Global solvability in a three-dimensional Keller-Segel-Stokes system involving arbitrary superlinear logistic degradation

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Abstract

The Keller-Segel-Stokes system

$$\left\{\begin{array}{rcll} n_t + u\cdot\nabla n &=& \Delta n - \nabla\cdot(n\nabla c) + \rho n - \mu n^\alpha, & \\ c_t + u\cdot\nabla c &=& \Delta c - c + n, & \qquad (\star) \\ u_t &=& \Delta u + \nabla P - n\nabla\Lambda, \qquad \nabla\cdot u = 0, & \end{array}\right.$$

is considered in a bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, with parameters $\rho\geq 0$, $\mu>0$ and $\alpha>1$, and with a given gravitational potential $\Lambda\in W^{2,\infty}(\Omega)$.

It is shown that in this general setting, when posed under no-flux boundary conditions for n and c and homogeneous Dirichlet boundary conditions for u, and for any suitably regular initial data, an associated initial value problem possesses at least one globally defined solution in an appropriate generalized sense. Since it is well-known that in the absence of absorption, already the corresponding fluid-free subsystem with $u\equiv 0$ and $\mu=0$ admits some solutions blowing up in finite time, this particularly indicates that any power-type superlinear degradation of the form in $(\star)$ goes along with some significant regularizing effect.

Key words: chemotaxis; Stokes; logistic source; generalized solution
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1 Introduction

The damping role of logistic sources in classical Keller-Segel systems. Chemotaxis, the biased movement of individuals corresponding to concentration gradients of a chemical signal, is known to be a mechanism of great significance for pattern formation in numerous biological contexts. Concentrating on cross-diffusion and signal production through cells as the main ingredients of a corresponding feedback loop, the apparently most prominent mathematical description of essential aspects within such processes is achieved by the classical Keller-Segel model ([8]) in its minimal version, as given by

$$\left\{ \begin{array}{rcl} n_t &=& \Delta n - \nabla \cdot (n \nabla c), \\ c_t &=& \Delta c - c + n. \end{array} \right. \tag{1.1}$$

Indeed, several studies have revealed a distinct ability of this system to describe spontaneous aggregation in two- and higher-dimensional settings; for instance, it is well-known that the homogeneous Neumann problem associated with (1.1) in balls $\Omega \subset \mathbb{R}^N$ possesses some solutions blowing up in finite time when either $N \geq 3$, or when $N = 2$ and the total mass of cells is large ([5], [27]).

In cases when the respective application contexts involve large time scales, model refinements which appropriately account for proliferation and death of cells seem in order. Yet leaving further coupling unaddressed, Keller-Segel-growth systems of the form

$$\left\{ \begin{array}{rcl} n_t &=& \Delta n - \nabla \cdot (n \nabla c) + \rho n - \mu n^\alpha, \\ c_t &=& \Delta c - c + n, \end{array} \right. \tag{1.2}$$

provide modifications of (1.1) which incorporate such mechanisms through the inclusion of standard logistic proliferation terms, and hence in a manner quite common in biomathematical modeling. Mathematically, the introduction of such logistic effects can rule out the possibility of unboundedness phenomena in the style of those known to occur for (1.1); indeed, in the most prototypical framework related to the choice $\alpha = 2$, corresponding Neumann-type initial-boundary value problems admit bounded smooth solutions for essentially arbitrary initial data if either the spatial dimension N satisfies $N = 2$ and μ is any positive number ([13]), or $N \geq 3$ and $\mu \geq \mu_0$ with some suitably large $\mu_0 = \mu_0(\rho, \Omega) > 0$ ([25]; cf. also [7], [34], [35] and the references therein for some more studies concerned with global classical solvability of (1.2)).

Further findings in the literature, however, indicate that also the presence of weaker degradation terms may go along with significant relaxation effects in comparison with (1.1): For instance, in the weakly quadratic absorption case when in (1.2) we have $\alpha = 2$, $N \geq 3$ and $\mu \in (0, \mu_0)$, after all some global solutions can be seen to exist within a natural weak solution concept ([11]). By resorting to some yet weaker notions of solvability, global solutions could more recently also be constructed for systems merely containing certain subquadratic degradation terms, where first steps in this direction required that $\alpha \geq 2 - \frac{1}{N}$ when $N \geq 2$ ([20], [21]), and where subsequent approaches successively facilitated extensions to the wider ranges $\alpha > \frac{2N+4}{N+4}$ ([32]) and $\alpha > \min\{\frac{2N+4}{N+4}, 2 - \frac{2}{N}\}$ ([36]); very recently, a framework generalized solvability was designed which allows for the construction of global solutions even in quite arbitrary superlinearly dampened logistic-type Keller-Segel systems, including the choice of arbitrary $\alpha > 1$ in (1.2) whenever $N \geq 2$ ([33]). It might be noted that in general it seems indeed appropriate to suitably relax requirements on regularity of "solutions" to (1.2), as some caveats assert

the occurrence of blow-up, with respect to spatial L^∞ norms of n, in parabolic-elliptic variants of (1.2): It is known, for example, that such explosions occur in the variant of (1.2) in which the respective second equation is replaced with $0 = \Delta c - c + n$ when $\alpha < \frac{7}{6}$ for $N \in \{3, 4\}$, or $\alpha < 1 + \frac{1}{2(N-1)}$ for $N \geq 5$ ([30]; see also [3], [17], [26] for several precedents in this direction).

Keller-Segel-fluid systems involving logistic degradation. The object of the present study is the Keller-Segel-Stokes system

$$\left\{ \begin{array}{rcll} n_t + u \cdot \nabla n &=& \Delta n - \nabla \cdot (n \nabla c) + \rho n - \mu n^\alpha, & x \in \Omega, \ t > 0, \\ c_t + u \cdot \nabla c &=& \Delta c - c + n, & x \in \Omega, \ t > 0, \\ u_t &=& \Delta u + \nabla P - n \nabla \Lambda, \quad \nabla \cdot u = 0, & x \in \Omega, \ t > 0, \end{array} \right. \tag{1.3}$$

which quite in general can be viewed as an extension of (1.2) that accounts for interaction of proliferating and chemotactically migrating populations with liquid environments through transport and buoyancy in the style of the modeling approach in [19]. More specifically, this system can, in particular, be used to model effects of chemotaxis on processes of coral fertilization, as addressed in [9] and [10] in the context of a related parabolic-elliptic chemotaxis system involving a given solenoidal fluid field. Indeed, in [9] and [10] the chemotactically directed motion of spermatozoids toward eggs is described in the framework of a two-component model accounting for transport through a prescribed fluid, with a focus on the question how far taxis may affect the reaction mechanism, in that particular context corresponding to the choice $\rho = 0$ and hence to a merely sink-type reaction term of the form $-\mu n^\alpha$, by influencing the total fertilization rate, as measured by the quantity $\int_\Omega n(x,t)dx$. The model (1.3) now extends this by considering the fluid field as an additional unknown, the evolution of which is governed by the incompressible Stokes system forced by the population density variable due to buoyancy in a given external gravitational potential Λ.

Due to an accordingly increased mathematical complexity, systems of the form (1.3) seem much less thoroughly understood than associated chemotaxis-only counterparts; in particular, the literature in this direction seems limited to the case $\alpha = 2$ of quadratic degradation: A result on global existence of weak solutions to a no-flux/no-flux/Dirichlet initial-boundary value therefor in two-dimensional bounded domains was established in [1] when $\rho = 0$, and in [15] it was seen that actually global bounded classical solutions can be found, even in a corresponding variant of (1.3) involving the full Navier-Stokes equations rather than its Stokes simplification, and for arbitrary $\rho \geq 0$. The three-dimensional version of (1.3) with $\alpha = 2$ is known to possess global bounded smooth solutions in convex domains whenever $\mu > 23$ ([16]), whereas for arbitrary $\mu > 0$ at least some global generalized solutions exist, again even a corresponding Navier-Stokes framework ([31]). No existence result, however, seems available for any chemotaxis-(Navier-)Stokes systems of the form (1.3) which involves subquadratic degradation in that $\alpha < 2$.

Main results. In order to address this apparently open solvability question in the context of a

prototypically simple initial-boundary value problem for (1.3), we shall subsequently consider

$$\left\{\begin{array}{rcll} n_t + u\cdot\nabla n &=& \Delta n - \nabla\cdot(n\nabla c) + \rho n - \mu n^\alpha, & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &=& \Delta c - c + n, & x\in\Omega,\ t>0,\\ u_t &=& \Delta u + \nabla P - n\nabla\Lambda, \quad \nabla\cdot u = 0, & x\in\Omega,\ t>0,\\ \frac{\partial n}{\partial\nu} = \frac{\partial c}{\partial\nu} = 0, \quad u=0, & & x\in\partial\Omega,\ t>0,\\ n(x,0)=n_0(x), \quad c(x,0)=c_0(x), \quad u(x,0)=u_0(x), & & x\in\Omega, \end{array}\right. \tag{1.4}$$

in a bounded domain $\Omega\subset\mathbb{R}^3$ with smooth boundary, with parameters $\rho\geq 0$, $\mu>0$ and $\alpha>1$, and with a given gravitational potential $\Lambda\in W^{2,\infty}(\Omega)$.

Throughout our analysis, we shall assume that the initial data therein are such that

$$\left\{\begin{array}{l} n_0\in C^0(\overline{\Omega}) \quad \text{is nonnegative, that}\\ c_0\in W^{1,\infty}(\Omega) \quad \text{is nonnegative,} \qquad \text{and that}\\ u_0\in D(A^\vartheta) \quad \text{with some } \vartheta\in(\frac{3}{4},1), \end{array}\right. \tag{1.5}$$

where A denotes the realization of the Stokes operator in the solenoidal subspace $L^2_\sigma(\Omega)=\{\varphi\in L^2(\Omega)|\nabla\cdot\varphi=0\}$ of $L^2(\Omega)$.

In this framework, we shall see that also in this considerably more complex setting than the fluid-free one in (1.2), actually *any* $\alpha>1$ is sufficient to ensure global existence of certain solutions:

Theorem 1.1 *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with smooth boundary, and suppose that $\rho\geq 0$, $\mu>0$ and*

$$\alpha>1, \tag{1.6}$$

and that $\Lambda\in W^{2,\infty}(\Omega)$. Then for any choice of (n_0,c_0,u_0) fulfilling (1.5), the problem (1.4) possesses at least one global generalized solution (n,c,u) in the sense of Definition 2.5 below.

Main ideas and key steps. In view of the circumstance that for small values of $\alpha>1$ we apparently can only expect quite poor a priori information on solution regularity, a major challenge, newly arising in the analysis of (1.4) when compared to that of (1.2), will especially be linked to the question how far regularity features of the taxis gradient may be influenced by the considered fluid interaction. Unlike in the associated unperturbed case with $u\equiv 0$ in which some strong spatio-temporal L^q precompactness properties of ∇c can be derived solely from temporally uniform L^1 bounds for n on the basis of rather standard regularization features in the corresponding semilinear heat equation solved by c (see, e.g., [33, Lemma 5.1]), in the present situation a corresponding argument evidently needs to appropriately cope with the additional appearance of the nonlinear convective term $-u\cdot\nabla c$ in the second equation from (1.4). The core of our analysis will thus consist in making adequate use of some basic integrability properties of the fluid velocity field, as quite directly resulting from L^1 boundedness of n thanks to standard smoothing properties of the Stokes semigroup (Lemma 2.3), in order to derive certain compactness features of c and especially ∇c, where by referring to certain strong L^p topologies inter alia ensure a.e. pointwise convergence along some sequences of solutions to suitably regularized problems (cf. (2.1) and Lemma 5.1). A key step toward this will be accomplished in Lemma 3.7, in which an argument based on maximal Sobolev regularity in inhomogeneous linear heat

equations will be applied to a variant of $(-\Delta+1)^{-\frac{1}{2}+\delta}c$ for suitably small $\delta = \delta(\alpha) > 0$, and where said basic regularity features of n and u will be used to estimate correspondingly arising inhomogeneities through an appropriate interpolation inequality, to be prepared in Lemma 3.6. In the context of a suitably weak solution concept quite closely paralleling that pursued in the fluid-free counterpart in [33], this pointwise convergence property will form an essential ingredient in an adequate limit process, to be performed in Lemma 5.2.

