokes equations 

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#### Abstract

The fully parabolic Keller-Segel system is coupled to the incompressible Navier-Stokes equations through transport and buoyancy. It is shown that when posed with no-flux/no-flux/Dirichlet boundary conditions in smoothly bounded planar domains and along with appropriate assumptions on regularity of the initial data, under a smallness condition exclusively involving the total initial population mass $m$ an associated initial-boundary value problem admits a globally defined generalized solution; in particular, this hypothesis is fully explicit and independent of the initial size of further solution components. Moreover, the obtained solution is seen to enjoy a certain temporally averaged boundedness property which, inter alia, rules out any finite-time collapse into persistent Dirac-type measures, as well as convergence to such singular profiles in the large time limit. Apart from that, it is found that under a further restriction on the size of $m$ any such solution becomes eventually smooth and asymptotically approaches a spatially homogeneous equilibrium.


Key words: chemotaxis; critical mass; global existence; eventual regularity MSC (2010): 35B45 (primary); 35B40, 92C17, 35Q35, 35D99 (secondary)

## 1 Introduction

The interaction of microbial populations with liquid environments has been a focal point in an increasing number of both experimental and theoretical approaches to understand fundamental principles of self-organization, especially in contexts of chemotactically biased movement of individuals ([10], [13], [34], [44], [46]). Recent developments in the analytical literature have provided some examples in which adequately designed fluid flows indeed affect core features of some Keller-Segel type systems to a significant extent. In [26] and [27], for instance, some qualitative effects of certain fluid flows on spreading properties and efficiency of an additional absorbing reaction in a model for broadcast spawning have been discussed. Apart from this, based on concepts of so-called relaxation enhancing and near-optimal mixing flows ([11], [63]), a subtle construction in [28] showed that if, in dependence on arbitrarily prescribed initial data, some suitably chosen incompressible fluid velocity field is added to a classical parabolic-elliptic Keller-Segel system through a simple transport mechanism, then even blow-up phenomena, in the style of those known to occur in the corresponding unperturbed chemotaxis system, can be suppressed; according to [21], in the particular two-dimensional case the latter can in fact already be achieved, at least for some class of initial data, by the fully explicit potential flow given by the velocity field $u\left(x_{1}, x_{2}\right):=A\left(-x_{1}, x_{2}\right),\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$, for appropriately large $A>0$.
In contrast to this, situations in which the fluid flow itself is a system variable, potentially influenced by the considered populations e.g. through buoyancy, seem much less understood. Indeed, the accordingly augmented complexity of such coupled chemotaxis-fluid systems apparently goes along with a significant reduction of accessibility to methods well-established in the theory of chemotaxis systems, especially in cases near criticality with respect to either crucial system parameters, or initial data sizes. Correspondingly, analytical efforts in this field so far seem to mainly have concentrated on chemotaxis systems which with regard to their singularity-enforcing potential can be viewed as being suitably far from critical in an appropriate sense, and on identifying constellations in which certain types of essentially diffusion-dominated solution behavior in the respective fluid-free unperturbed system remain largely unaffected by coupling to either the Stokes or the Navier-Stokes equations through transport and gravitational effects.

An example quite intensely studied in this regard is the oxytaxis-Navier-Stokes model for swimming aerobic bacteria, which in a normalized and prototypical form is given by

$$
\begin{cases}n_{t}+u \cdot \nabla n & =\Delta n-\nabla \cdot(n \nabla c),  \tag{1.1}\\ c_{t}+u \cdot \nabla c & =\Delta c-n c \\ u_{t}+(u \cdot \nabla) u & =\Delta u+\nabla P+n \nabla \phi, \quad \nabla \cdot u=0\end{cases}
$$

and which has been proposed in [46] to describe the unknown population density $n$ of bacteria that are attracted by oxygen as their nutrient, represented through their concentration $c$. Mainly due to the absorptive contribution $-n c$ to the second equation, hence reflecting consumption of the chemoattractant by individuals, in the fluid-free version thereof, in which thus the fluid velocity and pressure variables $u$ and $P$ as well as the gravitational potential $\phi$ vanish identically, known results witness the complete absence of any unboundedness phenomena in corresponding boundary value problems in two-dimensional cases, and at least global existence of some weak solutions that eventually become smooth and classical in three-dimensional domains ([42]). Now since here the dissipative effect of the considered signal consumption process apparently implies quite a strong explosion-counteracting
mechanism, the latter type of unconditionally diffusion-driven solution behavior is in fact inherited not only by solutions to corresponding boundary-value problems for the full problem (1.1) ([14], [7], [54], [59], [56], [25], [60], [64]), but also for a considerable number of generalizations involving e.g. variants of diffusion and cross-diffusion rates ([65], [15], [31], [51], [43], [58]).

Approaching criticality in chemotaxis(-fluid) systems capable of enforcing blow-up. In contexts of self-enhanced chemotactic motion, such as those addressed by the classical Keller-Segel system

$$
\left\{\begin{array}{l}
n_{t}=\Delta n-\nabla \cdot(n \nabla c),  \tag{1.2}\\
c_{t}=\Delta c-c+n,
\end{array}\right.
$$

and its close relatives, however, an accordingly increased destabilizing potential of cross-diffusive interaction becomes manifest in results on the occurrence of some exploding solutions both in three- or higher-dimensional frameworks ([2], [1], [35], [55]), and in planar settings ([2], [35], [36], [23], [24]); unlike in the chemotaxis-consumption setting of (1.1), only upon certain regularizing modifications, e.g. involving appropriate strengthening of diffusion or weakening of cross-diffusion, such Keller-Segelproduction systems admit allow for comparably comprehensive results on global well-posedness for initial data with arbitrary size ([41], [8], [9], [53]), and only in such cases of globally subcritical systems some satisfactory extensions to cases involving fluid coupling seem available ([49], [50], [48], [32], [4], [66], [61]).
In the original Keller-Segel system (1.2) without such dissipation-enhancing changes, only suitably small initial data are known to evolve into globally defined smooth and bounded solutions ([37], [5]), and the question how far correspondingly generated blow-up dichotomies with respect to suitably measured sizes of initial data may be affected by fluid interaction, e.g. in the style of that in (1.1), remains widely unaddressed in the literature; available results in this direction apparently remain in some distance of critical situations especially in two-dimensional cases in which, namely, (1.2) seems to be particularly critical. Indeed, as one of its possibly most striking features the two-dimensional version of (1.2), e.g. posed in a smoothly bounded planar domain $\Omega$, exhibits a critical mass phenomenon in the sense that the total population mass $\int n_{0}$ of the initial data $\left(n_{0}, c_{0}\right)$ acts as a crucial quantity in this regard: Whenever $\left(n_{0}, c_{0}\right)$ is suitably regular with $\int_{\Omega} n_{0}<4 \pi$, an associated no-flux initial-boundary value problem admits a global bounded classical solution ([37]), while if $\Omega$ is simply connected, then for any $m \in(4 \pi, \infty) \backslash\{4 k \pi \mid k \in \mathbb{N}\}$ one can find smooth initial data ( $n_{0}, c_{0}$ ) with $\int_{\Omega} n_{0}=m$ which are such that the corresponding problem possesses a solution blowing up either in finite or infinite time ([24]; cf. also [36] for a yet slightly farther reaching analogue addressing a parabolic-elliptic variant of (1.2)).
Whereas these supercritical-mass blow-up properties at least admit some trivial extension to an associated Keller-Segel-fluid system through the consideration of vanishing fluid velocities, complementing results on global existence of subcritical-mass solutions seem far from obvious. In fact, the approach underlying the optimal-range global existence analysis from [37] relies in essential parts on quite a fragile energy structure of (1.2), no meaningful adaptation of which seems to persist upon the introduction of fluid couplings as in (1.1). In accordance with this, available results on global existence of small-data solutions for the corresponding Keller-Segel-Navier-Stokes extension of (1.2), when posed in all of $\mathbb{R}^{2}$ with initial data ( $n_{0}, c_{0}, u_{0}$ ), draw on assumptions substantially more restrictive than the above by not only requiring $\int n_{0}$ to lie below an unknown small number potentially far below $4 \pi$,
but by furthermore, and yet more drastically, involving smallness also of $\left\|c_{0}\right\|_{W^{1,2}}+\left\|c_{0}\right\|_{L^{\infty}}$ and of $\left\|u_{0}\right\|_{W^{1,2}}([29])$.
Main results. The present work addresses the latter question in the context of the initial-boundary value problem

$$
\begin{cases}n_{t}+u \cdot \nabla n=\Delta n-\nabla \cdot(n \nabla c), & x \in \Omega, t>0  \tag{1.3}\\ c_{t}+u \cdot \nabla c & =\Delta c-c+n, \\ u_{t}+(u \cdot \nabla) u=\Delta u+\nabla P+n \nabla \Phi, \quad \nabla \cdot u=0, & x \in \Omega, t>0 \\ \frac{\partial n}{\partial \nu}=\frac{\partial c}{\partial \nu}=0, \quad u=0, & x \in \partial \Omega, t>0 \\ n(x, 0)=n_{0}(x), \quad c(x, 0)=c_{0}(x), \quad u(x, 0)=u_{0}(x), & x \in \Omega\end{cases}
$$

under the standing hypotheses that $\Omega \subset \mathbb{R}^{2}$ is a bounded domain with smooth boundary, that

$$
\begin{equation*}
\Phi \in W^{2, \infty}(\Omega) \tag{1.4}
\end{equation*}
$$

and that

$$
\left\{\begin{array}{l}
n_{0} \in C^{0}(\bar{\Omega}) \quad \text { is nonnegative with } n_{0} \not \equiv 0,  \tag{1.5}\\
c_{0} \in W^{1, \infty}(\Omega) \quad \text { is nonnegative, and } \\
u_{0} \in W^{2,2}\left(\Omega ; \mathbb{R}^{2}\right) \cap W_{0, \sigma}^{1,2}(\Omega)
\end{array} \quad\right. \text { and that }
$$

where $W_{0, \sigma}^{1,2}(\Omega):=W_{0}^{1,2}\left(\Omega ; \mathbb{R}^{2}\right) \cap L_{\sigma}^{2}(\Omega)$, with $L_{\sigma}^{2}(\Omega):=\left\{\varphi \in L^{2}\left(\Omega ; \mathbb{R}^{2}\right) \mid \nabla \cdot \varphi=0\right.$ in $\left.\mathcal{D}(\Omega)\right\}$ denoting the space of all solenoidal vector fields in $L^{2}\left(\Omega ; \mathbb{R}^{2}\right)$.
Our main goal will be to develop an approach which, despite the apparent lack of any energy structure similar to that availabe for (1.2), is capable of establishing a theory of global existence and regularity under a smallness assumption merely involving the initial data through the quantity $\int_{\Omega} n_{0}$ of immediate biological relevance. Our analysis in this direction will, in its first step, be based on tracing the evolution of functionals of the form

$$
\begin{equation*}
-\int_{\Omega} \ln (n+1)+\frac{1}{2} \int_{\Omega} c^{2} \tag{1.6}
\end{equation*}
$$

which, thanks to solenoidality, is essentially unaffected by the fluid velocity field. An accordingly obtained quasi-entropy property thereof (Lemma 2.4) will, in particular, imply an a priori estimate for

$$
\int_{0}^{T} \int_{\Omega} n \ln \frac{n}{\bar{n}_{0}}, \quad \bar{n}_{0}:=\frac{1}{|\Omega|} \int_{\Omega} n_{0}
$$

which will, through the use of a functional inequality that seems to be novel in this context (Lemma 2.2 and Lemma 2.3), provide some regularity information on the forcing term appearing in the NavierStokes subsystem of (1.3) that turns out to be sufficient to allow for a favorable estimation of corresponding sources in the standard Navier-Stokes energy inequality (Lemma 2.6 and Lemma 2.7).
Resulting a priori bounds will thereby enable us to derive the following first of our main results that asserts global existence, along with some boundedness property in particular ruling out collapse into persistent Dirac-type singularities, of a solution to (1.3) in an appropriately generalized framework, under a smallness condition which indeed merely involves $\int_{\Omega} n_{0}$, and which moreover is fully explicit:

Theorem 1.1 Let $\Omega \subset \mathbb{R}^{2}$ be a bounded domain with smooth boundary and $\Phi$ comply with (1.4), and suppose that $n_{0}, c_{0}$ and $u_{0}$ satisfy (1.5) as well as

$$
\begin{equation*}
\int_{\Omega} n_{0}<2 \pi \tag{1.7}
\end{equation*}
$$

Then there exist functions

$$
\left\{\begin{array}{l}
n \in L^{\infty}\left((0, \infty) ; L^{1}(\Omega)\right),  \tag{1.8}\\
c \in L^{\infty}\left((0, \infty) ; L^{1}(\Omega)\right) \cap L_{l o c}^{2}\left([0, \infty) ; W^{1,2}(\Omega)\right) \quad \text { and } \\
u \in L_{l o c}^{\infty}\left([0, \infty) ; L_{\sigma}^{2}(\Omega)\right) \cap L_{l o c}^{2}\left([0, \infty) ; W_{0}^{1,2}\left(\Omega ; \mathbb{R}^{2}\right)\right)
\end{array}\right.
$$

such that $(n, c, u)$ forms a global generalized solution of (1.3) in the sense of Definition 2.13 below. Moreover, this solution has the property that with some $C>0$ we have

$$
\begin{equation*}
\frac{1}{T} \int_{0}^{T} \int_{\Omega} n \ln \frac{n}{\bar{n}_{0}} \leq C \quad \text { for all } T>1 \tag{1.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{T} \int_{0}^{T} \int_{\Omega}|\nabla c|^{2} \leq C \quad \text { for all } T>1 \tag{1.10}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\frac{1}{T} \int_{0}^{T} \int_{\Omega}|\nabla u|^{2} \leq C \quad \text { for all } T>1 \tag{1.11}
\end{equation*}
$$

and thus, in particular,

$$
\operatorname{ess} \liminf _{t \rightarrow \infty}\left\{\int_{\Omega} n(\cdot, t) \ln \frac{n(\cdot, t)}{\bar{n}_{0}}+\int_{\Omega}|\nabla c(\cdot, t)|^{2}+\int_{\Omega}|\nabla u(\cdot, t)|^{2}\right\}<\infty
$$

Relying on a careful tracking of constants in the above a priori estimates, we shall thereafter see that upon imposing a further restriction on the size of $\int_{\Omega} n_{0}$, one can even achieve that the quantity

$$
\int_{\Omega} n \ln \frac{n}{\bar{n}_{0}}+\frac{1}{2} \int_{\Omega}|\nabla c|^{2}+\frac{1}{2} \int_{\Omega}|u|^{2}
$$

eventually plays the role of a genuine energy functional for (1.3) (Lemma 3.1 and Lemma 3.2). Through a series of subsequent parabolic regularity arguments, this will lead us to the second of our main results, according to which any such solution eventually becomes smooth and stabilizes toward a spatially homogeneous equilibrium:

Theorem 1.2 Let $\Omega \subset \mathbb{R}^{2}$ be a bounded domain with smooth boundary. Then there exists $m_{\star} \in(0,2 \pi)$ such that whenever (1.4) and (1.5) hold with

$$
\begin{equation*}
\int_{\Omega} n_{0}<m_{\star} \tag{1.12}
\end{equation*}
$$

one can find $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ with the property that the functions $n, c$ and $u$ obtained in Theorem 1.1 satisfy

$$
\begin{equation*}
n \in C^{2,1}(\bar{\Omega} \times[T, \infty)), \quad c \in C^{2,1}(\bar{\Omega} \times[T, \infty)) \quad \text { and } \quad u \in C^{2,1}\left(\bar{\Omega} \times[T, \infty) ; \mathbb{R}^{2}\right) \tag{1.13}
\end{equation*}
$$

and that with some $P \in C^{1,0}(\Omega \times(T, \infty))$ the quadruple $(n, c, u, P)$ solves the boundary value problem in (1.3) in the classical sense in $\Omega \times(T, \infty)$. Furthermore,

$$
\begin{equation*}
n(\cdot, t) \rightarrow \bar{n}_{0} \text { in } C^{2}(\bar{\Omega}), \quad c(\cdot, t) \rightarrow \bar{n}_{0} \text { in } C^{2}(\bar{\Omega}) \quad \text { and } \quad u(\cdot, t) \rightarrow 0 \text { in } C^{2}(\bar{\Omega}) \quad \text { as } t \rightarrow \infty \tag{1.14}
\end{equation*}
$$

Interesting natural problems left open here are to clarify how far the explicit condition (1.7) indeed is optimal for the conclusion made in Theorem 1.1, and to determine the maximal size of the bound $m_{\star}$ appearing in Theorem 1.2. Indeed, the knowledge on blow-up features in the fluid-free system (1.2) ([24], [3]) suggests to conjecture that at each mass level $m>4 \pi$, integrable and mass-preserving global solutions of the considered form might not exist for appropriately chosen $\left(n_{0}, c_{0}, u_{0}\right)$ with $\int_{\Omega} n_{0}=m$, which would, inter alia, mean that (1.3) in fact inherits from (1.2) the presence of a corresponding critical mass phenomenon. However, Theorem 1.1 leaves open the possibility that, somewhat in line with the findings from [21] in the opposite direction, the fluid interaction mechanism in (1.3) might at least slightly affect the precise value of the critical blow-up mass from (1.2), in the sense of a reduction to a level between $2 \pi$ and $4 \pi$, if the initial data for the fluid velocity field satisfy appropriate constraints.

A further challenge consists in describing how far the biologically meaningful quantity $\int_{\Omega} n_{0}$ continues to play a mathematically decisive role also in three-dimensional versions or variants of (1.3); since even for fluid-free Keller-Segel systems precedent findings in this direction are yet restricted to quite strongly restricted settings, and inter alia limited to frameworks of radial symmetry which apparently cannot be created in the presence of nontrivial fluid flows ([62]), this significantly goes beyond the scope of this work, however.

## 2 Basic estimates and global solvability in the case $\int_{\Omega} n_{0}<2 \pi$

The purpose of this part is to provide some fundamental a priori estimates through an analysis of the functional in (1.6), which on the one hand will constitute our main ingredient in the construction of global solutions (see Lemma 2.14), but which on the other hand will furthermore serve as a starting point for our asymptotic analysis in Section 3. In order to have globally existing objects at hand, we shall subsequently consider the regularized versions of (1.3) given by

$$
\begin{cases}n_{\varepsilon t}+u_{\varepsilon} \cdot \nabla n_{\varepsilon}=\Delta n_{\varepsilon}-\nabla \cdot\left(\frac{n_{\varepsilon}}{1+\varepsilon n_{\varepsilon}} \nabla c_{\varepsilon}\right), & x \in \Omega, t>0  \tag{2.1}\\ c_{\varepsilon t}+u_{\varepsilon} \cdot \nabla c_{\varepsilon}=\Delta c_{\varepsilon}-c_{\varepsilon}+n_{\varepsilon}, & x \in \Omega, t>0 \\ u_{\varepsilon t}+\left(u_{\varepsilon} \cdot \nabla\right) u_{\varepsilon}=\Delta u_{\varepsilon}+\nabla P_{\varepsilon}+n_{\varepsilon} \nabla \Phi, \\ \frac{\partial n_{\varepsilon}}{\partial \nu}=\frac{\partial c_{\varepsilon}}{\partial \nu}=0, \quad u_{\varepsilon}=0, & x \in \Omega, t>0 \\ n_{\varepsilon}(x, 0)=u_{\varepsilon}(x), \quad c_{\varepsilon}(x, 0)=c_{0}(x), \quad u_{\varepsilon}(x, 0)=u_{0}(x), & x \in \partial \Omega, t>0 \\ & x \in \Omega\end{cases}
$$

for $\varepsilon \in(0,1)$, that indeed enjoy the following unconditional solvability property in which, as throughout the remaining part of this paper, we let $A=-\mathcal{P} \Delta$ denote the realization of the Stokes operator in $L_{\sigma}^{2}(\Omega)$ with domain $D(A)=W^{2,2}\left(\Omega ; \mathbb{R}^{2}\right) \cap W_{0, \sigma}^{1,2}(\Omega)$, and with $\mathcal{P}$ representing the Helmholtz projection on $L^{2}\left(\Omega ; \mathbb{R}^{2}\right)$, and in which $A^{\beta}, \beta \in \mathbb{R}$, denote the corresponding sectorial fractional powers.