2 Preliminaries

2.1 Appropriate solutions and basic properties thereof

As a conveniently regularized variant of (1.4), for $\varepsilon \in (0,1)$ we shall subsequently consider the approximate problem

$$\left\{ \begin{array}{rcll} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon &=& \Delta n_\varepsilon - \nabla \cdot (n_\varepsilon \nabla c_\varepsilon) + \rho n_\varepsilon - \mu n_\varepsilon^\alpha, & x \in \Omega,\ t > 0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon &=& \Delta c_\varepsilon - c_\varepsilon + \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}, & x \in \Omega,\ t > 0, \\ u_{\varepsilon t} &=& \Delta u_\varepsilon + \nabla P_\varepsilon - n_\varepsilon \nabla \Lambda, & x \in \Omega,\ t > 0, \\ \frac{\partial n_\varepsilon}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = 0, \quad u_\varepsilon = 0, & & & x \in \partial\Omega,\ t > 0, \\ n_\varepsilon(x,0) = n_0(x), \quad c_\varepsilon(x,0) = c_0(x), \quad u_\varepsilon(x,0) = u_0(x), & & & x \in \Omega, \end{array} \right. \tag{2.1}$$

that according to a well-established construction on the basis of the contraction mapping principle can be seen to admit local-in-time smooth solutions which, due to boundedness of $0 \le n \mapsto \frac{n}{1+\varepsilon n}$ for each $\varepsilon \in (0,1)$, can actually be extended so as to become global classical solutions (cf., e.g., [29] and [23] for details in closely related situations):

Lemma 2.1 *Let $\varepsilon \in (0,1)$. Then there exist uniquely determined functions*

$$\left\{ \begin{array}{l} n_\varepsilon \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)), \\ c_\varepsilon \in \bigcap_{q>3} C^0([0,\infty); W^{1,q}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \qquad \textit{and} \\ u_\varepsilon \in C^0([0,\infty); D(A^\vartheta)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)) \end{array} \right.$$

such that $n_\varepsilon \ge 0$ and $c_\varepsilon \ge 0$ in $\Omega \times (0,\infty)$, and that (2.1) is satisfied in the classical sense with some $P_\varepsilon \in C^{1,0}(\Omega \times (0,\infty))$.

The following basic and essentially well-known properties of these solutions are due to the presence of the degradation term in the first equation of (2.1).

Lemma 2.2 *Let $\alpha > 1$. Then*

$$\int_\Omega n_\varepsilon(\cdot,t) \le m := \max\left\{ \int_\Omega n_0 \,,\, \left(\frac{\rho}{\mu}\right)^{\frac{1}{\alpha-1}} |\Omega| \right\} \qquad \textit{for all } t > 0 \textit{ and } \varepsilon \in (0,1), \tag{2.2}$$

and moreover

$$\int_0^T \int_\Omega n_\varepsilon^\alpha \le \frac{(1+\rho T) \cdot m}{\mu} \qquad \textit{for all } T > 0 \textit{ and } \varepsilon \in (0,1). \tag{2.3}$$

PROOF. By integration of the first equation in (2.1) we see that

$$\frac{d}{dt}\int_\Omega n_\varepsilon = \rho\int_\Omega n_\varepsilon - \mu\int_\Omega n_\varepsilon^\alpha \qquad \text{for all } t>0, \tag{2.4}$$

so that since $\int_\Omega n_\varepsilon^\alpha \geq |\Omega|^{1-\alpha}\Big\{\int_\Omega n_\varepsilon\Big\}^\alpha$ for all $t>0$ by the Hölder inequality, (2.2) readily follows through a straightforward ODE comparison argument. Thereupon, a direct time integration in (2.4) shows that

$$\int_\Omega n_\varepsilon(\cdot,T) + \mu\int_0^T\int_\Omega n_\varepsilon^\alpha \leq \int_\Omega n_0 + \rho\int_0^T\int_\Omega n_\varepsilon \leq \int_\Omega n_0 + \rho T m \qquad \text{for all } T>0$$

and hence establishes (2.3). □

Through a standard argument based on well-known smoothing properties of the Stokes semigroup, the L^1 boundedness feature expressed in (2.2) can readily be seen to entail the following (cf. also [22, Lemma 2.5] and [29, Corollary 3.4]).

Lemma 2.3 *Let $\alpha>1$, and let $\delta\in(0,\frac{3}{2})$ and $T>0$. Then there exists $C(\delta,T)>0$ such that*

$$\|u_\varepsilon(\cdot,t)\|_{L^{3-\delta}(\Omega)} \leq C(\delta,T) \qquad \text{for all } t\in(0,T) \text{ and any } \varepsilon\in(0,1) \tag{2.5}$$

and

$$\|\nabla u_\varepsilon(\cdot,t)\|_{L^{\frac{3}{2}-\delta}(\Omega)} \leq C(\delta,T) \qquad \text{for all } t\in(0,T) \text{ and each } \varepsilon\in(0,1). \tag{2.6}$$

2.2 A generalized notion of solvability

In this section, we quite closely follow the approach in [33] to develop a generalized solution concept suitable for our purposes. This will be based on the following observation on how products of the form $\phi(n)\psi(c)$ evolve in time, along suitably smooth trajectories either of (2.1) for $\varepsilon\in(0,1)$, or of the original problem (1.4) – the existence of the latter being formally presupposed here only.

Lemma 2.4 *Let ϕ,ψ and Φ belong to $C^2([0,\infty))$ with $\phi\geq 0, \psi>0, \phi''>0$ and $\Phi'=\sqrt{\phi''}$ on $[0,\infty)$, and suppose that $\varepsilon\in[0,1)$ and $T\in(0,\infty]$, and that $n\in C^{2,1}(\overline{\Omega}\times(0,T)), c\in C^{2,1}(\overline{\Omega}\times(0,T)), u\in C^{2,1}(\overline{\Omega}\times(0,T);\mathbb{R}^3)$ and $P\in C^{1,0}(\Omega\times(0,T))$ are such that $n\geq 0$ and $c\geq 0$ and that (2.1) holds in $\Omega\times(0,T)$. Then for arbitrary $\varphi\in C^\infty(\overline{\Omega}\times(0,T))$,*

$$\begin{aligned}
&\int_\Omega \partial_t\Big\{\phi(n)\psi(c)\Big\}\cdot\varphi \\
&\quad= -\int_\Omega \left|\nabla\Big(\Phi(n)\sqrt{\psi(c)}\Big) + \Big\{\frac{\phi'(n)}{\sqrt{\phi''(n)}}\cdot\frac{\psi'(c)}{\sqrt{\psi(c)}} - \frac{1}{2}\Phi(n)\frac{\psi'(c)}{\sqrt{\psi(c)}} - \frac{1}{2}n\sqrt{\phi''(n)}\sqrt{\psi(c)}\Big\}\cdot\nabla c\right|^2\cdot\varphi \\
&\qquad -\int_\Omega\Big\{\phi(n)\psi''(c) - \frac{\phi'^2(n)}{\phi''(n)}\cdot\frac{\psi'^2(c)}{\psi(c)} - \frac{1}{4}n^2\phi''(n)\psi(c)\Big\}\cdot|\nabla c|^2\varphi \\
&\qquad -\int_\Omega \frac{\phi'(n)}{\sqrt{\phi''(n)}}\sqrt{\psi(c)}\nabla\Big(\Phi(n)\sqrt{\psi(c)}\Big)\cdot\nabla\varphi \\
&\qquad +\int_\Omega\Big\{n\phi'(n)\psi(c) - \phi(n)\psi'(c) + \frac{1}{2}\frac{\Phi(n)\phi'(n)}{\sqrt{\phi''(n)}}\psi'(c)\Big\}\nabla c\cdot\nabla\varphi + \int_\Omega\phi(n)\psi(c)u\cdot\nabla\varphi \\
&\qquad +\int_\Omega\Big\{(\rho n-\mu n^\alpha)\phi'(n)\psi(c) - c\phi(n)\psi'(c) + \frac{n}{1+\varepsilon n}\phi(n)\psi'(c)\Big\}\cdot\varphi.
\end{aligned} \tag{2.7}$$

PROOF. This result can be proved by quite a trivial modification of that from Lemma 3.1 in [33]. □

On the basis of a formal evaluation of (2.7) for $\varepsilon = 0$, we can thereby extend the solution concept specified in [33] to the present case involving fluid interaction in a straightforward manner (cf. also [24] and [12] for related precedents).

Definition 2.5 *Let*

$$\left\{ \begin{array}{l} n \in L^{\alpha}_{loc}(\overline{\Omega} \times [0,\infty)), \\ c \in L^{1}_{loc}([0,\infty); W^{1,1}(\Omega)) \qquad \textit{and} \\ u \in L^{1}_{loc}([0,\infty); W^{1,1}_{0}(\Omega; \mathbb{R}^3)) \end{array} \right. \tag{2.8}$$

be such that $n \geq 0, c \geq 0$ *and* $\nabla \cdot u = 0$ *a.e. in* $\Omega \times (0,\infty)$. *Then we will call* (n,c,u) *a* global generalized solution of (1.4) *if*

$$\int_\Omega n(\cdot,t) \leq \int_\Omega n_0 + \rho \int_0^t \int_\Omega n - \mu \int_0^t \int_\Omega n^\alpha \qquad \textit{for a.e. } t > 0 \tag{2.9}$$

and

$$-\int_0^\infty \int_\Omega c\varphi_t - \int_\Omega c_0 \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla c \cdot \nabla \varphi - \int_0^\infty \int_\Omega c\varphi + \int_0^\infty \int_\Omega n\varphi + \int_0^\infty \int_\Omega cu \cdot \nabla \varphi \tag{2.10}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, *if*

$$-\int_0^\infty \int_\Omega u \cdot \varphi_t - \int_\Omega u_0 \cdot \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla u \cdot \nabla \varphi - \int_0^\infty \int_\Omega n\varphi \cdot \nabla \Lambda \tag{2.11}$$

for all $\varphi \in C_0^\infty(\Omega \times [0,\infty))$ *fulfilling* $\nabla \cdot \varphi = 0$ *in* $\Omega \times (0,\infty)$, *and if moreover one can find functions* ϕ, ψ *and* Φ *which belong to* $C^2([0,\infty))$ *and satisfy*

$$\phi' < 0, \quad \psi > 0 \quad \textit{and} \quad \phi'' > 0 \qquad \textit{on } [0,\infty) \tag{2.12}$$

as well as

$$\Phi' = \sqrt{\phi''} \qquad \textit{on } [0,\infty), \tag{2.13}$$

such that

$$\begin{array}{l} \Big\{ \phi(n)\psi''(c) - \dfrac{\phi'^2(n)}{\phi''(n)} \cdot \dfrac{\psi'^2(c)}{\psi(c)} - \dfrac{1}{4} n^2 \phi''(n)\psi(c) \Big\} |\nabla c|^2, \quad n\phi'(n)\psi(c)|\nabla c|, \\ \phi(n)\psi'(c)|\nabla c|, \quad \dfrac{\Phi(n)\phi'(n)}{\sqrt{\phi''(n)}} \psi'(c)|\nabla c| \quad \textit{and} \quad \phi(n)\psi(c)|u| \qquad \textit{as well as} \\ n^\alpha \phi'(n)\psi(c), \quad c\phi(n)\psi'(c) \quad \textit{and} \quad n\phi(n)\psi'(c) \quad \textit{belong to } L^1_{loc}(\overline{\Omega} \times [0,\infty)), \end{array} \tag{2.14}$$

that

$$\Phi(n)\sqrt{\psi(c)} \in L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{2.15}$$

and that

$$
\begin{aligned}
&-\int_0^\infty\int_\Omega \phi(n)\psi(c)\varphi_t - \int_\Omega \phi(n_0)\psi(c_0)\varphi(\cdot,0)\\
&\quad\le\ -\int_0^\infty\int_\Omega \left|\nabla\Big(\Phi(n)\sqrt{\psi(c)}\Big) + \Big\{\frac{\phi'(n)}{\sqrt{\phi''(n)}}\cdot\frac{\psi'(c)}{\sqrt{\psi(c)}} - \frac{1}{2}\Phi(n)\frac{\psi'(c)}{\sqrt{\psi(c)}} - \frac{1}{2}n\sqrt{\phi''(n)}\cdot\sqrt{\psi(c)}\Big\}\nabla c\right|^2\varphi\\
&\qquad -\int_0^\infty\int_\Omega \Big\{\phi(n)\psi''(c) - \frac{\phi'^2(n)}{\phi''(n)}\cdot\frac{\psi'^2(c)}{\psi(c)} - \frac{1}{4}n^2\phi''(n)\psi(c)\Big\}\cdot|\nabla c|^2\varphi\\
&\qquad -\int_0^\infty\int_\Omega \frac{\phi'(n)}{\sqrt{\phi''(n)}}\sqrt{\psi(c)}\nabla\Big(\Phi(n)\sqrt{\psi(c)}\Big)\cdot\nabla\varphi\\
&\qquad +\int_0^\infty\int_\Omega \Big\{n\phi'(n)\psi(c) - \phi(n)\psi'(c) + \frac{1}{2}\frac{\Phi(n)\phi'(n)}{\sqrt{\phi''(n)}}\psi'(c)\Big\}\nabla c\cdot\nabla\varphi\\
&\qquad +\int_0^\infty\int_\Omega \phi(n)\psi(c)u\cdot\nabla\varphi\\
&\qquad +\int_0^\infty\int_\Omega \Big\{(\rho n-\mu n^\alpha)\phi'(n)\psi(c) - c\phi(n)\psi'(c) + n\phi(n)\psi'(c)\Big\}\cdot\varphi
\end{aligned}
\tag{2.16}
$$

for each nonnegative $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$.