Lemma 2.1 Assume (1.5), and let $\varepsilon \in(0,1)$. Then the problem (2.1) possesses a global classical solution ( $n_{\varepsilon}, c_{\varepsilon}, u_{\varepsilon}, P_{\varepsilon}$ ) with

$$
\left\{\begin{array}{l}
n_{\varepsilon} \in C^{0}(\bar{\Omega} \times[0, \infty)) \cap C^{2,1}(\bar{\Omega} \times(0, \infty)),  \tag{2.2}\\
c_{\varepsilon} \in \bigcap_{q>2} C^{0}\left([0, \infty) ; W^{1, q}(\Omega)\right) \cap C^{2,1}(\bar{\Omega} \times(0, \infty)) \quad \text { and } \\
u_{\varepsilon} \in \bigcap_{\beta \in\left(\frac{1}{2}, 1\right)} C^{0}\left([0, \infty) ; D\left(A^{\beta}\right)\right) \cap C^{2,1}(\bar{\Omega} \times(0, \infty)),
\end{array}\right.
$$

which is such that $n_{\varepsilon}>0$ and $c_{\varepsilon} \geq 0$ in $\bar{\Omega} \times(0, \infty)$. Moreover,

$$
\begin{equation*}
\int_{\Omega} n_{\varepsilon}(\cdot, t)=\int_{\Omega} n_{0} \quad \text { for all } t>0 \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\Omega} c_{\varepsilon}(\cdot, t)=\int_{\Omega} n_{0}+\left\{\int_{\Omega} c_{0}-\int_{\Omega} n_{0}\right\} \cdot e^{-t} \quad \text { for all } t>0 \tag{2.4}
\end{equation*}
$$

in particular,

$$
\begin{equation*}
\int_{\Omega} c_{\varepsilon}(\cdot, t) \leq \int_{\Omega} n_{0}+\left\{\int_{\Omega} c_{0}\right\} \cdot e^{-t} \quad \text { for all } t>0 \tag{2.5}
\end{equation*}
$$

Proof. The statements concerning global solvability and the claimed positivity propeties follow from [48]. The identities in (2.3) and (2.4) can directly be obtained by integrating the first two equations in (2.1) and using that $\nabla \cdot u_{\varepsilon}=0$, whereupon (2.5) is an evident consequence of (2.4).

### 2.1 Two functional inequalities resulting from the Moser-Trudinger inequality

Now in subsequently establishing bounds for these approximate solutions, following classical precedents we shall rely on a consequence of the Moser-Trudinger inequality which apparently goes back to the seminal work [37]. In order to be able to rather precisely control the lower-order expressions therein, and to moreover prepare the derivation of a further functional inequality from this, let us briefly recall the main steps from the argument in [37, Lemma 3.4] and thereby include a full proof here.

Lemma 2.2 Suppose that $\Omega \subset \mathbb{R}^{2}$ is a bounded domain with smooth boundary. Then for all $\varepsilon>0$ there exists $M=M(\varepsilon, \Omega)>0$ such that if $0 \not \equiv \phi \in C^{0}(\bar{\Omega})$ is nonnegative and $\psi \in W^{1,2}(\Omega)$, then for each $a>0$,

$$
\begin{equation*}
\int_{\Omega} \phi|\psi| \leq \frac{1}{a} \int_{\Omega} \phi \ln \frac{\phi}{\bar{\phi}}+\frac{(1+\varepsilon) a}{8 \pi} \cdot\left\{\int_{\Omega} \phi\right\} \cdot \int_{\Omega}|\nabla \psi|^{2}+M a \cdot\left\{\int_{\Omega} \phi\right\} \cdot\left\{\int_{\Omega}|\psi|\right\}^{2}+\frac{M}{a} \int_{\Omega} \phi \tag{2.6}
\end{equation*}
$$

where $\bar{\phi}:=\frac{1}{|\Omega|} \int_{\Omega} \phi$.
Proof. According to the Moser-Trudinger inequality ([37, Theorem 2.2]), for fixed $\varepsilon>0$ we can pick $C_{1}=C_{1}(\varepsilon, \Omega)>0$ such that

$$
\int_{\Omega} e^{|\chi|} \leq C_{1} \exp \left\{\frac{1+\varepsilon}{8 \pi} \int_{\Omega}|\nabla \chi|^{2}+C_{1}\left\{\int_{\Omega}|\chi|\right\}^{2}\right\} \quad \text { for all } \chi \in W^{1,2}(\Omega)
$$

whence given any positive $\phi \in C^{0}(\bar{\Omega})$ and an arbitrary $\psi \in W^{1,2}(\Omega)$ we can estimate

$$
\begin{equation*}
\ln \left\{\int_{\Omega} e^{a|\psi|}\right\} \leq \ln C_{1}+\frac{(1+\varepsilon) a^{2}}{8 \pi} \int_{\Omega}|\nabla \psi|^{2}+C_{1} a^{2}\left\{\int_{\Omega}|\psi|\right\}^{2} \quad \text { for all } a>0 . \tag{2.7}
\end{equation*}
$$

As, on the other hand, writing $m:=\int_{\Omega} \phi$ we can use Jensen's inequality to see that

$$
\begin{aligned}
\ln \left\{\int_{\Omega} e^{a|\psi|}\right\} & =\ln \left\{\int_{\Omega} \frac{e^{a|\psi|} m}{\phi} \cdot \frac{\phi}{m}\right\} \\
& \geq \int_{\Omega}\left\{\ln \frac{e^{a|\psi|} m}{\phi}\right\} \cdot \frac{\phi}{m} \\
& =\frac{a}{m} \int_{\Omega} \phi|\psi|-\frac{1}{m} \int_{\Omega} \phi \ln \frac{\phi}{\phi}+\ln |\Omega| \quad \text { for all } a>0,
\end{aligned}
$$

from (2.7) we infer that

$$
\int_{\Omega} \phi|\psi|-\frac{1}{a} \int_{\Omega} \phi \ln \frac{\phi}{\bar{\phi}} \leq \frac{m}{a} \ln \frac{C_{1}}{|\Omega|}+\frac{(1+\varepsilon) a m}{8 \pi} \int_{\Omega}|\nabla \psi|^{2}+C_{1} a m \cdot\left\{\int_{\Omega}|\psi|\right\}^{2} \quad \text { for all } a>0
$$

and that thus (2.6) holds with $M:=\max \left\{C_{1}, \ln \frac{C_{1}}{|\Omega|}\right\}$ for any such $\phi$.
If $\phi \in C^{0}(\bar{\Omega})$ is merely assumed to be nonnegative, then applying (2.6) to $\phi+\delta$ for $\delta>0$ and letting $\delta \searrow 0$ readily yields validity of (2.6) also in this case.
The latter lemma also entails the following functional inequality exclusively referring to a single function, and relating its size in $L \log L$ to the $H^{1}$ norm of some logarithmic derivative.
Lemma 2.3 Let $\Omega \subset \mathbb{R}^{2}$ be a bounded domain with smooth boundary, and let $0 \not \equiv \phi \in C^{0}(\bar{\Omega})$ be nonnegative. Then for any choice of $\varepsilon>0$,

$$
\begin{equation*}
\int_{\Omega} \phi \ln (\phi+1) \leq \frac{1+\varepsilon}{2 \pi} \cdot\left\{\int_{\Omega} \phi\right\} \cdot \int_{\Omega} \frac{|\nabla \phi|^{2}}{(\phi+1)^{2}}+4 M \cdot\left\{\int_{\Omega} \phi\right\}^{3}+\left\{M-\ln \left\{\frac{1}{|\Omega|} \int_{\Omega} \phi\right\}\right\} \cdot \int_{\Omega} \phi \tag{2.8}
\end{equation*}
$$

where $M=M(\varepsilon, \Omega)>0$ is as in Lemma 2.2.
Proof. An application of Lemma 2.2 to $\psi:=\ln (\phi+1)$ and $a:=2$ shows that again with $\bar{\phi}:=\frac{1}{\Omega} \int_{\Omega} \phi$ we have

$$
\begin{aligned}
\int_{\Omega} \phi \ln (\phi+1) \leq & \frac{1}{2} \int_{\Omega} \phi \ln \frac{\phi}{\bar{\phi}}+\frac{1+\varepsilon}{4 \pi} \cdot\left\{\int_{\Omega} \phi\right\} \cdot \int_{\Omega} \frac{|\nabla \phi|^{2}}{(\phi+1)^{2}} \\
& +2 M \cdot\left\{\int_{\Omega} \phi\right\} \cdot\left\{\int_{\Omega} \ln (\phi+1)\right\}^{2}+\frac{M}{2} \int_{\Omega} \phi
\end{aligned}
$$

Since herein

$$
\begin{aligned}
\frac{1}{2} \int_{\Omega} \phi \ln \frac{\phi}{\bar{\phi}} & =\frac{1}{2} \int_{\Omega} \phi \ln \phi-\frac{1}{2}\left\{\ln \left\{\frac{1}{|\Omega|} \int_{\Omega} \phi\right\}\right\} \cdot \int_{\Omega} \phi \\
& \leq \frac{1}{2} \int_{\Omega} \phi \ln (\phi+1)-\frac{1}{2}\left\{\ln \left\{\frac{1}{|\Omega|} \int_{\Omega} \phi\right\}\right\} \cdot \int_{\Omega} \phi
\end{aligned}
$$

and since moreover $\int_{\Omega} \ln (\phi+1) \leq \int_{\Omega} \phi$, upon a straightforward rearrangement this yields (2.8).

### 2.2 A fluid-independent quasi-energy structure

A first application of Lemma 2.2, involving some suitably chosen intermediate value of the parameter $a$ therein, will enable us to assert the following energy-like property of the functional in (1.6) under our overall assumption that $\int_{\Omega} n_{0}<2 \pi$. As some of our subsequently obtained inequalities from this section, in the non-decaying part of its right-hand side the estimate (2.9) will contain a contribution that can later on be adjusted so as to become conveniently small under possibly further restrictions on the size of $\int_{\Omega} n_{0}$.

Lemma 2.4 Let $m_{0} \in(0,2 \pi)$. Then there exists $K\left(m_{0}\right)>0$ with the property that whenever (1.5) holds with $\int_{\Omega} n_{0} \leq m_{0}$, one can find $C\left(n_{0}, c_{0}, u_{0}\right)>0$ such that for each $\varepsilon \in(0,1)$,

$$
\begin{gather*}
\frac{d}{d t}\left\{-\int_{\Omega} \ln \left(n_{\varepsilon}+1\right)+\frac{1}{2} \int_{\Omega} c_{\varepsilon}^{2}\right\}+\frac{1}{K\left(m_{0}\right)} \cdot\left\{\int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+\int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+\int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}\right\} \\
\leq K\left(m_{0}\right) m \ln \frac{4 \pi}{m}+C\left(n_{0}, c_{0}, u_{0}\right) e^{-2 t} \quad \text { for all } t>0, \tag{2.9}
\end{gather*}
$$

where $m:=\int_{\Omega} n_{0}$.
Proof. We first use that $\nabla \cdot u_{\varepsilon}=0$ when integrating by parts in the first equation from (2.1) to see that thanks to Young's inequality,

$$
\begin{aligned}
-\frac{d}{d t} \int_{\Omega} \ln \left(n_{\varepsilon}+1\right)+\int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}} & =\int_{\Omega} \frac{n_{\varepsilon}}{\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)^{2}} \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon} \\
& \leq \frac{1}{2} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+\frac{1}{2} \int_{\Omega} \frac{n_{\varepsilon}^{2}}{\left(1+\varepsilon n_{\varepsilon}\right)^{2}\left(n_{\varepsilon}+1\right)^{2}}\left|\nabla c_{\varepsilon}\right|^{2} \\
& \leq \frac{1}{2} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+\frac{1}{2} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2} \quad \text { for all } t>0 .
\end{aligned}
$$

Next, again by solenoidality of $u_{\varepsilon}$,

$$
\frac{1}{2} \frac{d}{d t} \int_{\Omega} c_{\varepsilon}^{2}+\int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}=-\int_{\Omega} c_{\varepsilon}^{2}+\int_{\Omega} n_{\varepsilon} c_{\varepsilon} \leq \int_{\Omega} n_{\varepsilon} c_{\varepsilon} \quad \text { for all } t>0
$$

so that

$$
\begin{equation*}
\frac{d}{d t}\left\{-\int_{\Omega} \ln \left(n_{\varepsilon}+1\right)+\frac{1}{2} \int_{\Omega} c_{\varepsilon}^{2}\right\}+\frac{1}{2} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+\frac{1}{2} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2} \leq \int_{\Omega} n_{\varepsilon} c_{\varepsilon} \quad \text { for all } t>0 \tag{2.10}
\end{equation*}
$$

In order to appropriately estimate the integral on the right-hand side herein, we rely on our assumption that $m_{0}<2 \pi$ in observing that $\frac{m_{0}}{\pi}<\frac{4 \pi}{m_{0}}$, whence it is possible to find $a>0$ such that

$$
\begin{equation*}
\frac{m_{0}}{\pi}<a<\frac{4 \pi}{m_{0}} \tag{2.11}
\end{equation*}
$$

As these two inequalities warrant that $\frac{1}{a} \cdot \frac{m_{0}}{2 \pi}<\frac{1}{2}$ and $\frac{a m_{0}}{8 \pi}<\frac{1}{2}$, we can choose $\eta \in\left(0, \frac{1}{a}\right)$ small enough fulfilling both

$$
\begin{equation*}
\left(\frac{1}{a}+\eta\right) \cdot \frac{(1+\eta) m_{0}}{2 \pi}<\frac{1}{2} \tag{2.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{(1+\eta) a m_{0}}{8 \pi}<\frac{1}{2} \tag{2.13}
\end{equation*}
$$

We now apply Lemma 2.2 to these values of $a$ and $\eta$ to see that with $M=M(\eta, \Omega)>0$ as provided there, due to (2.3) and our assumption that $m \leq m_{0}$ we have

$$
\int_{\Omega} n_{\varepsilon} c_{\varepsilon} \leq \frac{1}{a} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+\frac{(1+\eta) a m}{8 \pi} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+M a m \cdot\left\{\int_{\Omega} c_{\varepsilon}\right\}^{2}+\frac{M m}{a} \quad \text { for all } t>0,
$$

and that hence, by Lemma 2.3,

$$
\begin{align*}
\int_{\Omega} n_{\varepsilon} c_{\varepsilon}+\eta \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\overline{n_{0}}} \leq & \left(\frac{1}{a}+\eta\right) \cdot \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+\frac{(1+\eta) a m}{8 \pi} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+M a m \cdot\left\{\int_{\Omega} c_{\varepsilon}\right\}^{2}+\frac{M m}{a} \\
= & \left(\frac{1}{a}+\eta\right) \cdot \int_{\Omega} n_{\varepsilon} \ln n_{\varepsilon}-\left(\frac{1}{a}+\eta\right) \cdot m \ln \frac{m}{|\Omega|} \\
& +\frac{(1+\eta) a m}{8 \pi} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+M a m \cdot\left\{\int_{\Omega} c_{\varepsilon}\right\}^{2}+\frac{M m}{a} \\
\leq & \left(\frac{1}{a}+\eta\right) \cdot\left\{\frac{(1+\eta) m}{2 \pi} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+4 M m^{3}+\left(M-\ln \frac{m}{|\Omega|}\right) \cdot m\right\} \\
& -\left(\frac{1}{a}+\eta\right) \cdot m \ln \frac{m}{|\Omega|} \\
& +\frac{(1+\eta) a m}{8 \pi} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+M a m \cdot\left\{\int_{\Omega} c_{\varepsilon}\right\}^{2}+\frac{M m}{a} \\
= & \left(\frac{1}{a}+\eta\right) \cdot \frac{(1+\eta) m}{2 \pi} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+\left(\frac{1}{a}+\eta\right) \cdot 4 M m^{3}+\left(\frac{1}{a}+\eta\right) \cdot M m \\
& +\frac{(1+\eta) a m}{8 \pi} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+M a m \cdot\left\{\int_{\Omega} c_{\varepsilon}\right\}^{2}+\frac{M m}{a} \\
& +2\left(\frac{1}{a}+\eta\right) \cdot m \ln \frac{4 \pi}{m}+2\left(\frac{1}{a}+\eta\right) \cdot m \ln \frac{|\Omega|}{4 \pi} \quad \text { for all } t>0 . \tag{2.14}
\end{align*}
$$

Here by Lemma 2.1, (2.11) and Young's inequality,

$$
\begin{aligned}
\operatorname{Mam} \cdot\left\{\int_{\Omega} c_{\varepsilon}\right\}^{2} & \leq M a m \cdot\left\{m+\left\{\int_{\Omega} c_{0}\right\} \cdot e^{-t}\right\}^{2} \\
& \leq 4 \pi M \cdot\left\{2 m^{2}+\left\{\int_{\Omega} c_{0}\right\}^{2} \cdot e^{-2 t}\right\} \\
& \leq 8 \pi M m^{2}+8 \pi M \cdot\left\{\int_{\Omega} c_{0}\right\}^{2} \cdot e^{-2 t} \quad \text { for all } t>0
\end{aligned}
$$

while clearly, thanks to the inequality $\eta \leq \frac{1}{a}$ and again (2.11),

$$
\left(\frac{1}{a}+\eta\right) \cdot 4 M m^{3}+\left(\frac{1}{a}+\eta\right) \cdot M m+\frac{M m}{a} \leq \frac{8 M m}{a} \cdot m^{2}+\frac{3 M}{a} \cdot m \leq 8 \pi M m^{2}+\frac{3 \pi M}{m_{0}} \cdot m .
$$

Therefore, (2.14) together with (2.10) shows that once more since $m \leq m_{0}$,

$$
\begin{align*}
& \frac{d}{d t}\left\{-\int_{\Omega} \ln \left(n_{\varepsilon}+1\right)+\frac{1}{2} \int_{\Omega} c_{\varepsilon}^{2}\right\}+C_{1} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+C_{2} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+\eta \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}} \\
& \leq \\
& \quad 16 \pi M m^{2}+\frac{3 \pi M}{m_{0}} \cdot m+2\left(\frac{1}{a}+\eta\right) \cdot m \ln \frac{4 \pi}{m}+2\left(\frac{1}{a}+\eta\right) \cdot m \ln _{+} \frac{|\Omega|}{4 \pi}  \tag{2.15}\\
& \quad+8 \pi M \cdot\left\{\int_{\Omega} c_{0}\right\}^{2} \cdot e^{-2 t} \quad \text { for all } t>0,
\end{align*}
$$

where $C_{1} \equiv C_{1}\left(m_{0}\right):=\frac{1}{2}-\left(\frac{1}{a}+\eta\right) \cdot \frac{(1+\eta) m_{0}}{2 \pi}$ and $C_{2} \equiv C_{2}\left(m_{0}\right):=\frac{1}{2}-\frac{(1+\eta) a m_{0}}{8 \pi}$ are both positive by (2.12) and (2.13), and where $\ln _{+} \xi:=\max \{\ln \xi, 0\}$ for $\xi>0$.

It remains to note that our restriction $m \leq m_{0} \leq 2 \pi$ ensures that $\ln \frac{4 \pi}{m} \geq \ln 2$ and hence

$$
\begin{aligned}
16 \pi M m^{2} & +\frac{3 \pi M}{m_{0}} \cdot m+2\left(\frac{1}{a}+\eta\right) \cdot m \ln \frac{4 \pi}{m}+2\left(\frac{1}{a}+\eta\right) \cdot m \ln _{+} \frac{|\Omega|}{4 \pi} \\
& =\left\{\frac{16 \pi M m}{\ln \frac{4 \pi}{m}}+\frac{3 \pi M}{m_{0} \ln \frac{4 \pi}{m}}+2\left(\frac{1}{a}+\eta\right)+\frac{2\left(\frac{1}{a}+\eta\right) \ln _{+} \frac{|\Omega|}{4 \pi}}{\ln \frac{4 \pi}{m}}\right\} \cdot m \ln \frac{4 \pi}{m} \\
& \leq\left\{\frac{32 \pi^{2} M}{\ln 2}+\frac{3 \pi M}{m_{0} \ln 2}+2\left(\frac{1}{a}+\eta\right)+\frac{2\left(\frac{1}{a}+\eta\right) \ln _{+} \frac{|\Omega|}{4 \pi}}{\ln 2}\right\} \cdot m \ln \frac{4 \pi}{m}
\end{aligned}
$$

so that (2.9) becomes a consequence of (2.15) upon evident choices of $K\left(m_{0}\right)$ and $C\left(n_{0}, c_{0}, u_{0}\right)$.
A straighforward implication of the latter is contained in the following lemma, which in view of twofold application below we formulate in such a way that it does not only yield some $\varepsilon$-independent bounds for the integrated versions of the corresponding dissipation rate functionals in (2.9), but that moreover already some large-time stabilization of the first solution component to a potantially small level, depending on possible further smallness requirements on $\int_{\Omega} n_{0}$ and to be made more precise e.g. in Lemma 3.2, is foreshadowed.