Remark. Similar to a corresponding comment made in [33] for the associated fluid-free counterpart, quite simple adaptation of the arguments detailed in [12, Lemma 2.5] and [28, Lemma 2.1] shows that this concept is indeed consistent with that of classical solvability in the sense that if $n_0\in C^0(\overline{\Omega}), c_0\in C^0(\overline{\Omega})$ an $u_0\in C^0(\overline{\Omega};\mathbb{R}^3)$ are such that $u_0|_{\partial\Omega}=0$, and if $0\le n\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))$, $0\le c\in C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty))$ and $u\in C^0(\overline{\Omega}\times[0,\infty);\mathbb{R}^3)\cap C^{2,1}(\overline{\Omega}\times(0,\infty);\mathbb{R}^3)$ are such that (n,c,u) forms a global generalized solution of (1.4) in the above sense, then there exists $P\in C^{1,0}(\Omega\times(0,\infty))$ such that (n,c,u,p) in fact solves (1.4) in the classical sense.

3 A strong precompactness feature of c_ε

Next approaching the core of our analysis, in this section we intend to derive a strong precompactness feature of c_ε with respect to the norm in $L^1((0,T);W^{1,1}(\Omega))$ for arbitrary $T>0$ (see Lemma 3.9), which will play a key role not only in the course of a suitable limit process in the second equation of (2.1), but also in the final verification of the supersolution property in Lemma 5.2.

Our first step in this direction results from a standard testing procedure applied to the second equation from (2.1), which thanks to the space-time integrability property in (2.3) yields the following.

Lemma 3.1 *Let* $\alpha>1$ *and*

$$
p^{(\alpha)} := \begin{cases} \frac{3\alpha}{5-2\alpha} & \text{if } \alpha<\frac{5}{2},\\ +\infty & \text{if } \alpha\ge\frac{5}{2}. \end{cases}
\tag{3.1}
$$

Then for each $T>0$ *and any* $p\in(1,\infty)$ *fulfilling* $p\le p^{(\alpha)}$ *there exists* $C(p,T)>0$ *such that*

$$
\int_\Omega c_\varepsilon^p(\cdot,t)\le C(p,T)\qquad \text{for all } t\in(0,T) \text{ and } \varepsilon\in(0,1)
\tag{3.2}
$$

and

$$\int_0^T \int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2 \le C(p,T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{3.3}$$

PROOF. Since $\nabla \cdot u_\varepsilon = 0$, from the second equation in (2.1) we obtain that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega c_\varepsilon^p + (p-1)\int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2 + \int_\Omega c_\varepsilon^p = \int_\Omega n_\varepsilon c_\varepsilon^{p-1} \qquad \text{for all } t>0, \tag{3.4}$$

where by the Hölder inequality, writing $f_\varepsilon(t) := \int_\Omega n_\varepsilon^\alpha(\cdot,t)$ for $t>0$ and $\varepsilon \in (0,1)$ we see that

$$\begin{aligned}\int_\Omega n_\varepsilon c_\varepsilon^{p-1} &\le f_\varepsilon^{\frac{1}{\alpha}}(t)\cdot\left\{\int_\Omega c_\varepsilon^{\frac{(p-1)\alpha}{\alpha-1}}\right\}^{\frac{\alpha-1}{\alpha}} \\ &= f_\varepsilon^{\frac{1}{\alpha}}(t)\|c_\varepsilon^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^{\frac{2(p-1)\alpha}{p(\alpha-1)}}(\Omega)} \qquad \text{for all } t>0. \end{aligned} \tag{3.5}$$

Here since our assumption on p warrants that if $\alpha < \frac{5}{2}$ then

$$\frac{2(p-1)\alpha}{p(\alpha-1)} = \frac{2\alpha}{\alpha-1}\cdot\Big(1-\frac{1}{p}\Big) \le \frac{2\alpha}{\alpha-1}\cdot\Big(1-\frac{5-2\alpha}{3\alpha}\Big) = \frac{10}{3},$$

and that thus in both cases $\alpha < \frac{5}{2}$ and $\alpha \ge \frac{5}{2}$ we have $\frac{2(p-1)\alpha}{p(\alpha-1)} < 6$, we may invoke the Gagliardo-Nirenberg inequality to find $C_1 = C_1(p) > 0$ such that

$$\|c_\varepsilon^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^{\frac{2(p-1)\alpha}{p(\alpha-1)}}(\Omega)} \le C_1\|\nabla c_\varepsilon^{\frac{p}{2}}\|^{\frac{3(p-\alpha)}{p\alpha}}_{L^2(\Omega)}\|c_\varepsilon^{\frac{p}{2}}\|^{\frac{(2p+1)\alpha-3p}{p\alpha}}_{L^2(\Omega)} + C_1\|c_\varepsilon^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^2(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1),$$

noting that the latter conclusion is trivially valid when $\frac{2(p-1)\alpha}{p(\alpha-1)} \le 2$, that is, when $p \le \alpha$. Again due to the inequality $p \le p^{(\alpha)}$, we next see that if $\alpha < \frac{5}{2}$ then herein

$$\frac{3(p-\alpha)}{p\alpha} = \frac{3}{\alpha}\cdot\Big(1-\frac{\alpha}{p}\Big) \le \frac{3}{\alpha}\cdot\left(1-\frac{\alpha}{\frac{3\alpha}{5-2\alpha}}\right) = \frac{2(\alpha-1)}{\alpha} < 2,$$

so that regardless of the size of α we have $\frac{3(p-\alpha)}{p\alpha} < 2$. Accordingly, Young's inequality applies so as to yield $C_2 = C_2(p) > 0$ such that

$$\begin{aligned} f_\varepsilon^{\frac{1}{\alpha}}(t)\|c_\varepsilon^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^{\frac{2(p-1)\alpha}{p(\alpha-1)}}(\Omega)} &\le \frac{2(p-1)}{p^2}\|\nabla c_\varepsilon^{\frac{p}{2}}\|^2_{L^2(\Omega)} + C_2 f_\varepsilon^{\frac{2p}{(2p+3)\alpha-3p}}(t)\|c_\varepsilon^{\frac{p}{2}}\|^{2\cdot\frac{(2p+1)\alpha-3p}{(2p+3)\alpha-3p}}_{L^2(\Omega)} \\ &\quad + C_1 f_\varepsilon^{\frac{1}{\alpha}}(t)\|c_\varepsilon^{\frac{p}{2}}\|^{\frac{2(p-1)}{p}}_{L^2(\Omega)} \\ &= \frac{p-1}{2}\int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2 + C_2 f_\varepsilon^{\frac{2p}{(2p+3)\alpha-3p}}(t)\cdot\left\{\int_\Omega c_\varepsilon^p\right\}^{\frac{(2p+1)\alpha-3p}{(2p+3)\alpha-3p}} \\ &\quad + C_1 f_\varepsilon^{\frac{1}{\alpha}}(t)\cdot\left\{\int_\Omega c_\varepsilon^p\right\}^{\frac{p-1}{p}} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \end{aligned}$$

whence combining (3.5) with (3.4) shows that $y_\varepsilon(t) := \int_\Omega c_\varepsilon^p(\cdot,t)$, $t \geq 0$, $\varepsilon \in (0,1)$, and $g_\varepsilon(t) := \frac{p-1}{2}\int_\Omega c_\varepsilon^{p-2}(\cdot,t)|\nabla c_\varepsilon(\cdot,t)|^2$, $t>0$, $\varepsilon\in(0,1)$, satisfy

$$\frac{1}{p}y_\varepsilon'(t) + g_\varepsilon(t) \leq C_2 f_\varepsilon^{\frac{2p}{(2p+3)\alpha-3p}}(t) y_\varepsilon^{\frac{(2p+1)\alpha-3p}{(2p+3)\alpha-3p}}(t) + C_1 f_\varepsilon^{\frac{1}{\alpha}}(t) y_\varepsilon^{\frac{p-1}{p}}(t) \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1). \tag{3.6}$$

Only at this point, we now take full advantage of the condition $p \leq p^{(\alpha)}$, which namely ensures that

$$1 - \frac{(2p+3)\alpha-3p}{2p} = \frac{(5-2\alpha)p-3\alpha}{2p} \leq 0$$

and thus $\frac{2p}{(2p+3)\alpha-3p} \leq 1$. Since furthermore, clearly,

$$\frac{(2p+1)\alpha-3p}{(2p+3)\alpha-3p} < 1 \qquad \text{and} \qquad \frac{p-1}{p} < 1,$$

several applications of Young's inequality enable us to see that (3.6) actually entails the inequality

$$\frac{1}{p}y_\varepsilon'(t) + g_\varepsilon(t) \leq (C_1+C_2)\cdot(f_\varepsilon(t)+1)\cdot(y_\varepsilon(t)+1) \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1), \tag{3.7}$$

which by nonnegativity of g_ε firstly implies that

$$y_\varepsilon(t)+1 \leq \Big(y_\varepsilon(0)+1\Big)\cdot e^{p(C_1+C_2)\int_0^t (f_\varepsilon(s)+1)ds} \qquad \text{for all } t>0 \text{ and } \varepsilon\in(0,1).$$

As a consequence of Lemma 2.2, this shows that $\sup_{\varepsilon\in(0,1)}\sup_{t\in(0,T)}(y_\varepsilon(t)+1)$ is finite for all $T>0$, whereupon a direct integration in (3.7) reveals that therefore also $\sup_{\varepsilon\in(0,1)}\int_0^T g_\varepsilon(s)ds < \infty$ for all $T>0$. By definition of $(y_\varepsilon)_{\varepsilon\in(0,1)}$ and $(g_\varepsilon)_{\varepsilon\in(0,1)}$, both (3.2) and (3.3) have thereby been established. □

A straightforward interpolation turns (3.2) and (3.3) into the following.

Lemma 3.2 *Let $\alpha\in(1,\frac{5}{2})$ and $p^{(\alpha)}$ be as in (3.1), and let $q\in(p^{(\alpha)},3p^{(\alpha)}]$. Then for all $T>0$ there exists $C(q,T)>0$ such that*

$$\int_0^T \|c_\varepsilon(\cdot,t)\|_{L^q(\Omega)}^{\frac{2p^{(\alpha)}q}{3(q-p^{(\alpha)})}} dt \leq C(q,T) \qquad \textit{for all } \varepsilon\in(0,1). \tag{3.8}$$

Proof. We abbreviate $p := p^{(\alpha)}$ and observe that then our assumption on q guarantees that $\frac{2q}{p}\leq 6$ and $\frac{2q}{p}>2$, so that an application of the Gagliardo-Nirenberg inequality yields $C_1 = C_1(q)>0$ such that

$$\|\varphi\|_{L^{\frac{2q}{p}}(\Omega)}^{\frac{4q}{3(q-p)}} \leq C_1\|\nabla\varphi\|_{L^2(\Omega)}^2\|\varphi\|_{L^2(\Omega)}^{\frac{2(3p-q)}{3(q-p)}} + C_1\|\varphi\|_{L^2(\Omega)}^{\frac{4q}{3(q-p)}} \qquad \text{for all } \varphi\in W^{1,2}(\Omega).$$