Lemma 2.5 For each $m_{0} \in(0,2 \pi)$, there exists $K\left(m_{0}\right)>0$ such that if (1.5) holds with $\int_{\Omega} n_{0} \leq m_{0}$, then one can pick $C\left(n_{0}, c_{0}, u_{0}\right)>0$ such that

$$
\begin{equation*}
\frac{1}{T} \cdot\left\{\int_{0}^{T} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+\int_{0}^{T} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+\int_{0}^{T} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}\right\} \leq K\left(m_{0}\right) m \ln \frac{4 \pi}{m}+\frac{C\left(n_{0}, c_{0}, u_{0}\right)}{T} \tag{2.16}
\end{equation*}
$$

for all $T>0$ and $\varepsilon \in(0,1)$, where again $m:=\int_{\Omega} n_{0}$.
Proof. We only need to integrate (2.9) and note that $0 \leq \int_{\Omega} \ln \left(n_{\varepsilon}+1\right) \leq \int_{\Omega} n_{\varepsilon}=\int_{\Omega} n_{0}$ for all $t \geq 0$ according to (2.3), and that $\frac{1}{T} \int_{0}^{T} e^{-2 t} d t=\frac{1-e^{-2 T}}{2 T} \leq \frac{1}{2 T}$ for all $T>0$.
For our mere construction of global solutions, the particular dependences of the expressions on the right-hand side in (2.16) are actually irrelevant:

Lemma 2.6 Assume (1.5) with $\int_{\Omega} n_{0}<2 \pi$. Then for all $T>0$ there exists $C\left(n_{0}, c_{0}, u_{0}, T\right)>0$ such that

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}} \leq C\left(n_{0}, c_{0}, u_{0}, T\right) \quad \text { for all } \varepsilon \in(0,1) \tag{2.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}} \leq C\left(n_{0}, c_{0}, u_{0}, T\right) \quad \text { for all } \varepsilon \in(0,1) \tag{2.18}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2} \leq C\left(n_{0}, c_{0}, u_{0}, T\right) \quad \text { for all } \varepsilon \in(0,1) \tag{2.19}
\end{equation*}
$$

Proof. This is an immediate by-product of Lemma 2.5 and (2.3) due to the well-known fact that for all positive $\phi \in C^{0}(\bar{\Omega})$, once more writing $\bar{\phi}:=\frac{1}{|\Omega|} \int_{\Omega} \phi$ we have

$$
\frac{1}{|\Omega|} \int_{\Omega} \phi \ln \frac{\phi}{\bar{\phi}} d x=\int_{\Omega} \phi \ln \phi \frac{d x}{|\Omega|}-\left\{\int_{\Omega} \phi \frac{d x}{|\Omega|}\right\} \cdot \ln \left\{\int_{\Omega} \phi \frac{d x}{|\Omega|}\right\} \geq 0
$$

according to Jensen's inequality.

### 2.3 Basic regularity features of the fluid flow

Now an issue of fundamental importance for any analysis of the fluid interaction in (1.3) seems to consist in the problem of deriving suitable regularity properties of the forcing term in the corresponding Navier-Stokes subsystem, where especially in view of the presence of the nonlinear convective term therein, the basic mass conservation feature (2.3) seems insufficient for any meaningful approach. Accordingly left with an analysis of the standard Navier-Stokes energy inequality, due to apparently lacking further knowledge on regularity of $n_{\varepsilon}$ we need to exclusively rely on the sparse information provided by Lemma 2.6. Fortunately, a second application of Lemma 2.2, now to some suitably large $a>0$, will reveal that the $L \log L$ bound in (2.18), even though available only in a temporally integrated form, yields sufficient regularity in this regard:

Lemma 2.7 There exists $K>0$ such that whenever (1.5) holds and $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega}\left|u_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq K m \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+K m^{2} \quad \text { for all } t>0 \tag{2.20}
\end{equation*}
$$

with $m:=\int_{\Omega} n_{0}$.
Proof. We abbreviate $C_{1}:=\|\nabla \Phi\|_{L^{\infty}(\Omega)}$ and invoke the Poincaré inequality to fix $C_{2}>0$ fulfilling

$$
\begin{equation*}
\int_{\Omega}|\varphi|^{2} \leq C_{2} \int_{\Omega}|\nabla \varphi|^{2} \quad \text { for all } \varphi \in W_{0}^{1,2}\left(\Omega ; \mathbb{R}^{2}\right) \tag{2.21}
\end{equation*}
$$

Then taking $M:=M(1, \Omega)>0$ as provided by Lemma 2.2, given $\left(n_{0}, c_{0}, u_{0}\right)$ satisfying (1.5) with $m=\int_{\Omega} n_{0}$ we apply the latter to $a:=\frac{C_{3}}{m}$ with $C_{3}:=\frac{1}{2} \cdot \frac{1}{\frac{C_{1}}{2 \pi}+2 C_{1} C_{2} M|\Omega|}$ to see that since $u_{\varepsilon}=\left(u_{\varepsilon 1}, u_{\varepsilon 2}\right)$ satisfies

$$
\left\{\int_{\Omega}\left|u_{\varepsilon}\right|\right\}^{2} \leq|\Omega| \int_{\Omega}\left|u_{\varepsilon}\right|^{2} \leq C_{2}|\Omega| \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \quad \text { for all } t>0
$$

by the Cauchy-Schwarz inequality and (2.21), we have

$$
\begin{aligned}
\int_{\Omega} n_{\varepsilon} u_{\varepsilon} \cdot \nabla \Phi & \leq C_{1} \int_{\Omega} n_{\varepsilon}\left|u_{\varepsilon}\right| \\
& \leq C_{1} \int_{\Omega} n_{\varepsilon}\left|u_{\varepsilon 1}\right|+C_{1} \int_{\Omega} n_{\varepsilon}\left|u_{\varepsilon 2}\right| \\
& \leq \frac{2 C_{1}}{a} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+\frac{C_{1} a m}{2 \pi} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}+2 C_{1} M a m \cdot\left\{\int_{\Omega}\left|u_{\varepsilon}\right|\right\}^{2}+\frac{2 C_{1} M m}{a} \\
& \leq \frac{2 C_{1}}{a} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+\left\{\frac{C_{1} a m}{2 \pi}+2 C_{1} C_{2} M|\Omega| a m\right\} \cdot \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}+\frac{2 C_{1} M m}{a} \\
& =\frac{2 C_{1} m}{C_{3}} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+\frac{1}{2} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}+\frac{2 C_{1} M m^{2}}{C_{3}} \quad \text { for all } t>0
\end{aligned}
$$

Therefore, testing the third equation in (2.1) against $u_{\varepsilon}$ shows that

$$
\frac{d}{d t} \int_{\Omega}\left|u_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq \frac{4 C_{1} m}{C_{3}} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+\frac{4 C_{1} M m^{2}}{C_{3}} \quad \text { for all } t>0
$$

and thus establishes (2.20).
A first integration in (2.20) indeed yields $L^{2}$ bounds for $u_{\varepsilon}$.
Lemma 2.8 Suppose that (1.5) is satisfied with $\int_{\Omega} n_{0}<2 \pi$. Then for all $T>0$ there exists $C\left(n_{0}, c_{0}, u_{0}, T\right)>0$ such that

$$
\begin{equation*}
\int_{\Omega}\left|u_{\varepsilon}(\cdot, t)\right|^{2} \leq C\left(n_{0}, c_{0}, u_{0}, T\right) \quad \text { for all } t \in(0, T) \text { and } \varepsilon \in(0,1) \tag{2.22}
\end{equation*}
$$

Proof. Assuming (1.5), by means of Lemma 2.7 we can fix $C_{1}=C_{1}\left(n_{0}, c_{0}, u_{0}\right)>0$ such that for all $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega}\left|u_{\varepsilon}\right|^{2} \leq C_{1} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+C_{1} \quad \text { for all } t>0 \tag{2.23}
\end{equation*}
$$

while Lemma 2.6 says that given any $T>0$ we can find $C_{2}=C_{2}\left(n_{0}, c_{0}, u_{0}, T\right)>0$ satisfying

$$
\begin{equation*}
\int_{0}^{T} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}} \leq C_{2} \quad \text { for all } \varepsilon \in(0,1) \tag{2.24}
\end{equation*}
$$

An integration of (2.23) thus implies that for any such $T$ and each $\varepsilon \in(0,1)$,

$$
\int_{\Omega}\left|u_{\varepsilon}\right|^{2} \leq \int_{\Omega}\left|u_{0}\right|^{2}+C_{1} C_{2}+C_{1} T \quad \text { for all } t \in(0, T)
$$

and that hence $(2.22)$ is valid with some suitably large $C\left(n_{0}, c_{0}, u_{0}, T\right)>0$.
Secondly, by focusing on the dissipation rate therein we infer from (2.20) the following further and now again semi-quantitative information.

Lemma 2.9 Let $m_{0} \in(0,2 \pi)$. Then there exists $K\left(m_{0}\right)>0$ such that if (1.5) is valid with $\int_{\Omega} n_{0} \leq$ $m_{0}$, one can find $C\left(n_{0}, c_{0}, u_{0}\right)>0$ with the property that whenever $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\frac{1}{T} \int_{0}^{T} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq K\left(m_{0}\right) m^{2} \ln \frac{4 \pi}{m}+\frac{C\left(n_{0}, c_{0}, u_{0}\right)}{T} \quad \text { for all } T>0 \tag{2.25}
\end{equation*}
$$

where again $m:=\int_{\Omega} n_{0}$.
Proof. According to Lemma 2.7, there exists $K_{1}>0$ such that for arbitrary ( $n_{0}, c_{0}, u_{0}$ ) fulfilling (1.5) with $m=\int_{\Omega} n_{0}>0$ we have

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega}\left|u_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq K_{1} m \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+K_{1} m^{2} \quad \text { for all } t>0 \text { and } \varepsilon \in(0,1) \tag{2.26}
\end{equation*}
$$

Furthermore, relying on the assumption $m_{0}<2 \pi$ we may employ Lemma 2.5 to infer the existence of $K_{2}\left(m_{0}\right)>0$ such that whenever (1.5) holds with $m:=\int_{\Omega} n_{0}$ satisfying $m \leq m_{0}$, one can find $C_{1}\left(n_{0}, c_{0}, u_{0}\right)>0$ such that the solution of (2.1) has the property that

$$
\begin{equation*}
\frac{1}{T} \int_{0}^{T} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}} \leq K_{2}\left(m_{0}\right) m \ln \frac{4 \pi}{m}+\frac{C_{1}\left(n_{0}, c_{0}, u_{0}\right)}{T} \quad \text { for all } T>0 \text { and } \varepsilon \in(0,1) \tag{2.27}
\end{equation*}
$$

An integration of (2.26) hence imples that for any such $\left(n_{0}, c_{0}, u_{0}\right)$,

$$
\begin{aligned}
\frac{1}{T} \int_{\Omega}\left|u_{\varepsilon}(\cdot, T)\right|^{2}+\frac{1}{T} \int_{0}^{T} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} & \leq \frac{1}{T} \int_{\Omega}\left|u_{0}\right|^{2}+\frac{K_{1} m}{T} \int_{0}^{T} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+K_{1} m^{2} \\
& \leq \frac{1}{T} \int_{\Omega}\left|u_{0}\right|^{2}+K_{1} K_{2}\left(m_{0}\right) m^{2} \ln \frac{4 \pi}{m}+\frac{K_{1} m C_{1}\left(n_{0}, c_{0}, u_{0}\right)}{T}+K_{1} m^{2}
\end{aligned}
$$

for all $T>0$ and $\varepsilon \in(0,1)$. Since

$$
\frac{K_{1} m^{2}}{m^{2} \ln \frac{4 \pi}{m}}=\frac{K_{1}}{\ln \frac{4 \pi}{m}} \leq \frac{K_{1}}{\ln 2}
$$

due to our restriction that $m \leq m_{0} \leq 2 \pi$, this already entails (2.25) if we let $K\left(m_{0}\right):=K_{1} K_{2}\left(m_{0}\right)+\frac{K_{1}}{\ln 2}$ and $C\left(n_{0}, c_{0}, u_{0}\right):=\int_{\Omega}\left|u_{0}\right|^{2}+K_{1} m C_{1}\left(n_{0}, c_{0}, u_{0}\right)$.

### 2.4 Estimates for time derivatives

On the basis of the information gained in Lemma 2.6, Lemma 2.8 and Lemma 2.9, a derivation of some regularity features with respect to time can be achieved in quite a straightforward manner.

Lemma 2.10 Assume (1.5), and let $T>0$. Then there exists $C\left(n_{0}, c_{0}, u_{0}, T\right)>0$ such that

$$
\begin{equation*}
\int_{0}^{T}\left\|\partial_{t} \ln \left(n_{\varepsilon}(\cdot, t)+1\right)\right\|_{\left(W^{2,2}(\Omega)\right)^{\star}} d t \leq C\left(n_{0}, c_{0}, u_{0}, T\right) \quad \text { for all } \varepsilon \in(0,1) \tag{2.28}
\end{equation*}
$$

Proof. We pick $C_{1}>0$ such that $\|\varphi\|_{L^{\infty}(\Omega)}+\|\nabla \varphi\|_{L^{2}(\Omega)} \leq C_{1}\|\varphi\|_{W^{2,2}(\Omega)}$ for all $\varphi \in W^{2,2}(\Omega)$, and then use (2.1) to see that for fixed $\varphi \in C^{\infty}(\bar{\Omega})$ such that $\|\varphi\|_{W^{2,2}(\Omega)} \leq 1$, according to Young's inequality and the Cauchy-Schwarz inequality we have

$$
\begin{aligned}
\left|\int_{\Omega} \partial_{t} \ln \left(n_{\varepsilon}+1\right) \varphi\right|= & \left\lvert\, \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}} \varphi-\int_{\Omega} \frac{n_{\varepsilon}}{\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)^{2}}\left(\nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon}\right) \varphi\right. \\
& -\int_{\Omega} \frac{n_{\varepsilon}}{\left(n_{\varepsilon}+1\right)^{2}}\left(u_{\varepsilon} \cdot \nabla n_{\varepsilon}\right) \varphi-\int_{\Omega} \frac{1}{n_{\varepsilon}+1} \nabla n_{\varepsilon} \cdot \nabla \varphi \\
& \left.+\int_{\Omega} \frac{n_{\varepsilon}}{\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)} \nabla c_{\varepsilon} \cdot \nabla \varphi+\int_{\Omega} \frac{n_{\varepsilon}}{n_{\varepsilon}+1}\left(u_{\varepsilon} \cdot \nabla \varphi\right) \right\rvert\, \\
\leq & \left\{\int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}\right\} \cdot\|\varphi\|_{L^{\infty}(\Omega)} \\
& +\left\{\frac{1}{2} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+\frac{1}{2} \int_{\Omega} \frac{n_{\varepsilon}^{2}}{\left(1+\varepsilon n_{\varepsilon}\right)^{2}\left(n_{\varepsilon}+1\right)^{2}}\left|\nabla c_{\varepsilon}\right|^{2}\right\} \cdot\|\varphi\|_{L^{\infty}(\Omega)} \\
& +\left\{\frac{1}{2} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+\frac{1}{2} \int_{\Omega} \frac{n_{\varepsilon}^{2}}{\left(n_{\varepsilon}+1\right)^{2}}\left|u_{\varepsilon}\right|^{2}\right\} \cdot\|\varphi\|_{L^{\infty}(\Omega)} \\
& +\left\{\int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}\right\}^{\frac{1}{2}} \cdot\|\nabla \varphi\|_{L^{2}(\Omega)} \\
& +\left\{\int_{\Omega} \frac{n_{\varepsilon}^{2}}{\left(1+\varepsilon n_{\varepsilon}\right)^{2}\left(n_{\varepsilon}+1\right)^{2}}\left|\nabla c_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\|\nabla \varphi\|_{L^{2}(\Omega)} \\
& +\left\{\int_{\Omega} \frac{n_{\varepsilon}^{2}}{\left(n_{\varepsilon}+1\right)^{2}}\left|u_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\|\nabla \varphi\|_{L^{2}(\Omega)} \\
\leq & \frac{5 C_{1}}{2} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}}+C_{1} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+C_{1} \int_{\Omega}\left|u_{\varepsilon}\right|^{2}+\frac{3 C_{1}}{2}
\end{aligned}
$$

for all $t>0$ and $\varepsilon \in(0,1)$. In view of Lemma 2.6 and Lemma 2.8, the claim thus results upon integration.

Lemma 2.11 If (1.5) holds, then for all $T>0$ there exists $C\left(n_{0}, c_{0}, u_{0}, T\right)>0$ such that

$$
\begin{equation*}
\int_{0}^{T}\left\|c_{\varepsilon t}(\cdot, t)\right\|_{\left(W^{2,2}(\Omega)\right)^{*}} d t \leq C\left(n_{0}, c_{0}, u_{0}, T\right) \quad \text { for all } \varepsilon \in(0,1) \tag{2.29}
\end{equation*}
$$

Proof. Given any $\varphi \in C^{\infty}(\bar{\Omega})$, by means of (2.1), the Cauchy-Schwarz inequality and Young's inequality we can estimate

$$
\begin{aligned}
\left|\int_{\Omega} c_{\varepsilon t} \varphi\right|= & \left|-\int_{\Omega} \nabla c_{\varepsilon} \cdot \nabla \varphi-\int_{\Omega} c_{\varepsilon} \varphi+\int_{\Omega} n_{\varepsilon} \varphi-\int_{\Omega}\left(u_{\varepsilon} \cdot \nabla c_{\varepsilon}\right) \varphi\right| \\
\leq & \left\{\int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\|\nabla \varphi\|_{L^{2}(\Omega)}+\left\{\int_{\Omega} c_{\varepsilon}\right\} \cdot\|\varphi\|_{L^{\infty}(\Omega)} \\
& +\left\{\int_{\Omega} n_{\varepsilon}\right\} \cdot \varphi\left\|_{L^{\infty}(\Omega)}+\left\{\int_{\Omega}\left|u_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\left\{\int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\right\| \varphi \|_{L^{\infty}(\Omega)}
\end{aligned}
$$

$$
\begin{aligned}
\leq & \left\{\frac{1}{2} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+\frac{1}{2}\right\} \cdot\|\nabla \varphi\|_{L^{2}(\Omega)} \\
& +\left\{\int_{\Omega} c_{\varepsilon}+\int_{\Omega} n_{\varepsilon}+\frac{1}{2} \int_{\Omega}\left|u_{\varepsilon}\right|^{2}+\frac{1}{2} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}\right\} \cdot\|\varphi\|_{L^{\infty}(\Omega)} \quad \text { for all } t>0 \text { and } \varepsilon \in(0,1)
\end{aligned}
$$

Again since $W^{2,2}(\Omega) \hookrightarrow L^{\infty}(\Omega)$, due to Lemma 2.6, Lemma 2.1 and Lemma 2.8 this entails (2.29).
Lemma 2.12 Suppose that (1.5) holds, and that $T>0$. Then there exists $C\left(n_{0}, c_{0}, u_{0}, T\right)>0$ such that

$$
\begin{equation*}
\int_{0}^{T}\left\|u_{\varepsilon t}(\cdot, t)\right\|_{\left(W^{2,2}(\Omega) \cap W_{0, \sigma}^{1,2}(\Omega)\right)^{\star}} d t \leq C\left(n_{0}, c_{0}, u_{0}, T\right) \quad \text { for all } \varepsilon \in(0,1) \tag{2.30}
\end{equation*}
$$

Proof. We take $\varphi \in C_{0}^{\infty}\left(\Omega ; \mathbb{R}^{2}\right)$ such that $\nabla \cdot \varphi=0$, and use the third equation in (2.1) along with the Cauchy-Schwarz inequality and Young's inequality to estimate

$$
\begin{aligned}
\left|\int_{\Omega} u_{\varepsilon t} \cdot \varphi\right|= & \left|-\int_{\Omega} \nabla u_{\varepsilon} \cdot \nabla \varphi-\int_{\Omega}\left(u_{\varepsilon} \cdot \nabla\right) u_{\varepsilon} \cdot \varphi+\int_{\Omega} n_{\varepsilon}(\varphi \cdot \nabla \Phi)\right| \\
\leq & \left\{\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\|\nabla \varphi\|_{L^{2}(\Omega)}+\left\{\int_{\Omega}\left|u_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\left\{\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot\|\varphi\|_{L^{\infty}(\Omega)} \\
& +\|\nabla \Phi\|_{L^{\infty}(\Omega)} \cdot\left\{\int_{\Omega} n_{\varepsilon}\right\} \cdot\|\varphi\|_{L^{\infty}(\Omega)} \\
\leq & \left\{\frac{1}{2} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}+\frac{1}{2}\right\} \cdot\|\nabla \varphi\|_{L^{2}(\Omega)}+\left\{\frac{1}{2} \int_{\Omega}\left|u_{\varepsilon}\right|^{2}+\frac{1}{2} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right\} \cdot\|\varphi\|_{L^{\infty}(\Omega)} \\
& +\|\nabla \Phi\|_{L^{\infty}(\Omega)} \cdot\left\{\int_{\Omega} n_{\varepsilon}\right\} \cdot\|\varphi\|_{L^{\infty}(\Omega)} \quad \text { for all } t>0 \text { and } \varepsilon \in(0,1),
\end{aligned}
$$

so that (2.30) follows from Lemma 2.9, Lemma 2.8, (2.3) and, again, the continuity of the embedding $W^{2,2}(\Omega) \hookrightarrow L^{\infty}(\Omega)$.