Therefore,

$$\begin{aligned}
\int_0^T \|c_\varepsilon(\cdot,t)\|_{L^q(\Omega)}^{\frac{2pq}{3(q-p)}}\,dt &= \int_0^T \|c_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^{\frac{2q}{p}}(\Omega)}^{\frac{4q}{3(q-p)}}\,dt \\
&\le C_1\int_0^T \|\nabla c_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^2 \|c_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^{\frac{2(3p-q)}{3(q-p)}}\,dt + C_1\int_0^T \|c_\varepsilon^{\frac{p}{2}}(\cdot,t)\|_{L^2(\Omega)}^{\frac{4q}{3(q-p)}}\,dt \\
&= \frac{p^2C_1}{4}\int_0^T \left\{\int_\Omega c_\varepsilon^{p-2}(\cdot,t)|\nabla c_\varepsilon(\cdot,t)|^2\right\}\cdot\left\{\int_\Omega c_\varepsilon^p(\cdot,t)\right\}^{\frac{3p-q}{3(q-p)}}dt \\
&\quad + C_1\int_0^T \left\{\int_\Omega c_\varepsilon^p(\cdot,t)\right\}^{\frac{2q}{3(q-p)}}dt \\
&\le \frac{p^2C_1}{4}\cdot\left\{\sup_{t\in(0,T)}\int_\Omega c_\varepsilon^p(\cdot,t)\right\}^{\frac{3p-q}{3(q-p)}}\cdot\int_0^T\int_\Omega c_\varepsilon^{p-2}|\nabla c_\varepsilon|^2 \\
&\quad + C_1T\cdot\left\{\sup_{t\in(0,T)}\int_\Omega c_\varepsilon^p(\cdot,t)\right\}^{\frac{2q}{3(q-p)}} \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1),
\end{aligned}$$

whence the claim immediately results from Lemma 3.1. □

Again by interpolation, in light of Lemma 3.2 the estimate (3.3) moreover entails some weight-free spatio-temporal L^q-estimate involving an exponent $q>1$:

Lemma 3.3 *Let $\alpha>1$. Then for all $T>0$ there exists $C(T)>0$ such that*

$$\int_0^T\int_\Omega |\nabla c_\varepsilon|^{\frac{5}{4}} \le C(T) \qquad \textit{for all } \varepsilon\in(0,1). \tag{3.9}$$

Proof. If $\alpha\ge\frac{10}{7}$ and hence in (3.1) we have $p^{(\alpha)}\ge 2$, the claim immediately follows upon invoking Lemma 3.1 with $p:=2$.
If $\alpha<\frac{10}{7}$ and hence $p^{(\alpha)}<2$, however, Lemma 3.1 and Lemma 3.2 ensure that given $T>0$ we can pick $C_1(T)>0$ and $C_2(T)>0$ such that

$$\int_0^T\int_\Omega c_\varepsilon^{p^{(\alpha)}-2}|\nabla c_\varepsilon|^2 \le C_1(T) \qquad \text{for all } \varepsilon\in(0,1) \tag{3.10}$$

and

$$\int_0^T\int_\Omega c_\varepsilon^{\frac{5p^{(\alpha)}}{3}} \le C_2(T) \qquad \text{for all } \varepsilon\in(0,1). \tag{3.11}$$

According to Young's inequality, this implies that for any such T,

$$\begin{aligned}
\int_0^T\int_\Omega |\nabla c_\varepsilon|^{\frac{5p^{(\alpha)}}{p^{(\alpha)}+3}} &= \int_0^T\int_\Omega \left\{c_\varepsilon^{p^{(\alpha)}-2}|\nabla c_\varepsilon|^2\right\}^{\frac{5p^{(\alpha)}}{2(p^{(\alpha)}+3)}}\cdot c_\varepsilon^{\frac{5p^{(\alpha)}(2-p^{(\alpha)})}{2(p^{(\alpha)}+3)}} \\
&\le \int_0^T\int_\Omega c_\varepsilon^{p^{(\alpha)}-2}|\nabla c_\varepsilon|^2 + \int_0^T\int_\Omega c_\varepsilon^{\frac{5p^{(\alpha)}}{3}} \\
&\le C_1(T)+C_2(T) \qquad \text{for all } \varepsilon\in(0,1),
\end{aligned}$$

so that (3.9) results also in this case, because

$$\frac{5p^{(\alpha)}}{p^{(\alpha)}+3} = \frac{5}{1+\frac{3}{p^{(\alpha)}}} \geq \frac{5}{1+3} = \frac{5}{4}$$

due to the inequality $p^{(\alpha)} > 1$. □

In conjunction with the information on fluid integrability from Lemma 2.3, the weighted gradient estimate in (3.3) can be seen to furthermore entail that the nonlinear convection term in the second equation from (2.1) admits the following estimate which, as we underline here, involves some superlinear summability power with respect to spatial integration, but only some possibly small positive integrability exponent in time.

Lemma 3.4 *Let $\alpha > 1$. Then there exist $p > 1$ and $\lambda > 0$ with the property that for all $T > 0$ one can find $C(T) > 0$ such that*

$$\int_0^T \|u_\varepsilon(\cdot,t)\cdot\nabla c_\varepsilon(\cdot,t)\|^\lambda_{L^p(\Omega)}dt \leq C(T) \qquad \text{for all } \varepsilon \in (0,1). \tag{3.12}$$

Proof. We first consider the cane when $\alpha < \frac{10}{7}$, in which with $p^{(\alpha)}$ taken from (3.1) we observe that then $p^{(\alpha)} > \alpha$ and $p^{(\alpha)} < 2$. Moreover, since $2 - p^{(\alpha)} < p^{(\alpha)}$ due to the fact that $p^{(\alpha)} > 1$, it is possible to fix $\delta > 0$ small enough such that besides

$$\delta < \frac{1}{6}, \tag{3.13}$$

we can achieve that

$$q := \frac{3(1+2\delta)(2-p^{(\alpha)})}{1-6\delta} \tag{3.14}$$

satisfies

$$q \leq 3p^{(\alpha)} \tag{3.15}$$

and

$$\frac{\delta(2-p^{(\alpha)})}{2-\delta} \leq \frac{2p^{(\alpha)}q}{3(q-p^{(\alpha)})_+}. \tag{3.16}$$

A first application of the Hölder inequality thereupon shows that for all $T > 0$ and $\varepsilon \in (0,1)$,

$$\begin{aligned}\int_0^T \|u_\varepsilon(\cdot,t)\cdot\nabla c_\varepsilon(\cdot,t)\|^\delta_{L^{1+\delta}(\Omega)}dt &\leq \int_0^T \|u_\varepsilon(\cdot,t)\|^\delta_{L^{\frac{3(1+\delta)(1+2\delta)}{1+4\delta}}(\Omega)} \|\nabla c_\varepsilon(\cdot,t)\|^\delta_{L^{\frac{3(1+2\delta)}{2}}(\Omega)}dt \\ &\leq C_1(T)\int_0^T \|\nabla c_\varepsilon(\cdot,t)\|^\delta_{L^{\frac{3(1+2\delta)}{2}}(\Omega)}dt, \end{aligned} \tag{3.17}$$

where

$$C_1(T) := \sup_{\varepsilon\in(0,1)}\sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|^\delta_{L^{\frac{3(1+\delta)(1+2\delta)}{1+4\delta}}(\Omega)}$$

is finite according to Lemma 2.3, because

$$\frac{3(1+\delta)(1+2\delta)}{1+4\delta} = \frac{3(1+3\delta+2\delta^2)}{1+4\delta} < \frac{3(1+3\delta+\delta)}{1+4\delta} = 3$$

due to the inequality $\delta < \frac{1}{2}$ implied by (3.13).
Next, making full use of (3.13) we employ the Hölder inequality for a second time to see that the integrand on the right of (3.17) can be estimated according to

$$\begin{aligned}
\|\nabla c_\varepsilon\|^\delta_{L^{\frac{3(1+2\delta)}{2}}(\Omega)} &= \left\{ \int_\Omega |\nabla c_\varepsilon|^{\frac{3(1+2\delta)}{2}} \right\}^{\frac{2\delta}{3(1+2\delta)}} \\
&= \left\{ \int_\Omega (c_\varepsilon^{p^{(\alpha)}-2} |\nabla c_\varepsilon|^2)^{\frac{3(1+2\delta)}{4}} \cdot c_\varepsilon^{\frac{3(1+2\delta)(2-p^{(\alpha)})}{4}} \right\}^{\frac{2\delta}{3(1+2\delta)}} \\
&\le \left\{ \int_\Omega c_\varepsilon^{p^{(\alpha)}-2} |\nabla c_\varepsilon|^2 \right\}^{\frac{\delta}{2}} \cdot \left\{ \int_\Omega c_\varepsilon^{\frac{3(1+2\delta)(2-p^{(\alpha)})}{1-6\delta}} \right\}^{\frac{\delta(1-6\delta)}{6(1+2\delta)}} \\
&= \left\{ \int_\Omega c_\varepsilon^{p^{(\alpha)}-2} |\nabla c_\varepsilon|^2 \right\}^{\frac{\delta}{2}} \cdot \|c_\varepsilon\|^{\frac{\delta(2-p^{(\alpha)})}{2}}_{L^q(\Omega)} \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

because $p^{(\alpha)} < 2$. Thanks to the fact that $\delta < 2$, a final application of the Hölder inequality therefore shows that as a consequence of (3.17), for all $T > 0$ and $\varepsilon \in (0,1)$ we have

$$\begin{aligned}
\int_0^T \|u_\varepsilon(\cdot,t)\cdot\nabla c_\varepsilon(\cdot,t)\|^\delta_{L^{1+\delta}(\Omega)} dt &\le C_1(T) \int_0^T \left\{ \int_\Omega c_\varepsilon^{p^{(\alpha)}-2}(\cdot,t)|\nabla c_\varepsilon(\cdot,t)|^2 \right\}^{\frac{\delta}{2}} \cdot \|c_\varepsilon(\cdot,t)\|^{\frac{\delta(2-p^{(\alpha)})}{2}}_{L^q(\Omega)} \\
&\le C_1(T) \cdot \left\{ \int_0^T \int_\Omega c_\varepsilon^{p^{(\alpha)}-2}|\nabla c_\varepsilon|^2 \right\}^{\frac{\delta}{2}} \cdot \left\{ \int_0^T \|c_\varepsilon(\cdot,t)\|^{\frac{\delta(2-p^{(\alpha)})}{2-\delta}}_{L^q(\Omega)} dt \right\}^{\frac{2-\delta}{2}}.
\end{aligned}$$

For any such α, the conclusion thus follows upon observing that

$$\sup_{\varepsilon\in(0,1)} \int_0^T \int_\Omega c_\varepsilon^{p^{(\alpha)}-2}|\nabla c_\varepsilon|^2 < \infty$$

due to Lemma 3.1, and that if $q > p^{(\alpha)}$ then

$$\sup_{\varepsilon\in(0,1)} \int_0^T \|c_\varepsilon(\cdot,t)\|^{\frac{\delta(2-p^{(\alpha)})}{2-\delta}}_{L^q(\Omega)} dt < \infty$$

thanks to (3.15), (3.16) and Lemma 3.2, whereas if $q \le p^{(\alpha)}$ then even

$$\sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T)} \|c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} < \infty \qquad \text{for all } T>0$$

by Lemma 3.1 and, e.g., Young's inequality.
If $\alpha \ge \frac{10}{7}$, however, then in (3.1) we have $p^{(\alpha)} \ge 2$, so that Lemma 3.1 entails that

$$C_2(T) := \sup_{\varepsilon\in(0,1)} \int_0^T \int_\Omega |\nabla c_\varepsilon|^2 < \infty \qquad \text{for all } T>0,$$

while from Lemma 2.3 we know that

$$C_3(T) := \sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^{\frac{5}{2}}(\Omega)} < \infty \qquad \text{for all } T>0.$$

By means of the Hölder inequality we can thus estimate

$$\begin{aligned}\int_0^T \|u_\varepsilon(\cdot,t)\cdot\nabla c_\varepsilon(\cdot,t)\|^2_{L^{\frac{10}{9}}(\Omega)}dt &\le \int_0^T \|u_\varepsilon(\cdot,t)\|^2_{L^{\frac{5}{2}}(\Omega)}\|\nabla c_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}dt\\ &\le C_2(T)C_3^2(T) \qquad \text{for all } T>0 \text{ and } \varepsilon\in(0,1),\end{aligned}$$

from which the claim directly follows in this case. □

Apart from that, when rewritten in the form $u_\varepsilon\cdot\nabla c_\varepsilon = \nabla\cdot(c_\varepsilon u_\varepsilon)$ the convection term addressed above enjoys a further regularity property, now in a reflexive Lebesgue setting with regard to both the space and the time variable.

Lemma 3.5 *If $\alpha>1$, then there exists $p>1$ such that to each $T>0$ there corresponds some $C(T)>0$ satisfying*

$$\int_0^T\int_\Omega |c_\varepsilon u_\varepsilon|^p \le C(T) \qquad \text{for all } \varepsilon\in(0,1). \tag{3.18}$$

Proof. We fix $\delta>0$ small such that

$$\delta < \frac{1}{2}, \tag{3.19}$$

and that

$$\frac{(1+\delta)(1+6\delta)}{2(1+2\delta)} \le 1, \tag{3.20}$$

and let $q := \frac{3(1+2\delta)}{2}$. Then (3.19) ensures that, as in Lemma 3.4,

$$\frac{3(1+\delta)(1+2\delta)}{1+4\delta} < 3,$$

so that Lemma 2.3 applies so as to warrant that for each $T>0$,

$$C_1(T) := \sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^{\frac{3(1+\delta)(1+2\delta)}{1+4\delta}}(\Omega)} < \infty. \tag{3.21}$$

Apart from that, using Lemma 3.1 we see that if $q\le p^{(\alpha)}$, with $p^{(\alpha)}$ taken from (3.1), then also

$$C_2(T) := \sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T)} \|c_\varepsilon(\cdot,t)\|_{L^q(\Omega)} < \infty \qquad \text{for all } T>0, \tag{3.22}$$

whereas if $q>p^{(\alpha)}$, and hence necessarily $\alpha<\frac{5}{2}$, then Lemma 3.2 asserts that

$$C_3(T) := \sup_{\varepsilon\in(0,1)} \int_0^T \|c_\varepsilon(\cdot,t)\|^{1+\delta}_{L^q(\Omega)}dt < \infty \qquad \text{for all } T>0, \tag{3.23}$$

because then due to the fact that $p^{(\alpha)} > 1$, (3.19) ensures that

$$q < \frac{3 \cdot (1 + 2 \cdot \frac{1}{2})}{2} = 3 < 3p^{(\alpha)},$$

and because our restriction (3.20) guarantees that

$$(1+\delta) \cdot \frac{3(q-p^{(\alpha)})}{2p^{(\alpha)}q} = \frac{3(1+\delta)}{2} \cdot \Big(\frac{1}{p^{(\alpha)}} - \frac{1}{q}\Big) < \frac{3(1+\delta)}{2} \cdot \Big(1 - \frac{1}{q}\Big) = \frac{(1+\delta)(1+6\delta)}{2(1+2\delta)} \le 1$$

and thus $1 + \delta \le \frac{2p^{(\alpha)}q}{3(q-p^{(\alpha)})}$. Now since the Hölder inequality implies that according to our choice of q we have

$$\int_0^T \int_\Omega |c_\varepsilon u_\varepsilon|^{1+\delta} \le \int_0^T \|c_\varepsilon(\cdot,t)\|^{1+\delta}_{L^q(\Omega)} \|u_\varepsilon(\cdot,t)\|^{1+\delta}_{L^{\frac{3(1+\delta)(1+2\delta)}{1+4\delta}}(\Omega)} dt \qquad \text{for all } T>0 \text{ and } \varepsilon \in (0,1),$$

in the case $q \le p^{(\alpha)}$ we may use the definitions of $(C_1(T))_{T>0}$ and $(C_2(T))_{T>0}$ in (3.21) and (3.22) to see that

$$\int_0^T \int_\Omega |c_\varepsilon u_\varepsilon|^{1+\delta} \le C_1^{1+\delta}(T) C_2^{1+\delta}(T) \cdot T \qquad \text{for all } T>0 \text{ and } \varepsilon \in (0,1),$$

while if $q > p^{(\alpha)}$ then on the basis of (3.21) and (3.23) we can estimate

$$\int_0^T \int_\Omega |c_\varepsilon u_\varepsilon|^{1+\delta} \le C_1^{1+\delta}(T) C_3(T) \qquad \text{for all } T>0 \text{ and } \varepsilon \in (0,1),$$

and thus conclude on letting $p := 1 + \delta$. □

In order to prepare an appropriate exploitation of the latter two lemmata, let us state the following interpolation inequality in which, as throughout the remainder of this section, given $p > 1$ we let $B = B_p$ denote the realization of $-\Delta + 1$ under homogeneous Neumann boundary conditions in $L^p(\Omega)$.

Lemma 3.6 *Let $p > 1$ and $\delta \in (0, \frac{1}{2})$. Then for all $\eta \in (0, \frac{1}{2})$ there exists $C = C(p,\delta,\eta) > 0$ such that*

$$\|B^{-\frac{1}{2}+\delta} \nabla \cdot \varphi\|_{L^p(\Omega)} \le C \|\nabla \cdot \varphi\|^{\frac{2\delta+2\eta}{1+2\eta}}_{L^p(\Omega)} \|\varphi\|^{\frac{1-2\delta}{1+2\eta}}_{L^p(\Omega)} \qquad \text{for all } \varphi \in C^1(\overline{\Omega};\mathbb{R}^3) \text{ such that } \varphi \cdot \nu|_{\partial\Omega} = 0. \tag{3.24}$$

Proof. According to Theorem 14.1 in [2], let us first pick $C_1 = C_1(p,\delta,\eta) > 0$ such that

$$\|B^{-\frac{1}{2}+\delta}\psi\|_{L^p(\Omega)} \le C_1 \|\psi\|^{\frac{2\delta+2\eta}{1+2\eta}}_{L^p(\Omega)} \|B^{-\frac{1}{2}-\eta}\psi\|^{\frac{1-2\delta}{1+2\eta}}_{L^p(\Omega)} \qquad \text{for all } \psi \in C^0(\overline{\Omega}), \tag{3.25}$$

and observe that due to the topological equivalence of $D(B^{\frac{1}{2}-\eta})$ to $W^{1-2\eta,p}(\Omega)$ ([6, Theorem 1.6.1]) and the continuity of the embeddings $W^{1,p}(\Omega) \hookrightarrow W^{1-2\eta,p}(\Omega)$ and $D(B^{\frac{1}{2}-\eta}) \hookrightarrow D(B^{-\frac{1}{2}-\eta})$, by relying on a Poincaré inequality we can choose $C_2 = C_2(p,\eta) > 0$ and $C_3 = C_3(p,\eta) > 0$ such that

$$\|B^{\frac{1}{2}-\eta}\psi\|_{L^p(\Omega)} \le C_2 \|\nabla\psi\|_{L^p(\Omega)} \qquad \text{for all } \psi \in W^{1,p}(\Omega) \text{ such that } \int_\Omega \psi = 0 \tag{3.26}$$

and

$$\|B^{-\frac{1}{2}-\eta}\psi\|_{L^p(\Omega)} \le C_3\|B^{\frac{1}{2}-\eta}\psi\|_{L^p(\Omega)} \qquad \text{for all } \psi \in W^{1,p}(\Omega). \tag{3.27}$$

We moreover recall that the Helmholtz projection acts as a bounded operator on $L^p(\Omega;\mathbb{R}^3)$ ([14]), whence we can fix $C_4 = C_4(p) > 0$ fulfilling

$$\|\mathcal{P}\psi\|_{L^p(\Omega)} \le C_4\|\psi\|_{L^p(\Omega)} \qquad \text{for all } \psi \in L^p(\Omega;\mathbb{R}^3). \tag{3.28}$$

Consequently, given $\varphi \in C^1(\overline{\Omega};\mathbb{R}^3)$ we can find $\rho \in W^{1,p}(\Omega)$ such that $\int_\Omega \rho = 0$ and $\varphi = \mathcal{P}\varphi + \nabla\rho$, where $\nabla\cdot(\mathcal{P}\varphi) = 0$ in $\mathcal{D}'(\Omega)$ ([14]). Therefore, taking any $\varphi \in C^1(\overline{\Omega};\mathbb{R}^3)$ with $\varphi\cdot\nu|_{\partial\Omega} = 0$ and an arbitrary $\psi \in C_0^\infty(\Omega)$ we see that since B and all its fractional powers are self-adjoint in $L^2(\Omega)$, and since evidently $\int_{\partial\Omega}(\varphi\cdot\nu)B^{-\frac{1}{2}-\eta}\psi = 0$ and $\int_\Omega \mathcal{P}\varphi\cdot\nabla(B^{-\frac{1}{2}-\eta}\psi) = 0$ as well as $\int_{\partial\Omega}\rho\frac{\partial}{\partial\nu}(B^{-\frac{1}{2}-\eta}\psi) = 0$, we see that

$$\begin{aligned}
\int_\Omega (B^{-\frac{1}{2}-\eta}\nabla\cdot\varphi)\cdot\psi &= \int_\Omega (\nabla\cdot\varphi)\cdot B^{-\frac{1}{2}-\eta}\psi \\
&= -\int_\Omega \varphi\cdot\nabla(B^{-\frac{1}{2}-\eta}\psi) \\
&= -\int_\Omega \nabla\rho\cdot\nabla(B^{-\frac{1}{2}-\eta}\psi) \\
&= \int_\Omega \rho\Delta(B^{-\frac{1}{2}-\eta}\psi) \\
&= \int_\Omega \rho\cdot(-B+1)(B^{-\frac{1}{2}-\eta}\psi) \\
&= -\int_\Omega \rho\cdot B^{\frac{1}{2}-\eta}\psi + \int_\Omega \rho\cdot B^{-\frac{1}{2}-\eta}\psi \\
&= -\int_\Omega B^{\frac{1}{2}-\eta}\rho\cdot\psi + \int_\Omega B^{-\frac{1}{2}-\eta}\rho\cdot\psi \\
&\le \|B^{\frac{1}{2}-\eta}\rho\|_{L^p(\Omega)}\|\psi\|_{L^{\frac{p}{p-1}}(\Omega)} + \|B^{-\frac{1}{2}-\eta}\rho\|_{L^p(\Omega)}\|\psi\|_{L^{\frac{p}{p-1}}(\Omega)}
\end{aligned}$$

by the Hölder inequality, so that thanks to (3.27), (3.26) and (3.28),

$$\begin{aligned}
\|B^{-\frac{1}{2}-\eta}\nabla\cdot\varphi\|_{L^p(\Omega)} &= \sup_{\substack{\psi\in C_0^\infty(\Omega)\\ \|\psi\|_{L^{\frac{p}{p-1}}(\Omega)}\le 1}} \int_\Omega (B^{-\frac{1}{2}-\eta}\nabla\cdot\varphi)\cdot\psi \\
&\le \|B^{\frac{1}{2}-\eta}\rho\|_{L^p(\Omega)} + \|B^{-\frac{1}{2}-\eta}\rho\|_{L^p(\Omega)} \\
&\le (1+C_3)\|B^{\frac{1}{2}-\eta}\rho\|_{L^p(\Omega)} \\
&\le C_2(1+C_3)\|\nabla\rho\|_{L^p(\Omega)} \\
&= C_2(1+C_3)\|\varphi-\mathcal{P}\varphi\|_{L^p(\Omega)} \\
&\le C_2(1+C_3)(1+C_4)\|\varphi\|_{L^p(\Omega)}.
\end{aligned}$$

In view of (3.25), this shows that for any such φ we have

$$\|B^{-\frac{1}{2}+\delta}\nabla\cdot\varphi\|_{L^p(\Omega)} \le C_1\|\nabla\cdot\varphi\|_{L^p(\Omega)}^{\frac{2\delta+2\eta}{1+2\eta}}\|B^{-\frac{1}{2}-\eta}\nabla\cdot\varphi\|_{L^p(\Omega)}^{\frac{1-2\delta}{1+2\eta}}$$

$$\leq \quad C_1 \|\nabla \cdot \varphi\|_{L^p(\Omega)}^{\frac{2\delta+2\eta}{1+2\eta}} \cdot \Big\{ C_2(1+C_3)(1+C_4) \Big\}^{\frac{1-2\delta}{1+2\eta}} \|\varphi\|_{L^p(\Omega)}^{\frac{1-2\delta}{1+2\eta}},$$

and that thus (3.24) holds with an obvious choice of $C(p,\delta,\eta)$. □

We can thereby accomplish the main step of our analysis in this section by combining Lemma 3.4 and Lemma 3.5 to achieve the following estimate of c_ε in a space which is compactly embedded into $W^{1,1}(\Omega)$.