### 2.5 Passing to the limit. Global existence of generalized solutions

In order to design a solution framework compatible with the sparse regularity information gathered above, we resort to a concept which especially with respect to the crucial first solution component does not require substantially more than mere integrability, along with some very mild hypotheses concerning integrability of weighted relatives of $\nabla n$, quite in the style of those asserted e.g. by Lemma 2.6. The following notion of generalized solvability is oriented along precedents from related chemotaxis problems involving poor regularity information (cf. e.g. [57], [30], [47]), and in its most essential part it postulates $n$ to have some supersolution feature with regard to its sub-problem in (1.3), accompanied by an appropriate mass conservation property:

Definition 2.13 Let

$$
\left\{\begin{array}{l}
n \in L^{\infty}\left((0, \infty) ; L^{1}(\Omega)\right)  \tag{2.31}\\
c \in L_{l o c}^{\infty}\left((0, \infty) ; L^{1}(\Omega)\right) \cap L_{l o c}^{2}\left([0, \infty) ; W^{1,2}(\Omega)\right) \quad \text { and } \\
u \in L_{l o c}^{2}\left([0, \infty) ; W_{0, \sigma}^{1,2}(\Omega)\right)
\end{array}\right.
$$

be such that $n \geq 0$ and $c \geq 0$ a.e. in $\Omega \times(0, \infty)$. Then we call $(n, c, u)$ a global generalized solution of (1.3) if

$$
\begin{align*}
-\int_{0}^{\infty} \int_{\Omega} c \varphi_{t}-\int_{\Omega} c_{0} \varphi(\cdot, 0)= & -\int_{0}^{\infty} \int_{\Omega} \nabla c \cdot \nabla \varphi-\int_{0}^{\infty} \int_{\Omega} c \varphi+\int_{0}^{\infty} \int_{\Omega} n \varphi \\
& +\int_{0}^{\infty} \int_{\Omega} c(u \cdot \nabla \varphi) \tag{2.32}
\end{align*}
$$

for all $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty))$, if

$$
\begin{equation*}
-\int_{0}^{\infty} \int_{\Omega} u \cdot \varphi_{t}-\int_{\Omega} u_{0} \cdot \varphi(\cdot, 0)=-\int_{0}^{\infty} \int_{\Omega} \nabla u \cdot \nabla \varphi+\int_{0}^{\infty} \int_{\Omega}(u \otimes u) \cdot \nabla \varphi+\int_{0}^{\infty} \int_{\Omega} n(\varphi \cdot \nabla \Phi) \tag{2.33}
\end{equation*}
$$

for all $\varphi \in C_{0}^{\infty}\left(\Omega \times[0, \infty) ; \mathbb{R}^{2}\right)$ such that $\nabla \cdot \varphi=0$, and if there exist positive functions $\psi \in C^{2}([0, \infty))$ and $\rho \in C^{2}([0, \infty)) \cap W^{2, \infty}((0, \infty))$ such that $\psi^{\prime}<0$ on $[0, \infty)$, that

$$
\begin{equation*}
\sqrt{\left|\psi^{\prime \prime}(n)\right|} \nabla n, \psi^{\prime}(n) \nabla n, n \psi^{\prime \prime}(n) \nabla n \text { and } n \psi^{\prime}(n) \nabla c \quad \text { belong to } L_{l o c}^{2}\left(\bar{\Omega} \times[0, \infty) ; \mathbb{R}^{2}\right) \tag{2.34}
\end{equation*}
$$

that

$$
\begin{align*}
-\int_{0}^{\infty} \int_{\Omega} \psi & \psi(n) \rho(c) \varphi_{t}-\int_{\Omega} \psi\left(n_{0}\right) \rho\left(c_{0}\right) \varphi(\cdot, 0) \\
\leq & -\int_{0}^{\infty} \int_{\Omega} \psi^{\prime \prime}(n) \rho(c)|\nabla n|^{2} \varphi \\
& +\int_{0}^{\infty} \int_{\Omega}\left\{-2 \psi^{\prime}(n) \rho^{\prime}(c)+n \psi^{\prime \prime}(n) \rho(c)\right\}(\nabla n \cdot \nabla c) \varphi \\
& +\int_{0}^{\infty} \int_{\Omega}\left\{-\psi(n) \rho^{\prime \prime}(c)+n \psi^{\prime}(n) \rho^{\prime}(c)\right\}|\nabla c|^{2} \varphi \\
& -\int_{0}^{\infty} \int_{\Omega} \psi^{\prime}(n) \rho(c) \nabla n \cdot \nabla \varphi \\
& +\int_{0}^{\infty} \int_{\Omega}\left\{n \psi^{\prime}(n) \rho(c)-\psi(n) \rho^{\prime}(c)\right\} \nabla c \cdot \nabla \varphi \\
& +\int_{0}^{\infty} \int_{\Omega} \psi(n) \rho(c)(u \cdot \nabla \varphi) \\
& -\int_{0}^{\infty} \int_{\Omega} \psi(n) c \rho^{\prime}(c) \varphi+\int_{0}^{\infty} \int_{\Omega} n \psi(n) \rho^{\prime}(c) \varphi \tag{2.35}
\end{align*}
$$

for all nonnegative $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty))$, and that

$$
\begin{equation*}
\int_{\Omega} n(\cdot, t)=\int_{\Omega} n_{0} \quad \text { for a.e. } t>0 \tag{2.36}
\end{equation*}
$$

Here, for given vectors $v \in \mathbb{R}^{3}$ and $w \in \mathbb{R}^{3}$ we have defined the matrix $v \otimes w$ by letting $(v \otimes w)_{i j}:=v_{i} w_{j}$ for $i, j \in\{1,2,3\}$.

Remark. i) Based on a series of essentially well-known arguments, it can be verified that this concept is consistent with that of classical solvability in the sense that suitably smooth generalized solutions actually must solve (1.3) classically. Indeed, if $n, c$ and $u$ are suitably smooth in $\bar{\Omega} \times[0, \infty)$, then standard reasonings relying on the Du Bois-Reymond lemma firstly warrant that due to the identities in (2.32) and (2.33), the second and third sub-problems of (1.3) are satisfied with some adequately regular $P$. Secondly, upon choosing test functions $\varphi$ compactly supported in $\Omega \times(0, \infty)$ it can be verified that the same token turns (2.35) into the pointwise inequality

$$
\begin{aligned}
\partial_{t}(\psi(n) \rho(c)) \leq & -\psi^{\prime \prime}(n) \rho(c)|\nabla n|^{2}+\left\{-2 \psi^{\prime}(n) \rho^{\prime}(c)+n \psi^{\prime \prime}(n) \rho(c)\right\} \nabla n \cdot \nabla c \\
& +\left\{-\psi(n) \rho^{\prime \prime}(c)+n \psi^{\prime}(n) \rho^{\prime}(c)\right\}|\nabla c|^{2} \\
& +\nabla \cdot\left\{\psi^{\prime}(n) \rho(c) \nabla n\right\}-\nabla \cdot\left\{\left\{n \psi^{\prime}(n) \rho(c)-\psi(n) \rho^{\prime}(c)\right\} \nabla c\right\}-\nabla \cdot\{\psi(n) \rho(c) u\} \\
& -\psi(n) c \rho^{\prime}(c)+n \psi(n) \rho^{\prime}(c)
\end{aligned}
$$

which by straightforward computation making use of the second and third equations from (1.3) can readily be seen to be equivalent to

$$
\psi^{\prime}(n) \rho(c) n_{t}=\partial_{t}(\psi(n) \rho(c))-\psi(n) \rho^{\prime}(c) c_{t} \leq \psi^{\prime}(n) \rho(c) \cdot\{\Delta n-\nabla \cdot(n \nabla c)-u \cdot \nabla n\} .
$$

Now by strict negativity of $\psi^{\prime}(n) \rho(c)$, this ensures that

$$
\begin{equation*}
n_{t} \geq \Delta n-\nabla \cdot(n \nabla c)-u \cdot \nabla n \quad \text { in } \Omega \times(0, \infty), \tag{2.37}
\end{equation*}
$$

and by taking test functions $\varphi$ supported near $\partial \Omega$ and $t=0$, in quite a similar fashion relying on well-established conclusions one can derive that furthermore $\frac{\partial n}{\partial \nu} \geq 0$ on $\partial \Omega \times(0, \infty)$ and $n(\cdot, 0) \geq n_{0}$ in $\Omega$. Therefore, if the inequality in (2.37) was strict at some point $\left(x_{0}, t_{0}\right) \in \Omega \times(0, \infty)$, and hence in an open neigborhood thereof, then integrating (2.37) would entail that $\int_{\Omega} n\left(\cdot, t_{0}\right)>\int_{\Omega} n_{0}$ and thereby contradict (2.36). For more detailed arguments of this type in some closely related contexts, we may refer to [30, Lemma 2.5] and [57, Lemma 2.1], for instance.
ii) With regard to the construction of solutions, the concept from Definition 2.13 brings about the evident advantages of firstly referring, through appropriate choices of $\psi$ to possibly quite strongly growth-limited nonlinear versions of $n$ in the crucial first sub-problem of (1.3) (cf. the definition in (2.45) below), and of secondly requiring the latter to be satisfied merely in the above sense combining the simple identity (2.36) with the inequality in (2.35). In fact, this further relaxation will be made substantial use of in our passage to the limit in the proof of Theorem 1.1, where inequalities of said form will result from the weak $L^{2}$ compactness properties of $\left(\nabla n_{\varepsilon}\right)_{\varepsilon \in(0,1)}$ entailed by Lemma 2.6, and where an apparent lack of corresponding strong precompactness features seems to obstruct the derivation of the associated variant of (2.35) involving genuine equality.
Now a canonical aspirant for such a solution can be selected through straightforward extraction procedures based on our estimates gained above:

Lemma 2.14 Assume (1.5) with $\int_{\Omega} n_{0}<2 \pi$. Then there exist $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \subset(0,1)$ and functions $n, c$ and $u$ which are such that the inclusions in (1.8) as well as the identity in (2.36) and the inequalities in
(1.9)-(1.11) hold with some $C>0$, that $n \geq 0$ and $c \geq 0$ a.e. in $\Omega \times(0, \infty)$, that

$$
\begin{equation*}
\ln (n+1) \in L_{l o c}^{2}\left([0, \infty) ; W^{1,2}(\Omega)\right) \tag{2.38}
\end{equation*}
$$

that $\varepsilon_{j} \searrow 0$ as $j \rightarrow \infty$ and

$$
\begin{align*}
& n_{\varepsilon} \rightarrow n \quad \text { in } L_{l o c}^{1}(\bar{\Omega} \times[0, \infty)) \text { and a.e. in } \Omega \times(0, \infty)  \tag{2.39}\\
& \nabla \ln \left(n_{\varepsilon}+1\right) \rightharpoonup \nabla \ln (n+1) \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty))  \tag{2.40}\\
& c_{\varepsilon} \rightarrow c \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \text { and a.e. in } \Omega \times(0, \infty)  \tag{2.41}\\
& \nabla c_{\varepsilon} \rightharpoonup \nabla c \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty))  \tag{2.42}\\
& u_{\varepsilon} \rightarrow u \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \text { and a.e. in } \Omega \times(0, \infty)  \tag{2.43}\\
& \nabla u_{\varepsilon} \rightharpoonup \nabla u \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \tag{2.44}
\end{align*}
$$

as $\varepsilon=\varepsilon_{j} \searrow 0$, and such that (2.32) holds for all $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty))$ and that (2.33) is satisfied for any $\varphi \in C_{0}^{\infty}\left(\Omega \times[0, \infty) ; \mathbb{R}^{2}\right)$ fulfilling $\nabla \cdot \varphi=0$.

Proof. For arbitrary $T>0$, Lemma 2.6, Lemma 2.8 and Lemma 2.9 together with Lemma 2.1 guarantee boundedness of $\left(\ln \left(n_{\varepsilon}+1\right)\right)_{\varepsilon \in(0,1)},\left(c_{\varepsilon}\right)_{\varepsilon \in(0,1)}$ and $\left(u_{\varepsilon}\right)_{\varepsilon \in(0,1)}$ in $L^{2}\left((0, T) ; W^{1,2}(\Omega)\right)$, while Lemma 2.10, Lemma 2.11 and Lemma 2.12 ensure that $\left(\partial_{t} \ln \left(n_{\varepsilon}+1\right)\right)_{\varepsilon \in(0,1)}$ and $\left(c_{\varepsilon t}\right)_{\varepsilon \in(0,1)}$ are bounded in $L^{1}\left((0, T) ;\left(W^{2,2}(\Omega)\right)^{\star}\right)$, and that $\left(u_{\varepsilon t}\right)_{\varepsilon \in(0,1)}$ is bounded in $L^{1}\left((0, T) ;\left(W^{2,2}(\Omega) \cap W_{0, \sigma}^{1,2}(\Omega)\right)^{\star}\right)$. Applications of an Aubin-Lions lemma ([45]) thus show that with some $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}} \subset(0,1)$ satisfying $\varepsilon_{j} \searrow 0$ as $j \rightarrow \infty$, and with some triple $(n, c, u)$ fulfilling (1.8) and (2.38), we have (2.40)-(2.44) as well as $\ln \left(n_{\varepsilon}+1\right) \rightarrow \ln (n+1)$ and hence also $n_{\varepsilon} \rightarrow n$ a.e. in $\Omega \times(0, \infty)$ as $\varepsilon=\varepsilon_{j} \searrow 0$. Since the $L \log L$ bound (2.18) moreover warrants uniform intebrability if $\left(n_{\varepsilon}\right)_{\varepsilon \in(0,1)}$ over $\Omega \times(0, T)$ for any $T>0$, the Vitali convergence theorem entails that indeed also (2.39) holds as $\varepsilon=\varepsilon_{j} \searrow 0$, and that hence (2.36) is satisfied.

The verification of (2.32) and (2.33) for arbitrary test functions from the indicated classes can thereafter be achieved on the basis of (2.1) and (2.39)-(2.44) in a straightforward manner (cf. [59, Lemma 4.1] for details in a closely related situation), whereas (1.9), (1.10) and (1.11) readily result from (2.16) and (2.25) through (2.39), (2.42) and (2.44).

For the derivation of our main result on global solvability, it thus essentially remains to verify (2.35) for some suitably chosen $\psi$ and $\rho$. In fact, this can be accomplished upon quite simple choices of suitably fast decreasing candidates therefor:

Proof of Theorem 1.1. Taking $(n, c, u)$ as provided by Lemma 2.14, in view of the outcome asserted by said lemma we only need to make sure that with some $\psi \in C^{2}([0, \infty))$ and $\rho \in C^{2}([0, \infty)) \cap$ $W^{2, \infty}((0, \infty))$ with $\psi>0, \psi^{\prime}<0$ and $\rho>0$ on $[0, \infty)$, the inclusions in (2.34) hold and the inequality in (2.35) is satisfied for arbitrary nonnegative $\varphi \in C_{0}^{\infty}((\bar{\Omega} \times[0, \infty))$. To achieve this, we let

$$
\begin{equation*}
\psi(s):=\frac{1}{s+1} \quad \text { and } \quad \rho(s):=e^{-s}, \quad s \geq 0 \tag{2.45}
\end{equation*}
$$

and first note that

$$
\begin{equation*}
\psi^{\prime}(s)=-\frac{1}{(s+1)^{2}}, \quad \psi^{\prime \prime}(s)=\frac{2}{(s+1)^{3}}, \quad \rho^{\prime}(s)=-e^{-s} \quad \text { and } \quad \rho^{\prime \prime}(s)=e^{-s}, \quad s \geq 0 \tag{2.46}
\end{equation*}
$$

in estimating

$$
\begin{aligned}
\int_{0}^{T} \int_{\Omega} & \left\{\left|\psi^{\prime \prime}(n)\right||\nabla n|^{2}+\psi^{\prime 2}(n)|\nabla n|^{2}+n^{2} \psi^{\prime \prime 2}(n)|\nabla n|^{2}+n^{2} \psi^{\prime 2}(n)|\nabla c|^{2}\right\} \\
& =\int_{0}^{T} \int_{\Omega}\left\{\frac{2}{(n+1)^{3}}|\nabla n|^{2}+\frac{1}{(n+1)^{4}}|\nabla n|^{2}+\frac{4 n^{2}}{(n+1)^{6}}|\nabla n|^{2}+\frac{n^{2}}{(n+1)^{4}}|\nabla c|^{2}\right\} \\
& \leq \int_{0}^{T} \int_{\Omega}\left\{7 \frac{|\nabla n|^{2}}{(n+1)^{2}}+|\nabla c|^{2}\right\} \quad \text { for all } T>0
\end{aligned}
$$

to see that (2.34) is implied by (1.8) and (2.38).
Now given any nonnegative $\varphi \in C_{0}^{\infty}(\bar{\Omega} \times[0, \infty)$ ), a lengthy but straightforward computation on the basis of several integrations by parts in (2.1) shows that again since $\nabla \cdot u_{\varepsilon}=0$,