Lemma 3.7 *If $\alpha > 1$, then there exist $p > 1$ and $\delta > 0$ such that for any choice of $\tau > 0$ and $T > \tau$ one can fix $C(\tau, T) > 0$ satisfying*

$$\int_\tau^T \int_\Omega |B^{\frac{1}{2}+\delta} c_\varepsilon|^p \leq C(\tau,T) \qquad \text{for all } \varepsilon \in (0,1). \tag{3.29}$$

Proof. We first invoke Lemma 3.4 and Lemma 3.5 to fix $p_0 > 1$ and $\lambda \in (0,2)$ with the property that for all $T > 0$ we can find $C_1(T) > 0$ and $C_2(T) > 0$ fulfilling

$$\int_0^T \|u_\varepsilon(\cdot,t) \cdot \nabla c_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)}^\lambda dt \leq C_1(T) \qquad \text{for all } \varepsilon \in (0,1) \tag{3.30}$$

and

$$\int_0^T \int_\Omega |c_\varepsilon u_\varepsilon|^{p_0} \leq C_2(T) \qquad \text{for all } \varepsilon \in (0,1), \tag{3.31}$$

and we then pick $p > 1$ such that

$$p < p_0, \quad p \leq p^{(\alpha)} \quad \text{and} \quad p \leq \alpha, \tag{3.32}$$

which is possible since $p^{(\alpha)} > 1$ and $\alpha > 1$. We thereafter use that $4p - 2\lambda > 4 \cdot 1 - 2 \cdot 2 = 0$ and $(1-2\delta)p\lambda - [\lambda - (4p-2\lambda)\delta] \cdot p_0 \to (p-p_0)\lambda < 0$ as $\delta \searrow 0$ in choosing some suitably small $\delta > 0$ such that

$$\delta < \frac{1}{2} \qquad \text{and} \qquad \delta < \frac{\lambda}{4p-2\lambda}, \tag{3.33}$$

and such that moreover

$$\frac{(1-2\delta)p\lambda}{\lambda - (4p-2\lambda)\delta} \leq p_0, \tag{3.34}$$

where the last inequality in (3.33) warrants that the operator $B^{-\frac{1}{2}+\delta}$ is bounded in $L^p(\Omega)$, whence with some $C_3 > 0$ we have

$$\|B^{-\frac{1}{2}+\delta}\varphi\|_{L^p(\Omega)}^p \leq C_3 \|\varphi\|_{L^p(\Omega)}^p \qquad \text{for all } \varphi \in L^p(\Omega). \tag{3.35}$$

Next, we invoke standard maximal Sobolev regularity theory in $L^p(\Omega)$ ([4]) to fix $C_4 > 0$ such that whenever $T > 0$, $w \in C^{2,1}(\overline{\Omega} \times [0,T])$ and $f \in C^0(\overline{\Omega} \times [0,T])$ are such that

$$\begin{cases} w_t = \Delta w - w + f(x,t), & x \in \Omega,\ t \in (0,T), \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t \in (0,T), \\ w(x,0) = 0, & x \in \Omega, \end{cases}$$

we have

$$\int_0^T \int_\Omega |Bw|^p \le C_4 \int_0^T \int_\Omega |f|^p. \tag{3.36}$$

We now let $\tau > 0$ and $T > 0$ be given and take any nondecreasing cut-off function $\zeta \in C^\infty([0,\infty))$ such that $\zeta \equiv 0$ in $[0, \frac{\tau}{2}]$ and $\zeta \equiv 1$ in $[\tau, \infty)$, and observe that then for each $\varepsilon \in (0,1)$,

$$w_\varepsilon(\cdot,t) := \zeta(t) \cdot B^{-\frac{1}{2}+\delta} c_\varepsilon(\cdot,t), \qquad t \in [0,T],$$

defines a function w_ε on $\overline{\Omega} \times [0,T]$ which since $c_\varepsilon(\cdot,t) \in D(B)$ for all $t > 0$, and since $B^{-\frac{1}{2}+\delta}$ maps $D(B)$ into itself, belongs to $C^{2,1}(\overline{\Omega} \times [0,T])$ and satisfies $\frac{\partial w_\varepsilon}{\partial \nu} = 0$ on $\partial\Omega \times (0,T)$ as well as $w_\varepsilon(\cdot,0) \equiv 0$ in Ω. Furthermore, using (2.1) we see that

$$\begin{aligned}
w_{\varepsilon t} &= \zeta(t) B^{-\frac{1}{2}+\delta} c_{\varepsilon t} + \zeta'(t) B^{-\frac{1}{2}+\delta} c_\varepsilon \\
&= \zeta(t) B^{-\frac{1}{2}+\delta} \Big\{ -Bc_\varepsilon + \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} - \nabla \cdot (c_\varepsilon u_\varepsilon) \Big\} + \zeta'(t) B^{-\frac{1}{2}+\delta} c_\varepsilon \\
&= -Bw_\varepsilon + \zeta(t) B^{-\frac{1}{2}+\delta} \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} - \zeta(t) B^{-\frac{1}{2}+\delta} \nabla \cdot (c_\varepsilon u_\varepsilon) + \zeta'(t) B^{-\frac{1}{2}+\delta} c_\varepsilon \qquad \text{in } \Omega \times (0,T),
\end{aligned}$$

so that (3.36) applies so as to warrant that

$$\begin{aligned}
\int_0^T \int_\Omega \zeta^p(t) |B^{\frac{1}{2}+\delta} c_\varepsilon|^p &= \int_0^T \int_\Omega |Bw_\varepsilon|^p \\
&\le C_4 \int_0^T \int_\Omega \Big| \zeta(t) B^{-\frac{1}{2}+\delta} \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} - \zeta(t) B^{-\frac{1}{2}+\delta} \nabla \cdot (c_\varepsilon u_\varepsilon) + \zeta'(t) B^{-\frac{1}{2}+\delta} c_\varepsilon \Big|^p \\
&\le 3^p C_4 \int_0^T \int_\Omega \Big| B^{-\frac{1}{2}+\delta} \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \Big|^p \\
&\quad + 3^p C_4 \int_0^T \int_\Omega \Big| B^{-\frac{1}{2}+\delta} \nabla \cdot (c_\varepsilon u_\varepsilon) \Big|^p \\
&\quad + 3^p C_4 \|\zeta'\|^p_{L^\infty((0,T))} \int_0^T \int_\Omega |B^{-\frac{1}{2}+\delta} c_\varepsilon|^p \qquad \text{for all } \varepsilon \in (0,1),
\end{aligned} \tag{3.37}$$

because $0 \le \zeta \le 1$. Here by (3.35), Young's inequality and the third restriction in (3.32),

$$\begin{aligned}
\int_0^T \int_\Omega \Big| B^{-\frac{1}{2}+\delta} \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \Big|^p &\le C_3 \int_0^T \int_\Omega \Big| \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \Big|^p \\
&\le C_3 \int_0^T \int_\Omega n_\varepsilon^p \\
&\le C_3 \int_0^T \int_\Omega n_\varepsilon^\alpha + C_3 |\Omega| T \qquad \text{for all } \varepsilon \in (0,1),
\end{aligned} \tag{3.38}$$

whereas (3.35) together with Young's inequality and the second requirement in (3.32) shows that

$$\int_0^T \int_\Omega |B^{-\frac{1}{2}+\delta} c_\varepsilon|^p \le C_3 \int_0^T \int_\Omega c_\varepsilon^p$$

$$
\begin{aligned}
&\leq C_3 \int_0^T \int_\Omega c_\varepsilon^{p^{(\alpha)}} + C_3|\Omega|T \\
&\leq C_3 T \cdot \sup_{t\in(0,T)} \int_\Omega c_\varepsilon^{p^{(\alpha)}}(\cdot,t) + C_3|\Omega|T \qquad \text{for all } \varepsilon \in (0,1). \qquad (3.39)
\end{aligned}
$$

Apart from that, applying Lemma 3.6 to $\eta := \delta < \frac{1}{2}$ we obtain $C_5 > 0$ such that for all $\varepsilon \in (0,1)$,

$$
\int_0^T \int_\Omega \left| B^{-\frac{1}{2}+\delta} \nabla \cdot (c_\varepsilon u_\varepsilon) \right|^p \leq C_5 \int_0^T \left\| \nabla \cdot (c_\varepsilon(\cdot,t) u_\varepsilon(\cdot,t)) \right\|_{L^p(\Omega)}^{\frac{4p\delta}{1+2\delta}} \|c_\varepsilon(\cdot,t) u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}^{\frac{(1-2\delta)p}{1+2\delta}} dt,
$$

so that relying on the identity $\nabla \cdot (c_\varepsilon u_\varepsilon) = u_\varepsilon \cdot \nabla c_\varepsilon$, on the fact that $\frac{4p\delta}{1+2\delta} < \lambda$ by the second condition in (3.33), and on (3.34), we may twice again employ Young's inequality to see that

$$
\begin{aligned}
\int_0^T \int_\Omega \left| B^{-\frac{1}{2}+\delta} \nabla \cdot (c_\varepsilon u_\varepsilon) \right|^p &\leq C_5 \int_0^T \left\| u_\varepsilon(\cdot,t) \cdot \nabla c_\varepsilon(\cdot,t) \right\|_{L^p(\Omega)}^{\lambda} dt + C_5 \int_0^T \|c_\varepsilon(\cdot,t) u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}^{\frac{(1-2\delta)p\lambda}{\lambda-(4p-2\lambda)\delta}} dt \\
&\leq C_5 \int_0^T \left\| u_\varepsilon(\cdot,t) \cdot \nabla c_\varepsilon(\cdot,t) \right\|_{L^p(\Omega)}^{\lambda} dt \\
&\quad + C_5 \int_0^T \|c_\varepsilon(\cdot,t) u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}^{p_0} dt + C_5 T \qquad \text{for all } \varepsilon \in (0,1).
\end{aligned}
$$

Since the first restriction in (3.32) ensures that due to the Hölder inequality we have

$$
\|\varphi\|_{L^p(\Omega)} \leq |\Omega|^{\frac{p_0-p}{p_0 p}} \|\varphi\|_{L^{p_0}(\Omega)} \qquad \text{for all } \varphi \in L^{p_0}(\Omega),
$$

along with (3.30) and (3.31) this implies that for all $\varepsilon \in (0,1)$,

$$
\begin{aligned}
\int_0^T \int_\Omega \left| B^{-\frac{1}{2}+\delta} \nabla \cdot (c_\varepsilon u_\varepsilon) \right|^p &\leq C_5 |\Omega|^{\frac{(p_0-p)\lambda}{p_0 p}} \int_0^T \left\| u_\varepsilon(\cdot,t) \cdot \nabla c_\varepsilon(\cdot,t) \right\|_{L^{p_0}(\Omega)}^{\lambda} dt \\
&\quad + C_5 |\Omega|^{\frac{p_0-p}{p}} \int_0^T \|c_\varepsilon(\cdot,t) u_\varepsilon(\cdot,t)\|_{L^{p_0}(\Omega)}^{p_0} dt + C_5 T \\
&\leq C_6(T) := C_1(T) C_5 |\Omega|^{\frac{(p_0-p)\lambda}{p_0 p}} + C_2(T) C_5 |\Omega|^{\frac{p_0-p}{p}} + C_5 T,
\end{aligned}
$$

so that from (3.37)-(3.39) we infer that

$$
\begin{aligned}
\int_0^T \int_\Omega \zeta^p(t) |B^{\frac{1}{2}+\delta} c_\varepsilon|^p &\leq 3^p C_4 \cdot \Big\{ C_3 C_7(T) + C_3|\Omega|T \Big\} \\
&\quad + 3^p C_4 C_6(T) \\
&\quad + 3^p C_4 \cdot \Big\{ C_3 C_8(T) + C_3|\Omega|T \Big\} \cdot \|\zeta'\|_{L^\infty((0,T))}^p \qquad \text{for all } \varepsilon \in (0,1),
\end{aligned}
$$

where

$$
C_7(T) := \sup_{\varepsilon\in(0,1)} \int_0^T \int_\Omega n_\varepsilon^\alpha \qquad \text{and} \qquad C_8(T) := \sup_{\varepsilon\in(0,1)} \sup_{t\in(0,T)} \int_\Omega c_\varepsilon^{p^{(\alpha)}}(\cdot,t)
$$

are both finite due to Lemma 2.2 and Lemma 3.1. It remains to recall that $\zeta \equiv 1$ on $[\tau, T]$ to conclude (3.29) from this upon an evident choice of $C(\tau, T)$. □

In preparation of an Aubin-Lions type argument, we supplement the above by some information on regularity of time derivatives.