$$
\begin{align*}
&-\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right) \varphi_{t}-\int_{\Omega} \psi\left(n_{0}\right) \rho\left(c_{0}\right) \varphi(\cdot, 0) \\
&= \int_{0}^{\infty} \int_{\Omega} \partial_{t}\left\{\psi\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right)\right\} \varphi \\
&=-\int_{0}^{\infty} \int_{\Omega} \nabla\left\{\psi^{\prime}\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right) \varphi\right\} \cdot\left\{\nabla n_{\varepsilon}-\frac{n_{\varepsilon}}{1+\varepsilon n_{\varepsilon}} \nabla c_{\varepsilon}-n_{\varepsilon} u_{\varepsilon}\right\} \\
&-\int_{0}^{\infty} \int_{\Omega} \nabla\left\{\psi\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right) \varphi\right\} \cdot\left\{\nabla c_{\varepsilon}-c_{\varepsilon} u_{\varepsilon}\right\} \\
&+\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right) \cdot\left(-c_{\varepsilon}+n_{\varepsilon}\right) \cdot \varphi \\
&=-\int_{0}^{\infty} \int_{\Omega} \psi^{\prime \prime}\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right)\left|\nabla n_{\varepsilon}\right|^{2} \varphi \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{-2 \psi^{\prime}\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right)+\frac{n_{\varepsilon} \psi^{\prime \prime}\left(n_{\varepsilon}\right)}{1+\varepsilon n_{\varepsilon}} \rho\left(c_{\varepsilon}\right)\right\}\left(\nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon}\right) \varphi \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{-\psi\left(n_{\varepsilon}\right) \rho^{\prime \prime}\left(c_{\varepsilon}\right)+\frac{n_{\varepsilon} \psi^{\prime}\left(n_{\varepsilon}\right)}{1+\varepsilon n_{\varepsilon}} \rho^{\prime}\left(c_{\varepsilon}\right)\right\}\left|\nabla c_{\varepsilon}\right|^{2} \varphi \\
&-\int_{0}^{\infty} \int_{\Omega} \psi^{\prime}\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right) \nabla n_{\varepsilon} \cdot \nabla \varphi+\int_{0}^{\infty} \int_{\Omega}\left\{\frac{n_{\varepsilon} \psi^{\prime}\left(n_{\varepsilon}\right)}{1+\varepsilon n_{\varepsilon}} \rho\left(c_{\varepsilon}\right)-\psi\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right)\right\} \nabla c_{\varepsilon} \cdot \nabla \varphi \\
&+\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right)\left(u_{\varepsilon} \cdot \nabla \varphi\right) \\
&-\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) c_{\varepsilon} \rho^{\prime}\left(c_{\varepsilon}\right) \varphi+\int_{0}^{\infty} \int_{\Omega} n_{\varepsilon} \psi\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right) \varphi \tag{2.47}
\end{align*}
$$

for all $\varepsilon \in(0,1)$. Here we note that as a consequence of the pointwise convergence properties in (2.39) and (2.41), and of the boundedness of $0 \leq \xi \mapsto \xi e^{-\xi}$, the dominated convergence theorem ensures that if we let $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ be as in Lemma 2.14, then

$$
\frac{1}{n_{\varepsilon}+1} e^{-c_{\varepsilon}} \rightarrow \frac{1}{n+1} e^{-c} \quad \text { and } \quad \frac{n_{\varepsilon}}{\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)^{2}} e^{-c_{\varepsilon}} \rightarrow \frac{n}{(n+1)^{2}} e^{-c}
$$

as well as

$$
\frac{1}{n_{\varepsilon}+1} c_{\varepsilon} e^{-c_{\varepsilon}} \rightarrow \frac{1}{n+1} c e^{-c} \quad \text { and } \quad \frac{n_{\varepsilon}}{n_{\varepsilon}+1} e^{-c_{\varepsilon}} \rightarrow \frac{n}{n+1} e^{-c}
$$

in $L_{\text {loc }}^{2}(\bar{\Omega} \times[0, \infty))$ as $\varepsilon=\varepsilon_{j} \searrow 0$. Therefore, (2.40), (2.42) and (2.43) guarantee that

$$
\begin{align*}
-\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right) \varphi_{t} & =-\int_{0}^{\infty} \int_{\Omega} \frac{1}{n_{\varepsilon}+1} e^{-c_{\varepsilon}} \varphi_{t} \\
& \rightarrow-\int_{0}^{\infty} \int_{\Omega} \frac{1}{n+1} e^{-c} \varphi_{t} \\
& =-\int_{0}^{\infty} \int_{\Omega} \psi(n) \rho(c) \varphi_{t} \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{2.48}
\end{align*}
$$

and that

$$
\begin{align*}
&-\int_{0}^{\infty} \int_{\Omega} \psi^{\prime}\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right) \nabla n_{\varepsilon} \cdot \nabla \varphi+\int_{0}^{\infty} \int_{\Omega}\left\{\frac{n_{\varepsilon} \psi^{\prime}\left(n_{\varepsilon}\right)}{1+\varepsilon n_{\varepsilon}} \rho\left(c_{\varepsilon}\right)-\psi\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right)\right\} \nabla c_{\varepsilon} \cdot \nabla \varphi \\
&+\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right)\left(u_{\varepsilon} \cdot \nabla \varphi\right) \\
&-\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) c_{\varepsilon} \rho^{\prime}\left(c_{\varepsilon}\right) \varphi+\int_{0}^{\infty} \int_{\Omega} n_{\varepsilon} \psi\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right) \varphi \\
&= \int_{0}^{\infty} \int_{\Omega} \frac{1}{n_{\varepsilon}+1} e^{-c_{\varepsilon}} \nabla \ln \left(n_{\varepsilon}+1\right) \cdot \nabla \varphi \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{-\frac{n_{\varepsilon}}{\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)^{2}} e^{-c_{\varepsilon}}+\frac{1}{n_{\varepsilon}+1} e^{-c_{\varepsilon}}\right\} \nabla c_{\varepsilon} \cdot \nabla \varphi \\
&+\int_{0}^{\infty} \int_{\Omega} \frac{1}{n_{\varepsilon}+1} e^{-c_{\varepsilon}}\left(u_{\varepsilon} \cdot \nabla \varphi\right) \\
&+\int_{0}^{\infty} \int_{\Omega} \frac{1}{n_{\varepsilon}+1} c_{\varepsilon} e^{-c_{\varepsilon}} \varphi-\int_{0}^{\infty} \int_{\Omega} \frac{n_{\varepsilon}}{n_{\varepsilon}+1} e^{-c_{\varepsilon}} \varphi \\
& \rightarrow \int_{0}^{\infty} \int_{\Omega} \frac{1}{n+1} e^{-c} \nabla \ln (n+1) \cdot \nabla \varphi+\int_{0}^{\infty} \int_{\Omega}\left\{-\frac{n}{(n+1)^{2}} e^{-c}+\frac{1}{n+1} e^{-c}\right\} \nabla c \cdot \nabla \varphi \\
&+\int_{0}^{\infty} \int_{\Omega} \frac{1}{n+1} e^{-c}(u \cdot \nabla \varphi) \\
&+\int_{0}^{\infty} \int_{\Omega} \frac{1}{n+1} c e^{-c} \varphi-\int_{0}^{\infty} \int_{\Omega} \frac{n}{n+1} e^{-c} \varphi \\
&=-\int_{0}^{\infty} \int_{\Omega} \psi^{\prime}(n) \rho(c) \nabla n \cdot \nabla \varphi+\int_{0}^{\infty} \int_{\Omega}\left\{n \psi^{\prime}(n) \rho(c)-\psi(n) \rho^{\prime}(c)\right\} \nabla c \cdot \nabla \varphi \\
&+\int_{0}^{\infty} \int_{\Omega} \psi(n) \rho(c)(u \cdot \nabla \varphi) \\
&-\int_{0}^{\infty} \int_{\Omega} \psi(n) c \rho^{\prime}(c) \varphi+\int_{0}^{\infty} \int_{\Omega} n \psi(n) \rho^{\prime}(c) \varphi \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 . \tag{2.49}
\end{align*}
$$

Similarly, in the rearranged representation of the first three integrals from the right-hand side of (2.47), as given by

$$
\begin{align*}
&-\int_{0}^{\infty} \int_{\Omega} \psi^{\prime \prime}\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right)\left|\nabla n_{\varepsilon}\right|^{2} \varphi \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{-2 \psi^{\prime}\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right)+\frac{n_{\varepsilon} \psi^{\prime \prime}\left(n_{\varepsilon}\right)}{1+\varepsilon n_{\varepsilon}} \rho\left(c_{\varepsilon}\right)\right\}\left(\nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon}\right) \varphi \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{-\psi\left(n_{\varepsilon}\right) \rho^{\prime \prime}\left(c_{\varepsilon}\right)+\frac{n_{\varepsilon} \psi^{\prime}\left(n_{\varepsilon}\right)}{1+\varepsilon n_{\varepsilon}} \rho^{\prime}\left(c_{\varepsilon}\right)\right\}\left|\nabla c_{\varepsilon}\right|^{2} \varphi \\
&=-2 \int_{0}^{\infty} \int_{\Omega} \frac{1}{\left(n_{\varepsilon}+1\right)^{3}} e^{-c_{\varepsilon}\left|\nabla n_{\varepsilon}\right|^{2} \varphi} \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{-2+\frac{2 n_{\varepsilon}}{\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{2}} e^{-c_{\varepsilon}}\left(\nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon}\right) \varphi \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{-1+\frac{n_{\varepsilon}}{\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{n_{\varepsilon}+1} e^{-c_{\varepsilon}}\left|\nabla c_{\varepsilon}\right|^{2} \varphi \\
&=-2 \int_{0}^{\infty} \int_{\Omega} \left\lvert\, \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{3}{2}} e^{-\frac{c_{\varepsilon}}{2}} \nabla n_{\varepsilon}+\left.\left\{\frac{1}{2}-\frac{n_{\varepsilon}}{2\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon}\right|^{2} \varphi}\right. \\
&+2 \int_{0}^{\infty} \int_{\Omega}\left\{\frac{1}{2}-\frac{n_{\varepsilon}}{2\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\}^{2} \cdot \frac{1}{n_{\varepsilon}+1} e^{-c_{\varepsilon}}\left|\nabla c_{\varepsilon}\right|^{2} \varphi \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{-1+\frac{n_{\varepsilon}}{\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{n_{\varepsilon}+1} e^{-c_{\varepsilon}}\left|\nabla c_{\varepsilon}\right|^{2} \varphi \\
&=-2 \int_{0}^{\infty} \int_{\Omega} \left\lvert\, \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{3}{2}} e^{-\frac{c_{\varepsilon}}{2}} \nabla n_{\varepsilon}+\left.\left\{\frac{1}{2}-\frac{n_{\varepsilon}}{2\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon}\right|^{2} \varphi}\right. \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{-1+\frac{n_{\varepsilon}}{\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right. \\
&\left.+2 \cdot\left\{\frac{1}{4}-\frac{n_{\varepsilon}}{2\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}+\frac{n_{\varepsilon}^{2}}{4\left(1+\varepsilon n_{\varepsilon}\right)^{2}\left(n_{\varepsilon}+1\right)^{2}}\right\}\right\} \cdot \frac{1}{n_{\varepsilon}+1} e^{-c_{\varepsilon}\left|\nabla c_{\varepsilon}\right|^{2} \varphi} \\
&=-2 \int_{0}^{\infty} \int_{\Omega}\left|\frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{3}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla n_{\varepsilon}+\left\{\frac{1}{2}-\frac{n_{\varepsilon}}{2\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon}\right|^{2} \varphi \\
&-\int_{0}^{\infty} \int_{\Omega}\left|\left\{\frac{1}{2}-\frac{n_{\varepsilon}^{2}}{2\left(1+\varepsilon n_{\varepsilon}\right)^{2}\left(n_{\varepsilon}+1\right)^{2}}\right\}^{\frac{1}{2}} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon}\right|^{2} \varphi,  \tag{2.50}\\
& \varepsilon \in(0,1),
\end{align*}
$$

we claim that

$$
\begin{equation*}
\frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{3}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla n_{\varepsilon} \rightharpoonup \frac{1}{(n+1)^{\frac{3}{2}}} e^{-\frac{c}{2}} \nabla n \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{2.51}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{\frac{1}{2}-\frac{n_{\varepsilon}}{2\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon} \rightharpoonup \frac{1}{2(n+1)^{\frac{3}{2}}} e^{-\frac{c}{2}} \nabla c \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{2.52}
\end{equation*}
$$

as well as

$$
\begin{align*}
\left\{\frac{1}{2}-\right. & \left.\frac{n_{\varepsilon}^{2}}{2\left(1+\varepsilon n_{\varepsilon}\right)^{2}\left(n_{\varepsilon}+1\right)^{2}}\right\}^{\frac{1}{2}} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon} \\
& \quad-\left\{\frac{1}{2}-\frac{n^{2}}{2(n+1)^{2}}\right\}^{\frac{1}{2}} \cdot \frac{1}{(n+1)^{2}} e^{-\frac{c}{2}} \nabla c \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 . \tag{2.53}
\end{align*}
$$

Indeed, since again from (2.39), (2.41) and the dominated convergence theorem we know that

$$
\frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \rightarrow \frac{1}{(n+1)^{\frac{1}{2}}} e^{-\frac{c}{2}} \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0
$$

and

$$
\begin{equation*}
\left\{\frac{1}{2}-\frac{n_{\varepsilon}}{2\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \rightarrow \frac{1}{2(n+1)^{\frac{3}{2}}} e^{-\frac{c}{2}} \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0 \tag{2.54}
\end{equation*}
$$

and

$$
\begin{align*}
\left\{\frac{1}{2}-\right. & \left.\frac{n_{\varepsilon}^{2}}{\left.2\left(1+\varepsilon n_{\varepsilon}\right)^{2}\right)\left(n_{\varepsilon}+1\right)^{2}}\right\}^{\frac{1}{2}} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \\
& \rightarrow\left\{\frac{1}{2}-\frac{n^{2}}{2(n+1)^{2}}\right\}^{\frac{1}{2}} \cdot \frac{1}{(n+1)^{\frac{1}{2}}} e^{-\frac{c}{2}} \quad \text { in } L_{l o c}^{2}(\bar{\Omega} \times[0, \infty)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0, \tag{2.55}
\end{align*}
$$

in view of (2.38) it firstly follows that

$$
\begin{aligned}
\frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{3}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla n_{\varepsilon} & =\frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla \ln \left(n_{\varepsilon}+1\right) \\
& \rightharpoonup \frac{1}{(n+1)^{\frac{1}{2}}} e^{-\frac{c}{2}} \nabla \ln (n+1) \\
& =\frac{1}{(n+1)^{\frac{3}{2}}} e^{-\frac{c}{2}} \nabla n \quad \text { in } L_{l o c}^{1}(\bar{\Omega} \times[0, \infty)) \quad \text { as } \varepsilon=\varepsilon_{j} \searrow 0,
\end{aligned}
$$

so that due to Lemma 2.6, the observation that

$$
\int_{0}^{T} \int_{\Omega}\left|\frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{3}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla n_{\varepsilon}\right|^{2} \leq \int_{0}^{T} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{\left(n_{\varepsilon}+1\right)^{2}} \quad \text { for all } T>0 \text { and } \varepsilon \in(0,1)
$$

shows that in fact (2.51) holds; likewise, combining (2.54) with (2.42) and Lemma 2.6 yields (2.52) thanks to the fact that
$\int_{0}^{T} \int_{\Omega}\left|\left\{\frac{1}{2}-\frac{n_{\varepsilon}}{2\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon}\right|^{2} \leq \int_{0}^{T} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2} \quad$ for all $T>0$ and $\varepsilon \in(0,1)$, while (2.55) along with (2.42) and Lemma 2.6 implies (2.53), because also

$$
\int_{0}^{T} \int_{\Omega}\left|\left\{\frac{1}{2}-\frac{n_{\varepsilon}^{2}}{2\left(1+\varepsilon n_{\varepsilon}\right)^{2}\left(n_{\varepsilon}+1\right)^{2}}\right\}^{\frac{1}{2}} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon}\right|^{2} \leq \int_{0}^{T} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}
$$

for all $T>0$ and $\varepsilon \in(0,1)$. According to the nonnegativity of $\varphi$, by lower semicontinuity of the norm in $L^{2}(\operatorname{supp} \varphi)$ with respect to weak convergence we thus infer from (2.51)-(2.53) and (2.50) that

$$
\begin{aligned}
& \int_{0}^{\infty} \int_{\Omega} \psi^{\prime \prime}(n) \rho(c)|\nabla n|^{2} \varphi-\int_{0}^{\infty} \int_{\Omega}\left\{-2 \psi^{\prime}(n) \rho(c)+n \psi^{\prime \prime}(n) \rho(c)\right\}(\nabla n \cdot \nabla c) \varphi \\
&-\int_{0}^{\infty} \int_{\Omega}\left\{-\psi(n) \rho^{\prime \prime}(c)+n \psi^{\prime}(n) \rho^{\prime}(c)\right\}|\nabla c|^{2} \varphi \\
&= 2 \int_{0}^{\infty} \int_{\Omega}\left|\frac{1}{(n+1)^{\frac{3}{2}}} e^{-\frac{c}{2}} \nabla n+\frac{1}{2(n+1)^{\frac{3}{2}}} e^{-\frac{c}{2}} \nabla c\right|^{2} \varphi \\
&+\int_{0}^{\infty} \int_{\Omega}\left|\left\{\frac{1}{2}-\frac{n^{2}}{2(n+1)^{2}}\right\}^{\frac{1}{2}} \cdot \frac{1}{(n+1)^{\frac{1}{2}}} e^{-\frac{c}{2}} \nabla c\right|^{2} \varphi \\
& \leq \liminf _{\varepsilon=\varepsilon_{j} \searrow 0}\left\{2 \int_{0}^{\infty} \int_{\Omega}\left|\frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{3}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla n_{\varepsilon}+\left\{\frac{1}{2}-\frac{n_{\varepsilon}}{2\left(1+\varepsilon n_{\varepsilon}\right)\left(n_{\varepsilon}+1\right)}\right\} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon}\right|^{2} \varphi\right. \\
&\left.+\int_{0}^{\infty} \int_{\Omega}\left|\left\{\frac{1}{2}-\frac{n_{\varepsilon}^{2}}{2\left(1+\varepsilon n_{\varepsilon}\right)^{2}\left(n_{\varepsilon}+1\right)^{2}}\right\}^{\frac{1}{2}} \cdot \frac{1}{\left(n_{\varepsilon}+1\right)^{\frac{1}{2}}} e^{-\frac{c_{\varepsilon}}{2}} \nabla c_{\varepsilon}\right|^{2} \varphi\right\} \\
&= \liminf _{\varepsilon=\varepsilon_{j} \searrow 0}\left\{\int_{0}^{\infty} \int_{\Omega} \psi^{\prime \prime}\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right)\left|\nabla n_{\varepsilon}\right|^{2} \varphi\right. \\
&-\int_{0}^{\infty} \int_{\Omega}\left\{-2 \psi^{\prime}\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right)+\frac{n_{\varepsilon} \psi^{\prime \prime}\left(n_{\varepsilon}\right)}{1+\varepsilon n_{\varepsilon}} \rho\left(c_{\varepsilon}\right)\right\}\left(\nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon}\right) \varphi \\
&\left.\quad-\int_{0}^{\infty} \int_{\Omega}\left\{-\psi\left(n_{\varepsilon}\right) \rho^{\prime \prime}\left(c_{\varepsilon}\right)+\frac{n_{\varepsilon} \psi^{\prime}\left(n_{\varepsilon}\right)}{1+\varepsilon n_{\varepsilon}} \rho^{\prime}\left(c_{\varepsilon}\right)\right\}\left|\nabla c_{\varepsilon}\right|^{2} \varphi\right\}
\end{aligned}
$$

so that in view of the identity (2.47) as well as the convergence properties stated in (2.48) and (2.49) we obtain that

$$
\begin{aligned}
& \int_{0}^{\infty} \int_{\Omega} \psi^{\prime \prime}(n) \rho(c)|\nabla n|^{2} \varphi-\int_{0}^{\infty} \int_{\Omega}\left\{-2 \psi^{\prime}(n) \rho(c)+n \psi^{\prime \prime}(n) \rho(c)\right\}(\nabla n \cdot \nabla c) \varphi \\
&-\int_{0}^{\infty} \int_{\Omega}\left\{-\psi(n) \rho^{\prime \prime}(c)+n \psi^{\prime}(n) \rho^{\prime}(c)\right\}|\nabla c|^{2} \varphi \\
& \leq \liminf _{\varepsilon=\varepsilon_{j} \searrow 0}\left\{\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right) \varphi_{t}+\int_{\Omega} \psi\left(n_{0}\right) \rho\left(c_{0}\right) \varphi(\cdot, 0)\right. \\
&-\int_{0}^{\infty} \int_{\Omega} \psi^{\prime}\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right) \nabla n_{\varepsilon} \cdot \nabla \varphi \\
&+\int_{0}^{\infty} \int_{\Omega}\left\{\frac{n_{\varepsilon} \psi^{\prime}\left(n_{\varepsilon}\right)}{1+\varepsilon n_{\varepsilon}} \rho\left(c_{\varepsilon}\right)-\psi\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right)\right\} \nabla c_{\varepsilon} \cdot \nabla \varphi \\
&+\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) \rho\left(c_{\varepsilon}\right)\left(u_{\varepsilon} \cdot \nabla \varphi\right) \\
&\left.-\int_{0}^{\infty} \int_{\Omega} \psi\left(n_{\varepsilon}\right) c_{\varepsilon} \rho^{\prime}\left(c_{\varepsilon}\right) \varphi+\int_{0}^{\infty} \int_{\Omega} n_{\varepsilon} \psi\left(n_{\varepsilon}\right) \rho^{\prime}\left(c_{\varepsilon}\right) \varphi\right\}
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{0}^{\infty} \int_{\Omega} \psi(n) \rho(c) \varphi_{t}+\int_{\Omega} \psi\left(n_{0}\right) \rho\left(c_{0}\right) \varphi(\cdot, 0) \\
& -\int_{0}^{\infty} \int_{\Omega} \psi^{\prime}(n) \rho(c) \nabla n \cdot \nabla \varphi \\
& +\int_{0}^{\infty} \int_{\Omega}\left\{n \psi^{\prime}(n) \rho(c)-\psi(n) \rho^{\prime}(c)\right\} \nabla c \cdot \nabla \varphi \\
& +\int_{0}^{\infty} \int_{\Omega}^{\infty} \psi(n) \rho(c)(u \cdot \nabla \varphi) \\
& -\int_{0}^{\infty} \int_{\Omega} \psi(n) c \rho^{\prime}(c) \varphi+\int_{0}^{\infty} \int_{\Omega} n \psi(n) \rho^{\prime}(c) \varphi
\end{aligned}
$$

which is equivalent to the inequality in (2.35).

## 3 Eventual regularity and stabilization

### 3.1 A conditional Lyapunov functional

In this part we shall derive Theorem 1.2 on the basis of the following identification of another energy structure in (2.1) which is now genuine by involving a truly nonincreasing functional, but which is conditional in the sense that it relies on an additional smallness assumption which, at this stage, yet involves all three solution components.
Lemma 3.1 There exist $m_{0} \in(0,2 \pi), \delta \in\left(0, \frac{1}{2}\right), \kappa>0$ and $C>0$ with the following property: If (1.5) holds with $\int_{\Omega} n_{0}<m_{0}$, and if writing

$$
\begin{equation*}
\mathcal{F}_{\varepsilon}(t):=\int_{\Omega} n_{\varepsilon}(\cdot, t) \ln \frac{n_{\varepsilon}(\cdot, t)}{\bar{n}_{0}}+\frac{1}{2} \int_{\Omega}\left|\nabla c_{\varepsilon}(\cdot, t)\right|^{2}+\frac{1}{2} \int_{\Omega}\left|u_{\varepsilon}(\cdot, t)\right|^{2}, \quad t>0, \varepsilon \in(0,1) \tag{3.1}
\end{equation*}
$$

for some $\varepsilon \in(0,1)$ and $t_{0}>0$ we have

$$
\begin{equation*}
\mathcal{F}_{\varepsilon}\left(t_{0}\right) \leq \delta \tag{3.2}
\end{equation*}
$$

then

$$
\begin{equation*}
\mathcal{F}_{\varepsilon}(t) \leq \delta e^{-\kappa\left(t-t_{0}\right)} \quad \text { for all } t>t_{0} \tag{3.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{t_{0}}^{t} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+\int_{t_{0}}^{t} \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}+\int_{t_{0}}^{t} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq C \quad \text { for all } t>t_{0} \tag{3.4}
\end{equation*}
$$

Proof. According to the two-dimensional Sobolev inequality, we can pick $C_{1}>0$ such that again writing $\bar{\varphi}:=\frac{1}{|\Omega|} \int_{\Omega} \varphi$ for $\varphi \in L^{1}(\Omega)$, we have

$$
\begin{equation*}
\int_{\Omega}(\varphi-\bar{\varphi})^{2} \leq C_{1} \cdot\left\{\int_{\Omega}|\nabla \varphi|\right\}^{2} \quad \text { for all } \varphi \in W^{1,1}(\Omega) \tag{3.5}
\end{equation*}
$$

while a combination of the Gagliardo-Nirenberg inequality with standard elliptic regularity ([19]) yields $C_{2}>0$ fulfilling

$$
\begin{equation*}
\int_{\Omega}|\nabla \varphi|^{4} \leq C_{2} \cdot\left\{\int_{\Omega}|\nabla \varphi|^{2}\right\} \cdot \int_{\Omega}|\Delta \varphi|^{2} \quad \text { for all } \varphi \in W^{2,2}(\Omega) \text { such that } \frac{\partial \varphi}{\partial \nu}=0 \text { on } \partial \Omega \tag{3.6}
\end{equation*}
$$

We furthermore employ the Poincaré inequality to fix $C_{3}>0$ such that

$$
\begin{equation*}
\int_{\Omega}|\varphi|^{2} \leq C_{3} \int_{\Omega}|\nabla \varphi|^{2} \quad \text { for all } \varphi \in W_{0}^{1,2}\left(\Omega ; \mathbb{R}^{2}\right) \tag{3.7}
\end{equation*}
$$

and rely on a logarithmic Sobolev inequality ([20], [38]) in choosing $C_{4}>0$ satisfying

$$
\begin{equation*}
\int_{\Omega} \varphi \ln \frac{\varphi}{\bar{\varphi}} \leq C_{4} \int_{\Omega} \frac{|\nabla \varphi|^{2}}{\varphi} \quad \text { for all } \varphi \in C^{1}(\bar{\Omega}) \text { such that } \varphi>0 \text { in } \bar{\Omega} . \tag{3.8}
\end{equation*}
$$

Finally abbreviating $C_{5}:=\|\nabla \Phi\|_{L^{\infty}(\Omega)}$, we take $m_{0}=m_{0}(\Omega) \in(0,2 \pi)$ and $\delta=\delta(\Omega) \in\left(0, \frac{1}{2}\right)$ small enough such that

$$
\begin{equation*}
\frac{2 m_{0}}{|\Omega|} \leq \frac{1}{2} \tag{3.9}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{1} \cdot\left(1+2 C_{3} C_{5}^{2}\right) m_{0} \leq \frac{1}{8} \tag{3.10}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\delta<\frac{1}{12 C_{2}}, \tag{3.11}
\end{equation*}
$$

and let $\kappa=\kappa(\Omega)>0$ be suitably small fulfilling

$$
\begin{equation*}
\kappa \leq 1, \quad C_{4} \kappa \leq \frac{1}{8} \quad \text { and } \quad \frac{\kappa}{2} \leq \frac{1}{8 C_{3}} \tag{3.12}
\end{equation*}
$$

Then assuming (1.5) with $m:=\int_{\Omega} n_{0}<m_{0}$, and supposing that (3.2) holds for some $\varepsilon \in(0,1)$ and $t_{0}>0$, we claim that

$$
\begin{equation*}
\mathcal{F}_{\varepsilon}^{\prime}(t)+\kappa \mathcal{F}_{\varepsilon}(t)+\frac{1}{4} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+\frac{1}{4} \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}+\frac{1}{4} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq 0 \quad \text { for all } t>t_{0} \tag{3.13}
\end{equation*}
$$

which will clearly imply both (3.3) and (3.4) upon integration.
To verify (3.13), we use (2.1), the positivity of $n_{\varepsilon}$ in $\bar{\Omega} \times(0, \infty)$ and (2.3) to see that
$\frac{d}{d t} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}=\int_{\Omega} \ln n_{\varepsilon} \nabla \cdot\left(\nabla n_{\varepsilon}-\frac{n_{\varepsilon}}{1+\varepsilon n_{\varepsilon}} \nabla c_{\varepsilon}\right)=-\int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+\frac{1}{1+\varepsilon n_{\varepsilon}} \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon} \quad$ for all $t>0$, that

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2} & =-\int_{\Omega} n_{\varepsilon} \Delta c_{\varepsilon}+\left(u_{\varepsilon} \cdot \nabla c_{\varepsilon}\right) \Delta c_{\varepsilon} \\
& =\int_{\Omega} \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon}-\int_{\Omega} \nabla c_{\varepsilon} \cdot\left(\nabla u_{\varepsilon} \cdot \nabla c_{\varepsilon}\right) \quad \text { for all } t>0
\end{aligned}
$$

and that

$$
\frac{1}{2} \frac{d}{d t} \int_{\Omega}\left|u_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}=\int_{\Omega} n_{\varepsilon}\left(u_{\varepsilon} \cdot \nabla \Phi\right)=\int_{\Omega}\left(n_{\varepsilon}-\bar{n}_{0}\right)\left(u_{\varepsilon} \cdot \nabla \Phi\right) \quad \text { for all } t>0
$$

so that

$$
\begin{align*}
\mathcal{F}_{\varepsilon}^{\prime}(t)+ & \kappa \mathcal{F}_{\varepsilon}(t)+\int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+\int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \\
= & \int_{\Omega}\left(\frac{1}{1+\varepsilon n_{\varepsilon}}+1\right) \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon}-\int_{\Omega} \nabla c_{\varepsilon} \cdot\left(\nabla u_{\varepsilon} \cdot \nabla c_{\varepsilon}\right)+\int_{\Omega}\left(n_{\varepsilon}-\bar{n}_{0}\right)\left(u_{\varepsilon} \cdot \nabla \Phi\right) \\
& +\kappa \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+\frac{\kappa}{2} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}+\frac{\kappa}{2} \int_{\Omega}\left|u_{\varepsilon}\right|^{2} \quad \text { for all } t>0 . \tag{3.14}
\end{align*}
$$

Here by Young's inequality,

$$
\begin{aligned}
\int_{\Omega}\left(\frac{1}{1+\varepsilon n_{\varepsilon}}+1\right) \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon} & \leq 2 \int_{\Omega}\left|\nabla n_{\varepsilon}\right| \cdot\left|\nabla c_{\varepsilon}\right| \\
& \leq \frac{1}{2} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+2 \int_{\Omega} n_{\varepsilon}\left|\nabla c_{\varepsilon}\right|^{2} \\
& =\frac{1}{2} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+2 \int_{\Omega}\left(n_{\varepsilon}-\bar{n}_{0}\right)\left|\nabla c_{\varepsilon}\right|^{2}+\frac{2 m}{|\Omega|} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2} \\
& \leq \frac{1}{2} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+\int_{\Omega}\left(n_{\varepsilon}-\bar{n}_{0}\right)^{2}+\int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{4}+\frac{2 m}{|\Omega|} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}
\end{aligned}
$$

and

$$
-\int_{\Omega} \nabla c_{\varepsilon} \cdot\left(\nabla u_{\varepsilon} \cdot \nabla c_{\varepsilon}\right) \leq \frac{1}{2} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}+\frac{1}{2} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{4}
$$

as well as

$$
\begin{aligned}
\int_{\Omega}\left(n_{\varepsilon}-\bar{n}_{0}\right)\left(u_{\varepsilon} \cdot \nabla \Phi\right) & \leq C_{5} \int_{\Omega}\left|n_{\varepsilon}-\bar{n}_{0}\right| \cdot\left|u_{\varepsilon}\right| \\
& \leq \frac{1}{8 C_{3}} \int_{\Omega}\left|u_{\varepsilon}\right|^{2}+2 C_{3} C_{5}^{2} \int_{\Omega}\left(n_{\varepsilon}-\bar{n}_{0}\right)^{2}
\end{aligned}
$$

for all $t>0$, so that from (3.14) we obtain that since $\frac{\kappa}{2}+\frac{2 m}{|\Omega|} \leq \frac{1}{2}+\frac{1}{2}=1$ by (3.12) and (3.9),

$$
\begin{align*}
& \mathcal{F}_{\varepsilon}^{\prime}(t)+\kappa \mathcal{F}_{\varepsilon}(t)+\frac{1}{2} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+\frac{1}{2} \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}+\frac{1}{2} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \\
& \leq\left(1+2 C_{3} C_{5}^{2}\right) \int_{\Omega}\left(n_{\varepsilon}-\bar{n}_{0}\right)^{2}+\kappa \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}} \\
&+\frac{3}{2} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{4}+\left(\frac{1}{8 C_{3}}+\frac{\kappa}{2}\right) \int_{\Omega}\left|u_{\varepsilon}\right|^{2} \quad \text { for all } t>0 . \tag{3.15}
\end{align*}
$$

Now by (3.5), the Cauchy-Schwarz inequality and (3.10),

$$
\begin{aligned}
\left(1+2 C_{3} C_{5}^{2}\right) \int_{\Omega}\left(n_{\varepsilon}-\bar{n}_{0}\right)^{2} & \leq C_{1} \cdot\left(1+2 C_{3} C_{5}^{2}\right) \cdot\left\{\int_{\Omega}\left|\nabla n_{\varepsilon}\right|\right\}^{2} \\
& \leq C_{1} \cdot\left(1+2 C_{3} C_{5}^{2}\right) \cdot\left\{\int_{\Omega} n_{\varepsilon}\right\} \cdot \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}
\end{aligned}
$$

$$
\begin{align*}
& \leq C_{1} \cdot\left(1+2 C_{3} C_{5}^{2}\right) \cdot m \cdot \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}} \\
& \leq \frac{1}{8} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}} \quad \text { for all } t>0 \tag{3.16}
\end{align*}
$$

while (3.8) together with (3.12) ensures that

$$
\begin{equation*}
\kappa \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}} \leq C_{4} \kappa \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}} \leq \frac{1}{8} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}} \quad \text { for all } t>0 . \tag{3.17}
\end{equation*}
$$

Moreover, (3.6) along with the nonnegativity of $\int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}$ for $t>0$ warrants that

$$
\begin{align*}
\frac{3}{2} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{4} & \leq \frac{3 C_{2}}{2} \cdot\left\{\int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}\right\} \cdot \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2} \\
& \leq 3 C_{2} \mathcal{F}_{\varepsilon}(t) \cdot \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2} \quad \text { for all } t>0 \tag{3.18}
\end{align*}
$$

whereas finally from (3.12) and (3.7) we obtain that

$$
\left(\frac{1}{8 C_{3}}+\frac{\kappa}{2}\right) \int_{\Omega}\left|u_{\varepsilon}\right|^{2} \leq \frac{1}{4 C_{3}} \int_{\Omega}\left|u_{\varepsilon}\right|^{2} \leq \frac{1}{4} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \quad \text { for all } t>0
$$

In conjunction with (3.16)-(3.18), this shows that (3.15) entails that
$\mathcal{F}_{\varepsilon}^{\prime}(t)+\kappa \mathcal{F}_{\varepsilon}(t)+\frac{1}{4} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+\left\{\frac{1}{2}-3 C_{2} \mathcal{F}_{\varepsilon}(t)\right\} \cdot \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}+\frac{1}{4} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq 0 \quad$ for all $t>0$,
whence introducing

$$
T_{0}:=\sup \left\{T>t_{0} \left\lvert\, \mathcal{F}_{\varepsilon}(t)<\frac{1}{12 C_{2}}\right. \text { for all } t \in\left(t_{0}, T\right)\right\}
$$

by means of a straightforward contradiction-based argument we infer that since $\mathcal{F}_{\varepsilon}$ is continuous with $\mathcal{F}_{\varepsilon}\left(t_{0}\right) \leq \delta<\frac{1}{12 C_{2}}$ by (3.11), we actually must have $T_{0}=+\infty$, because as a particular consequence of (3.19), we know that $\mathcal{F}_{\varepsilon}^{\prime}(t) \leq 0$ for all $t \in\left(t_{0}, T_{0}\right)$. As thus $\frac{1}{2}-3 C_{2} \mathcal{F}_{\varepsilon}(t) \geq \frac{1}{4}$ for all $t>t_{0}$, the claimed inequality in (3.13) immediately results from (3.19).
For solutions emanating from initial data having their total mass $m:=\int_{\Omega} n_{0}$ suitably small, however, making use of the quantitative dependence on $m$ of our estimates from Section 2 confirms that the hypothesis underlying the latter lemma can fortunately be fulfilled upon waiting suitably long:

Lemma 3.2 There exist $m_{\star} \in(0,2 \pi), \kappa>0$ and $C>0$ such that if (1.5) holds with $\int_{\Omega} n_{0}<m_{\star}$, then one can find $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ with the property that

$$
\begin{equation*}
\int_{\Omega} n_{\varepsilon}(\cdot, t) \ln \frac{n_{\varepsilon}(\cdot, t)}{\bar{n}_{0}} \leq e^{-\kappa(t-T)} \quad \text { for all } t>T \text { and } \varepsilon \in(0,1) \tag{3.20}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{\Omega}\left|\nabla c_{\varepsilon}(\cdot, t)\right|^{2} \leq e^{-\kappa(t-T)} \quad \text { for all } t>T \text { and } \varepsilon \in(0,1) \tag{3.21}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\int_{\Omega}\left|u_{\varepsilon}(\cdot, t)\right|^{2} \leq e^{-\kappa(t-T)} \quad \text { for all } t>T \text { and } \varepsilon \in(0,1) \text {, } \tag{3.22}
\end{equation*}
$$

and that

$$
\begin{equation*}
\int_{T}^{\infty} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}} \leq C \quad \text { for all } \varepsilon \in(0,1) \tag{3.23}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{T}^{\infty} \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2} \leq C \quad \text { for all } \varepsilon \in(0,1) \tag{3.24}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\int_{T}^{\infty} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq C \quad \text { for all } \varepsilon \in(0,1) . \tag{3.25}
\end{equation*}
$$

Proof. We fix $m_{0} \in(0,2 \pi), \delta \in\left(0, \frac{1}{2}\right)$ and $\kappa>0$ as given by Lemma 3.1, and then employ Lemma 2.5 and Lemma 2.9 to find $K_{1}=K_{1}\left(m_{0}\right)>0$ and $K_{2}=K_{2}\left(m_{0}\right)>0$ such that whenever ( $n_{0}, c_{0}, u_{0}$ ) satisfies (1.5) with $\int_{\Omega} n_{0}<m_{0}$, there exist $C_{i}\left(n_{0}, c_{0}, u_{0}\right)>0, i \in\{1,2\}$, such that for each $\varepsilon \in(0,1)$ we have

$$
\begin{equation*}
\frac{1}{T} \cdot\left\{\int_{0}^{T} \int_{\Omega} n_{\varepsilon} \ln \frac{n_{\varepsilon}}{\bar{n}_{0}}+\int_{0}^{T} \int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2}\right\} \leq K_{1} m \ln \frac{4 \pi}{m}+\frac{C_{1}\left(n_{0}, c_{0}, u_{0}\right)}{T} \quad \text { for all } T>0 \tag{3.26}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{T} \int_{0}^{T} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq K_{2} m^{2} \ln \frac{4 \pi}{m}+\frac{C_{2}\left(n_{0}, c_{0}, u_{0}\right)}{T} \quad \text { for all } T>0 . \tag{3.27}
\end{equation*}
$$

Once more invoking the Poincaré inequality in choosing $C_{3}>0$ such that

$$
\begin{equation*}
\int_{\Omega}|\varphi|^{2} \leq C_{3} \int_{\Omega}|\nabla \varphi|^{2} \quad \text { for all } \varphi \in W_{0}^{1,2}\left(\Omega ; \mathbb{R}^{2}\right) \tag{3.28}
\end{equation*}
$$

we then firstly fix $m_{\star} \in\left(0, m_{0}\right)$ suitably small such that

$$
\frac{3}{2} K_{1} m \ln \frac{4 \pi}{m}+\frac{C_{3}}{2} K_{2} m^{2} \ln \frac{4 \pi}{m} \leq \frac{\delta}{2} \quad \text { for all } m \in\left(0, m_{\star}\right),
$$

and thereupon, given any $\left(n_{0}, c_{0}, u_{0}\right)$ fulfilling (1.5) with $\int_{\Omega} n_{0}<m_{\star}$, take $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ large enough such that

$$
\frac{3 C_{1}\left(n_{0}, c_{0}, u_{0}\right)}{2 T}+\frac{C_{2}\left(n_{0}, c_{0}, u_{0}\right) C_{3}}{2 T} \leq \frac{\delta}{2} .
$$

Then, namely, for any ( $n_{0}, c_{0}, u_{0}$ ) complying with (1.5) and fulfilling $m:=\int_{\Omega} n_{0}<m_{\star}$, and for each $\varepsilon \in(0,1)$, the function $\mathcal{F}_{\varepsilon}$ correspondingly defined through (3.1) satisfies

$$
\begin{aligned}
\frac{1}{T} \int_{0}^{T} \mathcal{F}_{\varepsilon}(t) d t & \leq \frac{3}{2} \cdot\left\{K_{1} m \ln \frac{4 \pi}{m}+\frac{C_{1}\left(n_{0}, c_{0}, u_{0}\right)}{T}\right\}+\frac{C_{3}}{2} \cdot\left\{K_{2} m^{2} \ln \frac{4 \pi}{m}+\frac{C_{2}\left(n_{0}, c_{0}, u_{0}\right)}{T}\right\} \\
& \leq \frac{\delta}{2}+\frac{\delta}{2}=\delta
\end{aligned}
$$

by (3.26), (3.28) and (3.27), so that with some $t_{0}=t_{0}\left(n_{0}, c_{0}, u_{0}, \varepsilon\right) \in(0, T)$ we must have $\mathcal{F}_{\varepsilon}\left(t_{0}\right) \leq \delta$. Using that $t_{0} \leq T$ and that $\delta \leq \frac{1}{2}$, on applying Lemma 3.1 we readily obtain both (3.20)-(3.22) and (3.23)-(3.25) with some appropriately large $C>0$.