Lemma 3.8 *Let $\alpha > 1$. Then there exists an integer $m \geq 3$ such that for all $T > 0$ one can find $C(T) > 0$ fulfilling*

$$\int_0^T \|c_{\varepsilon t}(\cdot,t)\|_{(W_0^{m,2}(\Omega))^\star} dt \leq C(T) \qquad \text{for all } \varepsilon \in (0,1). \tag{3.40}$$

Proof. We let $p > 1$ be as provided by Lemma 3.5 and take $m \in \{3, 4, ...\}$ such that $m \geq \frac{6-p}{2p}$, which ensures that $W_0^{m,2}(\Omega)$ is continuously embedded into both $W^{2,\infty}(\Omega)$ and $W^{1,\frac{p}{p-1}}(\Omega)$, and that thus there exist positive constants C_1, C_2 and C_3 such that $\|\Delta\psi\|_{L^\infty(\Omega)} \leq C_1\|\psi\|_{W^{m,2}(\Omega)}$, $\|\psi\|_{L^\infty(\Omega)} \leq C_2\|\psi\|_{W^{m,2}(\Omega)}$ and $\|\nabla\psi\|_{L^{\frac{p}{p-1}}(\Omega)} \leq C_3\|\psi\|_{W^{m,2}(\Omega)}$ for all $\psi \in C_0^\infty(\Omega)$. Given any such ψ and an arbitrary $t > 0$, on the basis of (2.1) we can therefore estimate

$$\begin{aligned}
\left| \int_\Omega c_{\varepsilon t}(\cdot,t)\psi \right| &= \left| \int_\Omega c_\varepsilon \Delta\psi - \int_\Omega c_\varepsilon \psi + \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\psi + \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla\psi \right| \\
&\leq \|c_\varepsilon\|_{L^1(\Omega)}\|\Delta\psi\|_{L^\infty(\Omega)} + \|c_\varepsilon\|_{L^1(\Omega)}\|\psi\|_{L^\infty(\Omega)} + \|n_\varepsilon\|_{L^1(\Omega)}\|\psi\|_{L^\infty(\Omega)} \\
&\quad + \|c_\varepsilon u_\varepsilon\|_{L^p(\Omega)}\|\nabla\psi\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&\leq \Big\{ (C_1+C_2)\|c_\varepsilon\|_{L^1(\Omega)} + C_2\|n_\varepsilon\|_{L^1(\Omega)} + C_3\|c_\varepsilon u_\varepsilon\|_{L^p(\Omega)} \Big\} \|\psi\|_{W^{m,2}(\Omega)}
\end{aligned}$$

for $\varepsilon \in (0,1)$. Therefore,

$$\begin{aligned}
\int_0^T \|c_{\varepsilon t}(\cdot,t)\|_{(W_0^{m,2}(\Omega))^\star} dt &\leq (C_1+C_2)T \cdot \sup_{t\in(0,T)} \|c_\varepsilon(\cdot,t)\|_{L^1(\Omega)} + C_2 T \cdot \sup_{t\in(0,T)} \|n_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \\
&\quad + C_3 \int_0^T \|c_\varepsilon(\cdot,t)u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} dt \qquad \text{for all } T > 0 \text{ and } \varepsilon \in (0,1),
\end{aligned}$$

so that the claim results from Lemma 3.1, Lemma 2.2 and Lemma 3.5. □

We can thereby derive the main result of this section in quite a straightforward manner from Lemma 3.7:

Lemma 3.9 *Let $\alpha > 1$. Then for all $T > 0$,*

$$(c_\varepsilon)_{\varepsilon\in(0,1)} \text{ is relatively compact with respect to the strong topology in } L^1((0,T);W^{1,1}(\Omega)). \tag{3.41}$$

Proof. According to Lemma 3.7, we can find $p > 1$ and $\delta > 0$ such that

$$(c_\varepsilon)_{\varepsilon\in(0,1)} \text{ is bounded in } L^p((0,T);D(B_p^{\frac{1}{2}+\delta})) \quad \text{for all } T > 0,$$

whereas Lemma 3.8 provides an integer $m \geq 3$ with the property that

$$(c_{\varepsilon t})_{\varepsilon\in(0,1)} \text{ is bounded in } L^1((0,T);(W_0^{m,2}(\Omega))^\star) \quad \text{for all } T > 0.$$

Since $D(B_p^{\frac{1}{2}+\delta})$ is continuously embedded into $W^{1,p}(\Omega)$ ([6, Theorem 1.6.1]), an Aubin-Lions lemma ([18]) becomes applicable so as to guarantee that for all $T > 0$,

$(c_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact with respect to the strong topology in $L^p((0,T);W^{1,p}(\Omega))$.

As $p \geq 1$, this clearly entails (3.41). □

4 Compactness properties of $\left((n_\varepsilon+1)^{-p}e^{-\kappa c_\varepsilon}\right)_{\varepsilon\in(0,1)}$ for $p > 0$ and large κ

This section is devoted to an essentially straightforward adaptation of the reasoning from [33, Section 6], pursuing the goal to derive relative compactness of $\left((n_\varepsilon+1)^{-p}e^{-\kappa c_\varepsilon}\right)_{\varepsilon\in(0,1)}$ with respect to both the weak topology in $L^2((0,T);W^{1,2}(\Omega))$ and the strong topology in $L^2(\Omega\times(0,T))$ for arbitrary $T > 0$, each $p > 0$ and any suitably large $\kappa > 0$. This will be achieved on the basis of Lemma 2.4, in which we will choose ϕ and ψ as specified and described in the following statement imported from [33, Lemma 6.1].

Lemma 4.1 *Let $p > 0$ and $\kappa > 0$, and define*

$$\phi(s) := (s+1)^{-p}, \quad \Phi(s) := -2\sqrt{\frac{p+1}{p}}(s+1)^{-\frac{p}{2}} \quad \text{and} \quad \psi(\overline{s}) := e^{-\kappa\overline{s}}, \qquad s \geq 0,\ \overline{s} \geq 0. \tag{4.1}$$

Then

$$\Phi'(s) = \sqrt{\phi''(s)} \qquad \text{for all } s \geq 0, \tag{4.2}$$

and for any $s \geq 0$ and $\overline{s} \geq 0$ we have

$$\frac{\phi'(s)}{\sqrt{\phi''(s)}}\cdot\frac{\psi'(\overline{s})}{\sqrt{\psi(\overline{s})}} - \frac{1}{2}\Phi(s)\frac{\psi'(\overline{s})}{\sqrt{\psi(\overline{s})}} - \frac{1}{2}s\sqrt{\phi''(s)}\cdot\sqrt{\psi(\overline{s})} = -\frac{2\kappa + p(p+1)\frac{s}{s+1}}{2\sqrt{p(p+1)}}(s+1)^{-\frac{p}{2}}e^{-\frac{\kappa\overline{s}}{2}} \tag{4.3}$$

and

$$\phi(s)\psi''(\overline{s}) - \frac{\phi'^2(s)}{\phi''(s)}\cdot\frac{\psi'^2(\overline{s})}{\psi(\overline{s})} - \frac{1}{4}s^2\phi''(s)\psi(\overline{s}) = \frac{4\kappa^2 - p(p+1)^2\frac{s^2}{(s+1)^2}}{4(p+1)}(s+1)^{-p}e^{-\kappa\overline{s}} \tag{4.4}$$

as well as

$$\frac{\phi'(s)}{\sqrt{\phi''(s)}}\sqrt{\psi(\overline{s})} = -\frac{p}{\sqrt{p(p+1)}}(s+1)^{-\frac{p}{2}}e^{-\frac{\kappa\overline{s}}{2}} \tag{4.5}$$

and

$$s\phi'(s)\psi(\overline{s}) - \phi(s)\psi'(\overline{s}) + \frac{1}{2}\frac{\Phi(s)\phi'(s)}{\sqrt{\phi''(s)}}\psi'(\overline{s}) = -ps(s+1)^{-p-1}e^{-\kappa\overline{s}}. \tag{4.6}$$

When substantiated according to the latter choices, for $\varepsilon \in (0,1)$ Lemma 2.4 indeed takes the following form.

Corollary 4.2 *Let $p > 0$ and $\kappa > 0$. Then whenever $\varphi \in C^\infty(\overline{\Omega} \times (0,\infty))$,*

$$\begin{aligned}
&\int_\Omega \partial_t \Big\{ (n_\varepsilon+1)^{-p} e^{-\kappa c_\varepsilon} \Big\} \cdot \varphi \\
&\quad = -\frac{4(p+1)}{p} \int_\Omega \left| \nabla \Big\{ (n_\varepsilon+1)^{-\frac{p}{2}} e^{-\frac{\kappa c_\varepsilon}{2}} \Big\} + \frac{2\kappa + p(p+1)\frac{n_\varepsilon}{n_\varepsilon+1}}{4(p+1)} (n_\varepsilon+1)^{-\frac{p}{2}} e^{-\frac{\kappa c_\varepsilon}{2}} \nabla c_\varepsilon \right|^2 \varphi \\
&\qquad - \int_\Omega \frac{4\kappa^2 - p(p+1)^2 \frac{n_\varepsilon^2}{(n_\varepsilon+1)^2}}{4(p+1)} (n_\varepsilon+1)^{-p} e^{-\kappa c_\varepsilon} |\nabla c_\varepsilon|^2 \varphi \\
&\qquad -2 \int_\Omega (n_\varepsilon+1)^{-\frac{p}{2}} e^{-\frac{\kappa c_\varepsilon}{2}} \nabla \Big\{ (n_\varepsilon+1)^{-\frac{p}{2}} e^{-\frac{\kappa c_\varepsilon}{2}} \Big\} \cdot \nabla \varphi \\
&\qquad -p \int_\Omega n_\varepsilon (n_\varepsilon+1)^{-p-1} e^{-\kappa c_\varepsilon} \nabla c_\varepsilon \cdot \nabla \varphi \\
&\qquad + \int_\Omega (n_\varepsilon+1)^{-p} e^{-\kappa c_\varepsilon} u_\varepsilon \cdot \nabla \varphi \\
&\qquad -p \int_\Omega (n_\varepsilon+1)^{-p-1} (\rho n_\varepsilon - \mu n_\varepsilon^\alpha) e^{-\kappa c_\varepsilon} \varphi \\
&\qquad +\kappa \int_\Omega (n_\varepsilon+1)^{-p} c_\varepsilon e^{-\kappa c_\varepsilon} \varphi - \kappa \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} (n_\varepsilon+1)^{-p} e^{-\kappa c_\varepsilon} \varphi
\end{aligned} \tag{4.7}$$

for all $t > 0$ and $\varepsilon \in (0,1)$.

Proof. We only need to combine Lemma 2.4 with Lemma 4.1. □

Using that the factor $4\kappa^2 - p(p+1)^2 \frac{n_\varepsilon^2}{(n_\varepsilon+1)^2}$ appearing in the second integrand on the right-hand side of (4.7) has a uniform positive lower bound whenever $\kappa^2 > \frac{p(p+1)^2}{4}$, the following can readily be derived from the latter.

Lemma 4.3 *If $p > 0$ and $\kappa > 0$ satisfy*

$$\kappa > \frac{\sqrt{p} \cdot (p+1)}{2}, \tag{4.8}$$

then for all $T > 0$ there exists $C = C(T,p,\kappa) > 0$ such that

$$\int_0^T \int_\Omega \left| \nabla \Big\{ (n_\varepsilon+1)^{-\frac{p}{2}} e^{-\frac{\kappa c_\varepsilon}{2}} \Big\} \right|^2 \le C \qquad \text{for all } \varepsilon \in (0,1) \tag{4.9}$$

and

$$\int_0^T \int_\Omega (n_\varepsilon+1)^{-p} e^{-\kappa c_\varepsilon} |\nabla c_\varepsilon|^2 \le C \qquad \text{for all } \varepsilon \in (0,1). \tag{4.10}$$

Proof. Upon choosing $\varphi \equiv 1$ in Corollary 4.2, on the basis of Lemma 2.2 this can be seen by copying almost word by word the proof of Lemma 6.3 in [33]. □

In a straightforward manner, this also entails some time regularity feature of said coupled quantities:

Lemma 4.4 *Let $p > 0$ and $\kappa > \frac{\sqrt{p}\cdot(p+1)}{2}$. Then for all $T > 0$ there exists $C = C(T, p, \kappa, m) > 0$ such that*

$$\int_0^T \left\| \partial_t \Big\{ \Big(n_\varepsilon(\cdot,t) + 1 \Big)^{-p} e^{-\kappa c_\varepsilon(\cdot,t)} \Big\} \right\|_{(W^{2,2}(\Omega))^\star} dt \le C \qquad \text{for all } \varepsilon \in (0,1). \tag{4.11}$$