### 3.2 Higher ultimate regularity properties

In this section, an appropriately organized bootstrap procedure will show that the bounds provided by Lemma 3.2 are actually sufficient to entail the statements on both eventual smoothness and stabilization in Theorem 1.2. In our first step in this direction we perform a standard $L^{2}$ testing procedure to draw the following conclusion from (3.21), (3.23) and (3.24).

Lemma 3.3 There exists $C>0$ such that if $m_{\star} \in(0,2 \pi)$ is as in Lemma 3.2, and if (1.5) holds with $\int_{\Omega} n_{0}<m_{\star}$, then one can find $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ such that for all $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\int_{\Omega} n_{\varepsilon}^{2}(\cdot, t) \leq C \quad \text { for all } t>T \tag{3.29}
\end{equation*}
$$

Proof. We first apply Lemma 3.2 to find $C_{1}>0$ such that for any ( $n_{0}, c_{0}, u_{0}$ ) fulfilling (1.5) with $\int_{\Omega} n_{0}<m_{\star}$ we can find $T_{1}=T_{1}\left(n_{0}, c_{0}, u_{0}\right)>0$ such that

$$
\begin{equation*}
\int_{\Omega}\left|\nabla c_{\varepsilon}\right|^{2} \leq 1 \quad \text { for all } t>T_{1} \text { and } \varepsilon \in(0,1), \tag{3.30}
\end{equation*}
$$

that

$$
\begin{equation*}
\int_{T_{1}}^{\infty} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}} \leq C_{1} \quad \text { for all } \varepsilon \in(0,1) \tag{3.31}
\end{equation*}
$$

and that

$$
\begin{equation*}
\int_{T_{1}}^{\infty} \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2} \leq C_{1} \quad \text { for all } \varepsilon \in(0,1) \tag{3.32}
\end{equation*}
$$

and to make appropriate use of (3.31), we once more employ a Sobolev inequality and the GagliardoNirenberg inequality along with elliptic regularity theory to find $C_{2}>0, C_{3}>0$ and $C_{4}>0$ fulfilling

$$
\begin{equation*}
\int_{\Omega} \varphi^{2} \leq C_{2} \cdot\left\{\int_{\Omega}|\nabla \varphi|\right\}^{2}+C_{2} \cdot\left\{\int_{\Omega}|\varphi|\right\}^{2} \quad \text { for all } \varphi \in W^{1,1}(\Omega) \tag{3.33}
\end{equation*}
$$

and

$$
\begin{equation*}
\|\varphi\|_{L^{4}(\Omega)}^{2} \leq C_{3}\|\nabla \varphi\|_{L^{2}(\Omega)}\|\varphi\|_{L^{2}(\Omega)}+C_{3}\|\varphi\|_{L^{2}(\Omega)}^{2} \quad \text { for all } \varphi \in W^{1,2}(\Omega) \tag{3.34}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\|\nabla \varphi\|_{L^{4}(\Omega)}^{2} \leq C_{4}\|\Delta \varphi\|_{L^{2}(\Omega)}\|\nabla \varphi\|_{L^{2}(\Omega)} \quad \text { for all } \varphi \in W^{2,2}(\Omega) \text { such that } \frac{\partial \varphi}{\partial \nu}=0 \text { on } \partial \Omega \text {. } \tag{3.35}
\end{equation*}
$$

Then, namely, for arbitrary $\varepsilon \in(0,1)$ and each $t_{1}>T_{1}+1$ taking $t_{0}=t_{0}\left(n_{0}, c_{0}, u_{0}, \varepsilon\right) \in\left(t_{1}-1, t_{1}\right)$ in such a way that in accordance with (3.31) we have

$$
\int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\left(\cdot, t_{0}\right)\right|^{2}}{n_{\varepsilon}\left(\cdot, t_{0}\right)} \leq C_{1},
$$

by means of (3.33) and the Cauchy-Schwarz inequality we obtain that on the one hand,

$$
\begin{align*}
\int_{\Omega} n_{\varepsilon}^{2}\left(\cdot, t_{0}\right) & \leq C_{2} \cdot\left\{\int_{\Omega}\left|\nabla n_{\varepsilon}\left(\cdot, t_{0}\right)\right|\right\}^{2}+C_{2} \cdot\left\{\int_{\Omega} n_{\varepsilon}\left(\cdot, t_{0}\right)\right\}^{2} \\
& \leq C_{2} \cdot\left\{\int_{\Omega} n_{\varepsilon}\left(\cdot, t_{0}\right)\right\} \cdot \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\left(\cdot, t_{0}\right)\right|^{2}}{n_{\varepsilon}\left(\cdot, t_{0}\right)}+C_{2} \cdot\left\{\int_{\Omega} n_{\varepsilon}\left(\cdot, t_{0}\right)\right\}^{2} \\
& \leq C_{5}:=2 \pi C_{1} C_{2}+4 \pi^{2} C_{2} \tag{3.36}
\end{align*}
$$

because $\int_{\Omega} n_{\varepsilon}(\cdot, t) \leq m_{\star} \leq 2 \pi$ by (2.3).
On the other hand, testing the first equation in (2.1) by $n_{\varepsilon}$ and once more using that $\nabla \cdot u_{\varepsilon}=0$ shows that

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} \int_{\Omega} n_{\varepsilon}^{2}+\int_{\Omega}\left|\nabla n_{\varepsilon}\right|^{2} & =\int_{\Omega} \frac{n_{\varepsilon}}{1+\varepsilon n_{\varepsilon}} \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon} \\
& \leq \frac{1}{2} \int_{\Omega}\left|\nabla n_{\varepsilon}\right|^{2}+\frac{1}{2} \int_{\Omega} n_{\varepsilon}^{2}\left|\nabla c_{\varepsilon}\right|^{2} \quad \text { for all } t>0 \text { and } \varepsilon \in(0,1) \tag{3.37}
\end{align*}
$$

where thanks to the Cauchy-Schwarz inequality, (3.34), (3.35), (3.30) and Young's inequality,

$$
\begin{aligned}
\frac{1}{2} \int_{\Omega} n_{\varepsilon}^{2}\left|\nabla c_{\varepsilon}\right|^{2} & \leq \frac{1}{2}\left\|n_{\varepsilon}\right\|_{L^{4}(\Omega)}^{2}\left\|\nabla c_{\varepsilon}\right\|_{L^{4}(\Omega)}^{2} \\
& \leq \frac{C_{3} C_{4}}{2} \cdot\left\{\left\|\nabla n_{\varepsilon}\right\|_{L^{2}(\Omega)}\left\|n_{\varepsilon}\right\|_{L^{2}(\Omega)}+\left\|n_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2}\right\} \cdot\left\|\Delta c_{\varepsilon}\right\|_{L^{2}(\Omega)}\left\|\nabla c_{\varepsilon}\right\|_{L^{2}(\Omega)} \\
& \leq \frac{C_{3} C_{4}}{2}\left\|\nabla n_{\varepsilon}\right\|_{L^{2}(\Omega)}\left\|n_{\varepsilon}\right\|_{L^{2}(\Omega)}\left\|\Delta c_{\varepsilon}\right\|_{L^{2}(\Omega)}+\frac{C_{3} C_{4}}{2}\left\|n_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2}\left\|\Delta c_{\varepsilon}\right\|_{L^{2}(\Omega)} \\
& \leq \frac{1}{2} \int_{\Omega}\left|\nabla n_{\varepsilon}\right|^{2}+\frac{C_{3}^{2} C_{4}^{2}}{8} \cdot\left\{\int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}\right\} \cdot \int_{\Omega} n_{\varepsilon}^{2}+\frac{C_{3} C_{4}}{2} \cdot\left\{\int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}\right\}^{\frac{1}{2}} \cdot \int_{\Omega} n_{\varepsilon}^{2} \\
& \leq \frac{1}{2} \int_{\Omega}\left|\nabla n_{\varepsilon}\right|^{2}+\frac{C_{3}^{2} C_{4}^{2}}{4} \cdot\left\{\int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}\right\} \cdot \int_{\Omega} n_{\varepsilon}^{2}+\frac{1}{2} \int_{\Omega} n_{\varepsilon}^{2}
\end{aligned}
$$

for all $t>T_{1}$ and $\varepsilon \in(0,1)$. Therefore, from (3.37) we obtain that

$$
\frac{d}{d t} \int_{\Omega} n_{\varepsilon}^{2} \leq\left\{\frac{C_{3}^{2} C_{4}^{2}}{2} \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}+1\right\} \cdot \int_{\Omega} n_{\varepsilon}^{2} \quad \text { for all } t>T_{1} \text { and } \varepsilon \in(0,1)
$$

so that an integration relying on (3.36) reveals that

$$
\begin{aligned}
\int_{\Omega} n_{\varepsilon}^{2}\left(\cdot, t_{1}\right) & \leq\left\{\int_{\Omega} n_{\varepsilon}^{2}\left(\cdot, t_{0}\right)\right\} \cdot \exp \left\{\frac{C_{3}^{2} C_{4}^{2}}{2} \int_{t_{0}}^{t_{1}} \int_{\Omega}\left|\Delta c_{\varepsilon}\right|^{2}+\left(t_{1}-t_{0}\right)\right\} \\
& \leq C_{5} \cdot e^{\frac{C_{1} C_{3}^{2} C_{4}^{2}}{2}+1} \quad \text { for all } \varepsilon \in(0,1)
\end{aligned}
$$

because $t_{1}-t_{0} \leq 1$. As $t_{1}>T_{1}+1$ was arbitrary, this establishes (3.29) with $T:=T_{1}+1$ and $C:=C_{5} \cdot e^{\frac{C_{1} C_{3}^{2} C_{4}^{2}}{2}+1}$.
Together with (3.22) and (3.25), this in turn provides suitable regularity for a standard $H^{1}$ testing approach to the third sub-problem of (2.1), now interpreted as a linear Stokes evolution equation with its forcing term not only originating from the influence of $n_{\varepsilon}$, but also from the nonlinear convection term $\left(u_{\varepsilon} \cdot \nabla\right) u_{\varepsilon}$.

Lemma 3.4 There exists $C>0$ such that with $m_{\star} \in(0,2 \pi)$ taken from Lemma 3.2, if we are given $\left(n_{0}, c_{0}, u_{0}\right)$ fulfilling (1.5) with $\int_{\Omega} n_{0}<m_{\star}$, then we can fix $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ such that for all $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\int_{\Omega}\left|\nabla u_{\varepsilon}(\cdot, t)\right|^{2} \leq C \quad \text { for all } t>T \tag{3.38}
\end{equation*}
$$

Proof. In view of Lemma 3.2 and Lemma 3.3, we can find $C_{1}>0$ such that whenever (1.5) holds with $\int_{\Omega} n_{0}<m_{\star}$, we can find $T_{1}=T_{1}\left(n_{0}, c_{0}, u_{0}\right)>0$ such that

$$
\begin{equation*}
\int_{\Omega}\left|u_{\varepsilon}\right|^{2} \leq 1 \quad \text { for all } t>T_{1} \text { and } \varepsilon \in(0,1) \tag{3.39}
\end{equation*}
$$

and that

$$
\begin{equation*}
\int_{T_{1}}^{\infty} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq C_{1} \quad \text { for all } \varepsilon \in(0,1) \tag{3.40}
\end{equation*}
$$

as well as

$$
\begin{equation*}
\int_{\Omega} n_{\varepsilon}^{2} \leq C_{1} \quad \text { for all } t>T_{1} \text { and } \varepsilon \in(0,1) \tag{3.41}
\end{equation*}
$$

while using the Gagliardo-Nirenberg inequality along with well-known regularity features of the Stokes operator ([39]) we can pick $C_{2}>0$ in such a way that

$$
\begin{equation*}
\|\varphi\|_{L^{\infty}(\Omega)}^{2} \leq C_{2}\|A \varphi\|_{L^{2}(\Omega)}\|\varphi\|_{L^{2}(\Omega)} \quad \text { for all } \varphi \in W^{2,2}(\Omega) \cap W_{0, \sigma}^{1,2}(\Omega) \tag{3.42}
\end{equation*}
$$

Now assuming (1.5) with $\int_{\Omega} n_{0}<m_{\star}$, taking $T_{1}=T_{1}\left(n_{0}, c_{0}, u_{0}\right)$ as above and fixing an arbitrary $t_{1}>T_{1}+1$ we can rely on (3.40) in verifying that for each $\varepsilon \in(0,1)$ we can choose $t_{0}=t_{0}\left(n_{0}, c_{0}, u_{0}, \varepsilon\right)$ such that

$$
\begin{equation*}
\int_{\Omega}\left|\nabla u_{\varepsilon}\left(\cdot, t_{0}\right)\right|^{2} \leq C_{1} \tag{3.43}
\end{equation*}
$$

Next, using the third equation in (2.1) together with Young's inequality the fact that $\mathcal{P}$ acts as a projection on $L^{2}\left(\Omega ; \mathbb{R}^{2}\right)$ we can estimate

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}+\int_{\Omega}\left|A u_{\varepsilon}\right|^{2} & =-\int_{\Omega} A u_{\varepsilon} \cdot \mathcal{P}\left[\left(u_{\varepsilon} \cdot \nabla\right) u_{\varepsilon}\right]+\int_{\Omega} A u_{\varepsilon} \cdot \mathcal{P}\left[n_{\varepsilon} \nabla \Phi\right] \\
& \leq \frac{1}{2} \int_{\Omega}\left|A u_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\mathcal{P}\left[\left(u_{\varepsilon} \cdot \nabla\right) u_{\varepsilon}\right]\right|^{2}+\int_{\Omega}\left|\mathcal{P}\left[n_{\varepsilon} \nabla \Phi\right]\right|^{2} \\
& \leq \frac{1}{2} \int_{\Omega}\left|A u_{\varepsilon}\right|^{2}+\int_{\Omega}\left|\left(u_{\varepsilon} \cdot \nabla\right) u_{\varepsilon}\right|^{2}+\int_{\Omega}\left|n_{\varepsilon} \nabla \Phi\right|^{2} \tag{3.44}
\end{align*}
$$

for all $t>0$ and $\varepsilon \in(0,1)$. Here, following a standard argument (see e.g. [56, Lemma 6.2]) we can combine (3.42) with Young's inequality to see that

$$
\begin{aligned}
\int_{\Omega}\left|\left(u_{\varepsilon} \cdot \nabla\right) u_{\varepsilon}\right|^{2} & \leq\left\|u_{\varepsilon}\right\|_{L^{\infty}(\Omega)}^{2}\left\|\nabla u_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2} \\
& \leq C_{2}\left\|A u_{\varepsilon}\right\|_{L^{2}(\Omega)}\left\|u_{\varepsilon}\right\|_{L^{2}(\Omega)}\left\|\nabla u_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2} \\
& \leq C_{2}\left\|A u_{\varepsilon}\right\|_{L^{2}(\Omega)}\left\|\nabla u_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2} \\
& \leq \frac{1}{2} \int_{\Omega}\left|A u_{\varepsilon}\right|^{2}+\frac{C_{2}^{2}}{2} \cdot\left\{\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right\}^{2} \quad \text { for all } t>T_{1} \text { and } \varepsilon \in(0,1),
\end{aligned}
$$

while (3.41) asserts that

$$
\int_{\Omega}\left|n_{\varepsilon} \nabla \Phi\right|^{2} \leq C_{3}:=C_{1}\|\nabla \Phi\|_{L^{\infty}(\Omega)}^{2} \quad \text { for all } t>T_{1} \text { and } \varepsilon \in(0,1)
$$

From (3.44) we therefore conclude that

$$
\frac{d}{d t} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2} \leq C_{2}^{2} \cdot\left\{\int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right\}^{2}+2 C_{3} \quad \text { for all } t>T_{1} \text { and } \varepsilon \in(0,1)
$$

so that according to $(3.43),(3.40)$ and the fact that $t_{1}-t_{0} \leq 1$,

$$
\begin{aligned}
\int_{\Omega}\left|\nabla u_{\varepsilon}\left(\cdot, t_{1}\right)\right|^{2} & \leq\left\{\int_{\Omega}\left|\nabla u_{\varepsilon}\left(\cdot, t_{0}\right)\right|^{2}\right\} \cdot \exp \left\{C_{2}^{2} \int_{t_{0}}^{t_{1}} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right\}+2 C_{3} \int_{t_{0}}^{t_{1}} \exp \left\{C_{2}^{2} \int_{s}^{t_{1}} \int_{\Omega}\left|\nabla u_{\varepsilon}\right|^{2}\right\} d s \\
& \leq C_{1} \cdot e^{C_{1} C_{2}^{2}}+2 C_{3} \cdot e^{C_{1} C_{2}^{2}}
\end{aligned}
$$

which entails (3.38) with $T:=T_{1}+1$ and $C:=\left(C_{1}+2 C_{3}\right) e^{C_{1} C_{2}^{2}}$.
This information can be used so as to immediately improve itself through an argument now based on $L^{p}-L^{q}$ estimates for the Stokes semigroup.

Lemma 3.5 Let $\beta \in\left(\frac{1}{2}, 1\right)$. Then one can find $C(\beta)>0$ such that if $m_{\star} \in(0,2 \pi)$ is as in Lemma 3.2 and $\left(n_{0}, c_{0}, u_{0}\right)$ satisfies (1.5) with $\int_{\Omega} n_{0}<m_{\star}$, then there exists $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ such that for all $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\int_{\Omega}\left|A^{\beta} u_{\varepsilon}(\cdot, t)\right|^{2} \leq C(\beta) \quad \text { for all } t>T \tag{3.45}
\end{equation*}
$$

Proof. Given $\beta \in\left(\frac{1}{2}, 1\right)$, using that $\beta<1$ we can fix $p \in(1,2)$ sitably close to 2 such that $\beta+\frac{1}{p}-\frac{1}{2}<1$. Then relying on well-known smoothing and continuity properties of the Stokes semigroup $\left(e^{-t A}\right)_{t \geq 0}$ and the Helmholtz projection in $L^{p}\left(\Omega ; \mathbb{R}^{2}\right)\left(\left[18\right.\right.$, p. 201], [17]), we can find $C_{1}>0$ and $C_{2}>0$ such that whenever (1.5) holds, for each $t>1$ and any $\varepsilon \in(0,1)$ we have

$$
\begin{aligned}
\left\|A^{\beta} u_{\varepsilon}(\cdot, t)\right\|_{L^{2}(\Omega)}= & \| A^{\beta} e^{-A} u_{\varepsilon}(\cdot, t-1)-\int_{t-1}^{t} A^{\beta} e^{-(t-s) A} \mathcal{P}\left[\left(u_{\varepsilon}(\cdot, s) \cdot \nabla\right) u_{\varepsilon}(\cdot, s)\right] d s \\
& \quad+\int_{t-1}^{t} A^{\beta} e^{-(t-s) A_{\mathcal{P}}}\left[n_{\varepsilon}(\cdot, s) \nabla \Phi\right] d s \|_{L^{2}(\Omega)} \\
\leq & C_{1}\left\|u_{\varepsilon}(\cdot, t-1)\right\|_{L^{2}(\Omega)}+C_{1} \int_{t-1}^{t}(t-s)^{-\beta-\frac{1}{p}+\frac{1}{2}}\left\|_{\mathcal{P}}\left[\left(u_{\varepsilon}(\cdot, s) \cdot \nabla\right) u_{\varepsilon}(\cdot, s)\right]\right\|_{L^{p}(\Omega)} d s \\
& +C_{1} \int_{t-1}^{t}(t-s)^{-\beta}\left\|\mathcal{P}\left[n_{\varepsilon}(\cdot, s) \nabla \Phi\right]\right\|_{L^{2}(\Omega)} d s \\
\leq & C_{1}\left\|u_{\varepsilon}(\cdot, t-1)\right\|_{L^{2}(\Omega)}+C_{2} \int_{t-1}^{t}(t-s)^{-\beta-\frac{1}{p}+\frac{1}{2}}\left\|\left(u_{\varepsilon}(\cdot, s) \cdot \nabla\right) u_{\varepsilon}(\cdot, s)\right\|_{L^{p}(\Omega)} d s \\
& +C_{1} \int_{t-1}^{t}(t-s)^{-\beta}\left\|n_{\varepsilon}(\cdot, s) \nabla \Phi\right\|_{L^{2}(\Omega)} d s \\
\leq & C_{1}\left\|u_{\varepsilon}(\cdot, t-1)\right\|_{L^{2}(\Omega)}+C_{2} \int_{t-1}^{t}(t-s)^{-\beta-\frac{1}{p}+\frac{1}{2}}\left\|u_{\varepsilon}(\cdot, s)\right\|_{L^{\frac{2 p}{2-p}(\Omega)}}\left\|\nabla u_{\varepsilon}(\cdot, s)\right\|_{L^{2}(\Omega)} d s \\
& +C_{1}\|\nabla \Phi\|_{L^{\infty}(\Omega)} \int_{t-1}^{t}(t-s)^{-\beta}\left\|n_{\varepsilon}(\cdot, s)\right\|_{L^{2}(\Omega)} d s
\end{aligned}
$$

thanks to the Hölder inequality. Since $-\beta-\frac{1}{p}+\frac{1}{2}>-1$, and since $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2 p}{2-p}}(\Omega)$, the claim therefore results by making use of the temporally uniform boundedness features asserted by Lemma 3.2, Lemma 3.4 and Lemma 3.3.

Therefore, the fluid field eventually becomes regular enough to allow for ultimately estimating the taxis gradient in Lebesgue spaces with arbitrarily high integrability powers.

Lemma 3.6 Let $q \in(2, \infty)$. Then there exists $C(q)>0$ such that if $m_{\star} \in(0,2 \pi)$ is as in Lemma 3.2, whenever $\left(n_{0}, c_{0}, u_{0}\right)$ satisfies (1.5) with $\int_{\Omega} n_{0}<m_{\star}$, there exists $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ such that for all $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\int_{\Omega}\left|\nabla c_{\varepsilon}(\cdot, t)\right|^{q} \leq C(q) \quad \text { for all } t>T \tag{3.46}
\end{equation*}
$$

Proof. For fixed $q \in(2, \infty)$, an application of known regularization estimates for the Neumann heat semigroup $\left(e^{t \Delta}\right)_{t \geq 0}$ on $\Omega([52])$ shows that with some $C_{1}>0$, whenever (1.5) holds we have

$$
\begin{aligned}
\left\|\nabla c_{\varepsilon}(\cdot, t)\right\|_{L^{q}(\Omega)}= & \left\|\nabla e^{\Delta-1} c_{\varepsilon}(\cdot, t-1)+\int_{t-1}^{t} \nabla e^{(t-s)(\Delta-1)}\left\{n_{\varepsilon}(\cdot, s)-u_{\varepsilon}(\cdot, s) \cdot \nabla c_{\varepsilon}(\cdot, s)\right\} d s\right\|_{L^{q}(\Omega)} \\
\leq & C_{1}\left\|c_{\varepsilon}(\cdot, t-1)\right\|_{L^{1}(\Omega)} \\
& +C_{1} \int_{t-1}^{t}(t-s)^{-1+\frac{1}{q}}\left\{\left\|n_{\varepsilon}(\cdot, s)\right\|_{L^{2}(\Omega)}+\left\|u_{\varepsilon}(\cdot, s)\right\|_{L^{\infty}(\Omega)}\left\|\nabla c_{\varepsilon}\right\|_{L^{2}(\Omega)}\right\} d s
\end{aligned}
$$

for all $t>1$ and $\varepsilon \in(0,1)$. Since an application of Lemma 3.5 to an arbitrary $\beta \in\left(\frac{1}{2}, 1\right)$ in particular yields $C_{2}>0$ such that for any such $\left(n_{0}, c_{0}, u_{0}\right)$ we can find $T_{1}=T_{1}\left(n_{0}, c_{0}, u_{0}\right)>0$ fulfilling $\left\|u_{\varepsilon}(\cdot, t)\right\|_{L^{\infty}(\Omega)} \leq C_{2}$ for all $t>T_{1}$ due to continuity of the embedding $D\left(A^{\beta}\right) \hookrightarrow L^{\infty}(\Omega)([22])$, from (2.4), Lemma 3.3 and Lemma 3.2 we readily infer (3.46) with suitably large $T=T\left(n_{0}, c_{0}, u_{0}\right)>T_{1}+1$ and $C>0$.

An application of the latter to the particular exponent $q=4$ improves the information from Lemma 3.3 as follows.

Lemma 3.7 There exists $C>0$ such that if $m_{\star} \in(0,2 \pi)$ is as in Lemma 3.2, and if (1.5) holds with $\int_{\Omega} n_{0}<m_{\star}$, then one can find $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ such that for all $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\int_{\Omega} n_{\varepsilon}^{4}(\cdot, t) \leq C \quad \text { for all } t>T \tag{3.47}
\end{equation*}
$$

Proof. By (2.1), Young's inequality, the Gagliardo-Nirenberg inequality and (2.3), we can find $C_{1}>0, C_{2}>0$ and $C_{3}>0$ such that whenever (1.5) holds with $\int_{\Omega} n_{0} \leq 2 \pi$,

$$
\begin{aligned}
\frac{1}{4} \frac{d}{d t} \int_{\Omega} n_{\varepsilon}^{4}+\frac{3}{4} \int_{\Omega}\left|\nabla n_{\varepsilon}^{2}\right|^{2} & =3 \int_{\Omega} \frac{n_{\varepsilon}^{3}}{1+\varepsilon n_{\varepsilon}} \nabla n_{\varepsilon} \cdot \nabla c_{\varepsilon} \\
& \leq \frac{3}{8} \int_{\Omega}\left|\nabla n_{\varepsilon}^{2}\right|^{2}+\frac{3}{2} \int_{\Omega} n_{\varepsilon}^{4}\left|\nabla c_{\varepsilon}\right|^{2} \\
& \leq \frac{3}{8} \int_{\Omega}\left|\nabla n_{\varepsilon}^{2}\right|^{2}+\frac{3}{2}\left\|\nabla c_{\varepsilon}\right\|_{L^{4}(\Omega)}^{2}\left\|n_{\varepsilon}^{2}\right\|_{L^{4}(\Omega)}^{2}
\end{aligned}
$$

$$
\begin{aligned}
& \leq \frac{3}{8} \int_{\Omega}\left|\nabla n_{\varepsilon}^{2}\right|^{2}+C_{1}\left\|\nabla c_{\varepsilon}\right\|_{L^{4}(\Omega)}^{2} \cdot\left\{\left\|\nabla n_{\varepsilon}^{2}\right\|_{L^{2}(\Omega)}^{2}\left\|n_{\varepsilon}^{2}\right\|_{L^{\frac{1}{2}(\Omega)}}^{\frac{2}{7}}+\left\|n_{\varepsilon}^{2}\right\|_{L^{\frac{1}{2}}(\Omega)}^{\frac{16}{7}}\right\}^{\frac{7}{8}} \\
& \leq \frac{3}{8} \int_{\Omega}\left|\nabla n_{\varepsilon}^{2}\right|^{2}+C_{2}\left\|\nabla c_{\varepsilon}\right\|_{L^{4}(\Omega)}^{2} \cdot\left\{\left\|\nabla n_{\varepsilon}^{2}\right\|_{L^{2}(\Omega)}^{2}+1\right\}^{\frac{7}{8}} \\
& \leq \frac{3}{8} \int_{\Omega}\left|\nabla n_{\varepsilon}^{2}\right|^{2}+\frac{3}{8} \cdot\left\{\left\|\nabla n_{\varepsilon}\right\|_{L^{2}(\Omega)}^{2}+1\right\}+C_{3}\left\|\nabla c_{\varepsilon}\right\|_{L^{4}(\Omega)}^{16}
\end{aligned}
$$

that is,

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega} n_{\varepsilon}^{4} \leq \frac{3}{2}+4 C_{3}\left\|\nabla c_{\varepsilon}\right\|_{L^{4}(\Omega)}^{16} \quad \text { for all } t>0 \text { and } \varepsilon \in(0,1) \tag{3.48}
\end{equation*}
$$

Since another application of the Gagliardo-Nirenberg inequality together with (2.3) provides $C_{4}>0$ and $C_{5}>0$ such that for any such $\left(n_{0}, c_{0}, u_{0}\right)$ we have

$$
\begin{aligned}
\int_{t}^{t+1}\left\|n_{\varepsilon}(\cdot, t)\right\|_{L^{4}(\Omega)}^{\frac{4}{3}} d t & =\int_{t}^{t+1}\left\|n_{\varepsilon}^{\frac{1}{2}}(\cdot, t)\right\|_{L^{8}(\Omega)}^{\frac{8}{3}} d t \\
& \leq C_{4} \int_{t}^{t+1}\left\|\nabla n_{\varepsilon}^{\frac{1}{2}}(\cdot, t)\right\|_{L^{2}(\Omega)}^{2}\left\|n_{\varepsilon}^{\frac{1}{2}}(\cdot, t)\right\|_{L^{2}(\Omega)}^{\frac{2}{3}} d t+C_{4} \int_{t}^{t+1}\left\|n_{\varepsilon}^{\frac{1}{2}}(\cdot, t)\right\|_{L^{2}(\Omega)}^{\frac{8}{3}} d t \\
& \leq C_{5} \int_{t}^{t+1} \int_{\Omega} \frac{\left|\nabla n_{\varepsilon}\right|^{2}}{n_{\varepsilon}}+C_{5} \quad \text { for all } t>0 \text { and } \varepsilon \in(0,1)
\end{aligned}
$$

and since thus Lemma 3.2 implies the existence of $C_{6}>0$ and $T_{1}=T_{1}\left(n_{0}, c_{0}, u_{0}\right)$ such that

$$
\int_{t}^{t+1}\left\|n_{\varepsilon}(\cdot, t)\right\|_{L^{4}(\Omega)}^{\frac{4}{3}} d t \leq C_{6} \quad \text { for all } t>T_{1} \text { and } \varepsilon \in(0,1)
$$

given any $t_{1}>T_{1}+1$ we can find $t_{0}=t_{0}\left(n_{0}, c_{0}, u_{0}, \varepsilon\right) \in\left(t_{1}-1, t_{1}\right)$ such that

$$
\int_{\Omega} n_{\varepsilon}^{4}\left(\cdot, t_{0}\right) \leq C_{6}^{3}
$$

so that integrating (3.48) yields

$$
\int_{\Omega} n_{\varepsilon}^{4}\left(\cdot, t_{1}\right) \leq C_{6}^{3}+4 C_{3} \sup _{t \in\left(t_{1}-1, t_{1}\right)}\left\|\nabla c_{\varepsilon}(\cdot, t)\right\|_{L^{4}(\Omega)}^{16}
$$

Employing Lemma 3.6 with $q:=4$ thus shows that (3.47) can be achieved on choosing $C>0$ and $T=T\left(n_{0}, c_{0}, u_{0}\right)>T_{1}+1$ suitably large.
Together with Lemma 3.6 and Lemma 3.4, this asserts eventual boundedness in the first solution component:

Lemma 3.8 There exists $C>0$ such that if $m_{\star} \in(0,2 \pi)$ is as in Lemma 3.2, and if (1.5) is satisfied with $\int_{\Omega} n_{0}<m_{\star}$, then there exists $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ such that for all $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\int_{\Omega}\left\|n_{\varepsilon}(\cdot, t)\right\|_{L^{\infty}(\Omega)} \leq C \quad \text { for all } t>T \tag{3.49}
\end{equation*}
$$

Proof. We take any $q \in(2,4)$ and once more invoke standard smoothing estimates for the Neumann heat semigroup $\left(e^{t \Delta}\right)_{t \geq 0}([16])$ to find $C_{1}>0$ such that whenever (1.5) holds, we have

$$
\begin{aligned}
\left\|n_{\varepsilon}(\cdot, t)\right\|_{L^{\infty}(\Omega)}= & \left\|e^{\Delta} n_{\varepsilon}(\cdot, t-1)-\int_{t-1}^{t} e^{(t-s) \Delta} \nabla \cdot\left\{\frac{n_{\varepsilon}(\cdot, s)}{1+\varepsilon n_{\varepsilon}(\cdot, s)} \nabla c_{\varepsilon}(\cdot, s)+n_{\varepsilon}(\cdot, s) u_{\varepsilon}(\cdot, s)\right\} d s\right\|_{L^{\infty}(\Omega)} \\
\leq & C_{1}\left\|n_{\varepsilon}(\cdot, t-1)\right\|_{L^{1}(\Omega)} \\
& +C_{1} \int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{1}{q}}\left\{\left\|\frac{n_{\varepsilon}(\cdot, s)}{1+\varepsilon n_{\varepsilon}(\cdot, s)} \nabla c_{\varepsilon}(\cdot, s)\right\|_{L^{q}(\Omega)}+\left\|n_{\varepsilon}(\cdot, s) u_{\varepsilon}(\cdot, s)\right\|_{L^{q}(\Omega)}\right\} d s \\
\leq & C_{1}\left\|n_{\varepsilon}(\cdot, t-1)\right\|_{L^{1}(\Omega)} \\
& +C_{1} \int_{t-1}^{t}(t-s)^{-\frac{1}{2}-\frac{1}{q}}\left\|n_{\varepsilon}(\cdot, s)\right\|_{L^{4}(\Omega)} \cdot\left\{\left\|\nabla c_{\varepsilon}(\cdot, s)\right\|_{L^{\frac{4 q}{4-q}}(\Omega)}+\left\|u_{\varepsilon}(\cdot, s)\right\|_{L^{\frac{4 q}{4-q}}(\Omega)}\right\} d s
\end{aligned}
$$

for all $t>1$ and $\varepsilon \in(0,1)$. Therefore, the claim is a consequence of Lemma 3.7, Lemma 3.6 and e.g. Lemma 3.4, because $W^{1,2}(\Omega) \hookrightarrow L^{\frac{4 q}{4-q}}(\Omega)$.

Based on the latter, the following $\varepsilon$-independent eventual smoothness property can be derived in a straightforward way.

Lemma 3.9 There exist $\theta \in(0,1)$ and $C>0$ with the property that if $m_{\star} \in(0,2 \pi)$ is taken from Lemma 3.2, and if (1.5) holds with $\int_{\Omega} n_{0}<m_{\star}$, then there exists $T=T\left(n_{0}, c_{0}, u_{0}\right)>0$ such that for all $\varepsilon \in(0,1)$,

$$
\left\|n_{\varepsilon}\right\|_{C^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times[t, t+1])}+\left\|c_{\varepsilon}\right\|_{C^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times[t, t+1])}+\left\|u_{\varepsilon}\right\|_{C^{2+\theta, 1+\frac{\theta}{2}}(\bar{\Omega} \times[t, t+1])} \leq C \quad \text { for all } t>T \text {. }
$$

Proof. This can be seen by employing a series of quite well-established bootstrap arguments to the three sub-problems of (2.1) on the basis of Lemma 3.8, Lemma 3.6, Lemma 3.5 and standard regularity theories for the inhomogeneous heat and Stokes evolution equations, respectively (cf. e.g. [60] for a precedent providing details in a closely related setting).
In view of Lemma 2.14, this immediately entails the first part of the claim made in Theorem 1.2:
Lemma 3.10 Let $m_{\star} \in(0,2 \pi)$ be as in Lemma 3.2, and suppose that (1.5) holds with $\int_{\Omega} n_{0}<m_{\star}$. Then there exist $T>0$ and $P \in C^{1,0}(\Omega \times(T, \infty))$ such that the functions $n$, $c$ and $u$ from Lemma 2.14) satisfy (1.13), and that $(n, c, u, P)$ forms a classical solution of the boundary value problem in (1.3) in $\Omega \times(T, \infty)$.

Proof. This is an evident consequence of Lemma 3.9, Lemma 2.14, (2.1) and a standard construction of associated pressures in the Navier-Stokes system ([39]).
Thanks to the ultimate $C^{2}$ compactness property of trajectories implied by Lemma 3.9 , we no only need to recall the decay statements contained in Lemma 3.2 to finally make sure that such solutions smoothly approach the equilibrium $\left(\bar{n}_{0}, \bar{n}_{0}, 0\right)$ in the large time limit.

Lemma 3.11 Assume (1.5) with $\int_{\Omega} n_{0}<m_{\star}$, where $m_{\star} \in(0,2 \pi)$ is as in Lemma 3.2. Then the triple $(n, c, u)$ gained in Lemma 2.14 has the convergence properties in (1.14).

Proof. According to Lemma 3.9, Lemma 2.14 and the Arzelà-Ascoli theorem, there exists $T>0$ such that
$(n(\cdot, t))_{t>T}, \quad(c(\cdot, t))_{t>T}$ and $(u(\cdot, t))_{t>T}$ are relatively compact with respect to the norm in $C^{2}(\bar{\Omega})$.
On the other hand, since a Cziszár-Kullback inequality ([12]) provides $C_{1}>0$ such that

$$
\left\|n_{\varepsilon}(\cdot, t)-\bar{n}_{0}\right\|_{L^{1}(\Omega)} \leq C_{1} \int_{\Omega} n_{\varepsilon}(\cdot, t) \ln \frac{n_{\varepsilon}(\cdot, t)}{\bar{n}_{0}} \quad \text { for all } t>0 \text { and } \varepsilon \in(0,1)
$$

from the decay properties in (3.20) and (3.22) we infer on letting $\varepsilon=\varepsilon_{j} \searrow 0$, with $\left(\varepsilon_{j}\right)_{j \in \mathbb{N}}$ as given by Lemma 2.14, that

$$
\begin{equation*}
\left\|n(\cdot, t)-\bar{n}_{0}\right\|_{L^{1}(\Omega)} \rightarrow 0 \quad \text { and } \quad\|u(\cdot, t)\|_{L^{2}(\Omega)} \rightarrow 0 \quad \text { as } t \rightarrow \infty \tag{3.51}
\end{equation*}
$$

Moreover, Lemma 2.1 along with Lemma 2.14 shows that

$$
\int_{\Omega} c(\cdot, t)=\int_{\Omega} n_{0}+\left\{\int_{\Omega} c_{0}-\int_{\Omega} n_{0}\right\} \cdot e^{-t} \quad \text { for all } t>T
$$

and that hence

$$
\overline{c(\cdot, t)} \rightarrow \bar{n}_{0} \quad \text { ast } \rightarrow \infty
$$

As a Poincaré inequality ensures the existence of $C_{2}>0$ such that

$$
\int_{\Omega}|c(\cdot, t)-\overline{c(\cdot, t)}|^{2} \leq C_{2} \int_{\Omega}|\nabla c(\cdot, t)|^{2} \quad \text { for all } t>T
$$

from (3.21) we thus infer that

$$
\begin{equation*}
\left\|c(\cdot, t)-\bar{n}_{0}\right\|_{L^{2}(\Omega)} \rightarrow 0 \quad \text { as } t \rightarrow \infty \tag{3.52}
\end{equation*}
$$

A straightforward combination of (3.51) and (3.52) with (3.50) readily yields (1.14).
Our main result on eventual smoothness and stabilization of small-mass solutions has thereby in fact already been achieved completely:
Proof of Theorem 1.2. We only need to combine Lemma 3.10 with Lemma 3.11 .

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