PROOF. For $\varepsilon \in (0,1)$, we abbreviate $a_\varepsilon := \nabla \Big\{ (n_\varepsilon+1)^{-\frac{p}{2}} e^{-\frac{\kappa c_\varepsilon}{2}} \Big\}$ and $b_\varepsilon := (n_\varepsilon+1)^{-\frac{p}{2}} e^{-\frac{\kappa c_\varepsilon}{2}} \nabla v_\varepsilon$ for $\varepsilon \in (0,1)$. An application of Corollary 4.2 to $\varphi(x,t) := \zeta(x)$, $(x,t) \in \overline{\Omega} \times (0,\infty)$, for fixed $\zeta \in C^\infty(\overline{\Omega})$, then shows that if we let $C_1 = C_1(p,\kappa) := \frac{8(p+1)}{p} \cdot \Big(\frac{2\kappa + p(p+1)}{4(p+1)} \Big)^2 + \frac{4\kappa^2 + p(p+1)}{4(p+1)}$, then

$$\begin{aligned}
&\Big| \int_\Omega \partial_t \Big\{ (n_\varepsilon+1)^{-p} e^{-\kappa c_\varepsilon} \Big\} \cdot \zeta \Big| \\
&\quad = \Big| - \frac{4(p+1)}{p} \int_\Omega \Big| a_\varepsilon + \frac{2\kappa + p(p+1)\frac{n_\varepsilon}{n_\varepsilon+1}}{4(p+1)} b_\varepsilon \Big|^2 \zeta \\
&\qquad - \int_\Omega \frac{4\kappa^2 - p(p+1)^2 \frac{n_\varepsilon^2}{(n_\varepsilon+1)^2}}{4(p+1)} |b_\varepsilon|^2 \zeta \\
&\qquad - 2 \int_\Omega (n_\varepsilon+1)^{-\frac{p}{2}} e^{-\frac{\kappa c_\varepsilon}{2}} a_\varepsilon \cdot \nabla\zeta \\
&\qquad - p \int_\Omega n_\varepsilon (n_\varepsilon+1)^{-\frac{p}{2}-1} e^{-\frac{\kappa c_\varepsilon}{2}} b_\varepsilon \cdot \nabla\zeta \\
&\qquad - p \int_\Omega (n_\varepsilon+1)^{-p-1} (\rho n_\varepsilon - \mu n_\varepsilon^\alpha) e^{-\kappa c_\varepsilon} \zeta \\
&\qquad + \int_\Omega (n_\varepsilon+1)^{-p} e^{-\kappa c_\varepsilon} u_\varepsilon \cdot \nabla\zeta \\
&\qquad + \kappa \int_\Omega (n_\varepsilon+1)^{-p} c_\varepsilon e^{-\kappa c_\varepsilon} \zeta - \kappa \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} (n_\varepsilon+1)^{-p} e^{-\kappa c_\varepsilon} \zeta \Big| \\
&\quad \le \frac{8(p+1)}{p} \cdot \Big\{ \int_\Omega |a_\varepsilon|^2 \Big\} \cdot \|\zeta\|_{L^\infty(\Omega)} + c_5 \cdot \Big\{ \int_\Omega |b_\varepsilon|^2 \Big\} \cdot \|\zeta\|_{L^\infty(\Omega)} \\
&\qquad + 2 \cdot \Big\{ \int_\Omega |a_\varepsilon|^2 \Big\}^{\frac{1}{2}} \cdot \|\nabla\zeta\|_{L^2(\Omega)} + p \cdot \Big\{ \int_\Omega |b_\varepsilon|^2 \Big\} \cdot \|\nabla\zeta\|_{L^2(\Omega)} \\
&\qquad + p\rho \cdot \Big\{ \int_\Omega n_\varepsilon \Big\} \cdot \|\zeta\|_{L^\infty(\Omega)} + p\mu \cdot \Big\{ \int_\Omega n_\varepsilon^\alpha \Big\} \cdot \|\zeta\|_{L^\infty(\Omega)} + \Big\{ \int_\Omega |u_\varepsilon|^2 \Big\} \cdot \|\nabla\zeta\|_{L^2(\Omega)} \\
&\qquad + \frac{|\Omega|}{e} \|\zeta\|_{L^\infty(\Omega)} + \kappa \cdot \Big\{ \int_\Omega n_\varepsilon \Big\} \cdot \|\zeta\|_{L^\infty(\Omega)}
\end{aligned} \tag{4.12}$$

for all $t > 0$ and $\varepsilon \in (0,1)$, because $\frac{n_\varepsilon}{n_\varepsilon+1} \le 1$, $(n_\varepsilon+1)^{-1} \le 1$, $e^{-\kappa c_\varepsilon} \le 1$ and $\kappa v_\varepsilon e^{-\kappa c_\varepsilon} \le \frac{1}{e}$ in $\Omega \times (0,\infty)$. Since from Lemma 4.3, Lemma 2.2 and 2.3 we know that for all $T > 0$ we have

$$\sup_{\varepsilon\in(0,1)} \Big\{ \int_0^T \int_\Omega |a_\varepsilon|^2 + \int_0^T \int_\Omega |b_\varepsilon|^2 + \int_0^T \int_\Omega n_\varepsilon^\alpha + \int_0^T \int_\Omega |u_\varepsilon|^2 \Big\} < \infty, \tag{4.13}$$

and since $W^{2,2}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$, from (4.12) we readily conclude (4.11 upon taking the supremum over all $\zeta \in C^\infty(\overline{\Omega})$ fulfilling $\|\zeta\|_{W^{2,2}(\Omega)} \leq 1$, and then integrating over $t \in (0,T)$ for fixed $T > 0$. □

In consequence, we infer the following.

Corollary 4.5 *Suppose that $p > 0$ and $\kappa > \frac{\sqrt{p}(p+1)}{2}$. Then for all $T > 0$, $((n_\varepsilon + 1)^{-p} e^{-\kappa c_\varepsilon})_{\varepsilon \in (0,1)}$ is relatively compact in $L^2((0,T); W^{1,2}(\Omega))$ with respect to the weak topology, and relatively compact in $L^2(\Omega \times (0,T))$ with respect to the strong topology.*

Proof. This can be derived from Lemma 4.3 and Lemma 4.4 by verbatim copying a corresponding argument detailed in [33, Lemma 7.1]. □

5 Passing to the limit. Proof of Theorem 1.1

Thanks to the boundedness and compactness features obtained so far, we are now in the position to construct a limit triple which satisfies the second and the third sub-problem in (1.4) in the spirit of Definition 2.5.

Lemma 5.1 *Let $\alpha > 1$. Then there exist $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1)$ and functions*

$$\begin{cases} n \in L^\alpha_{loc}(\overline{\Omega} \times [0,\infty)), \\ c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \qquad \text{and} \\ u \in L^1_{loc}([0,\infty); W^{1,1}_0(\Omega); \mathbb{R}^3)) \end{cases} \tag{5.1}$$

such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that $n \geq 0, c \geq 0$ and $\nabla \cdot u = 0$ a.e. in $\Omega \times (0,\infty)$, that

$$n_\varepsilon \to n \quad \text{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \tag{5.2}$$

$$c_\varepsilon \to c \quad \text{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty), \tag{5.3}$$

$$\nabla c_\varepsilon \to \nabla c \quad \text{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \text{ and a.e. in } \Omega \times (0,\infty) \qquad \text{as well as} \tag{5.4}$$

$$u_\varepsilon \rightharpoonup u \quad \text{in } L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \tag{5.5}$$

as $\varepsilon = \varepsilon_j \searrow 0$, and such that (2.9) holds, that (2.10) is satisfied for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, and that (2.11) is fulfilled for each $\varphi \in C_0^\infty(\Omega \times [0,\infty); \mathbb{R}^3)$ such that $\nabla \cdot \varphi = 0$.

Proof. We fix any $p > 0$ and $\kappa > \frac{\sqrt{p}(p+1)}{2}$, and let $w_\varepsilon := (n_\varepsilon + 1)^{-p} e^{-\kappa c_\varepsilon}$ for $\varepsilon \in (0,1)$. In view of Lemma 3.9 and Corollary 4.5, we can then find $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0,1)$ and nonnegative functions w an c on $\Omega \times (0,\infty)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and that as $\varepsilon = \varepsilon_j \searrow 0$ we have (5.3), (5.4) as well as

$$w_\varepsilon \to w \quad \text{a.e. in } \Omega \times (0,\infty).$$

Therefore,

$$n_\varepsilon = (e^{\kappa c_\varepsilon} w_\varepsilon)^{-\frac{1}{p}} - 1 \to n := (e^{\kappa c} w)^{-\frac{1}{p}} - 1 \quad \text{a.e. in } \Omega \times (0,\infty) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Recalling that (2.3) implies uniform integrability of $(n_\varepsilon)_{\varepsilon \in (0,1)}$ over $\Omega \times (0,T)$ for each $T > 0$, we obtain (5.2) as a consequence of the Vitali convergence theorem, while (5.5) directly results from Lemma 2.3,

and while the inclusions in (5.1) follow from the boundedness properties in (3.41), (2.5), (2.6) and (2.3) when combined with (5.2)-(5.5) and Fatou's lemma.
Apart from that, taking a null set $N \subset (0,\infty)$ such that in accordance with (5.2) and the Tonelli theorem we have $n_\varepsilon(\cdot,t) \to n(\cdot,t)$ a.e. in Ω for all $t \in (0,\infty) \setminus N$ as $\varepsilon = \varepsilon_j \searrow 0$ and hence

$$\int_\Omega n(\cdot,t) \le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_\Omega n_\varepsilon(\cdot,t) \qquad \text{for all } t \in (0,\infty) \setminus N, \tag{5.6}$$

in the identity

$$\int_\Omega n_\varepsilon(\cdot,t) + \mu \int_0^t \int_\Omega n_\varepsilon^\alpha = \int_\Omega n_0 + \rho \int_0^t \int_\Omega n_\varepsilon,$$

valid for all $t > 0$ and $\varepsilon \in (0,1)$ due to (2.1), we may again employ (5.2) along with Fatou's lemma to infer that the inequality in (2.9) indeed holds for each $t \in (0,\infty) \setminus N$.

Finally, given $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ we see from the second equation in (2.1) that

$$\begin{aligned} -\int_0^\infty \int_\Omega c_\varepsilon \varphi_t - \int_\Omega c_0 \varphi(\cdot,0) & \\ = \; -\int_0^\infty \int_\Omega \nabla c_\varepsilon \cdot \nabla \varphi - \int_0^\infty \int_\Omega c_\varepsilon \varphi + \int_0^\infty \int_\Omega \frac{n_\varepsilon}{1+n_\varepsilon} \varphi + \int_0^\infty \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla \varphi & \end{aligned} \tag{5.7}$$

for all $\varepsilon \in (0,1)$, where clearly, by (5.3) and (5.4),

$$\int_0^\infty \int_\Omega c_\varepsilon \varphi_t \to \int_0^\infty \int_\Omega c \varphi_t, \quad \int_0^\infty \int_\Omega \nabla c_\varepsilon \cdot \nabla \varphi \to \int_0^\infty \int_\Omega \nabla c \cdot \nabla \varphi \quad \text{and} \quad \int_0^\infty \int_\Omega c_\varepsilon \varphi \to \int_0^\infty \int_\Omega c \varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$. Furthermore, once more due to (5.2) we infer from the Vitali convergence theorem that also $\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \to n$ in $L^1_{loc}(\overline{\Omega} \times [0,\infty))$ and thus

$$\int_0^\infty \int_\Omega \frac{n_\varepsilon}{1+\varepsilon n_\varepsilon} \varphi \to \int_0^\infty \int_\Omega n \varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$, whereas Lemma 3.5 entails

$$\int_0^\infty \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla \varphi \to \int_0^\infty \int_\Omega c u \cdot \nabla \varphi$$

as $\varepsilon = \varepsilon_j \searrow 0$. The identity in (2.10) thus results from (5.7), and that in (2.11) can be verified in quite a similar way by relying on (5.2) and (5.5). □

Quite in the style of Lemma 8.2 in [33], we can now verify the remaining parts of Definition 2.5 by making use of Corollary 4.2 and the convergence properties gathered in Lemma 5.1:

Lemma 5.2 *Let $\alpha > 1$, and given $p > 0$ and $\kappa > \frac{\sqrt{p}(p+1)}{2}$, let ϕ, Φ and ψ be as accordingly defined by (4.1). Then (2.12)-(2.14) are satisfied, and (2.16) holds for any nonnegative $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.*

PROOF. For fixed $T > 0$, Lemma 4.3 implies that

$$\Big((n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^2((0,T);W^{1,2}(\Omega)),$$

and that both

$$\Big((n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\nabla c_\varepsilon\Big)_{\varepsilon\in(0,1)} \text{ and } \Big(n_\varepsilon(n_\varepsilon+1)^{-\frac{p}{2}-1}e^{-\frac{\kappa c_\varepsilon}{2}}\nabla c_\varepsilon\Big)_{\varepsilon\in(0,1)} \text{ are bounded in } L^2(\Omega\times(0,T)).$$

In view of Lemma 5.1 and Egorov's theorem, this implies that as $\varepsilon=\varepsilon_j\searrow 0$,

$$\nabla\Big\{(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\Big\}\rightharpoonup\nabla\Big\{(n+1)^{-\frac{p}{2}}e^{-\frac{\kappa c}{2}}\Big\}\qquad\text{in } L^2(\Omega\times(0,T)), \tag{5.8}$$

and that

$$(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\nabla c_\varepsilon\rightharpoonup(n+1)^{-\frac{p}{2}}e^{-\frac{\kappa c}{2}}\nabla c\qquad\text{in } L^2(\Omega\times(0,T)) \tag{5.9}$$

and

$$n_\varepsilon(n_\varepsilon+1)^{-\frac{p}{2}-1}e^{-\frac{\kappa c_\varepsilon}{2}}\nabla c_\varepsilon\rightharpoonup n(n+1)^{-\frac{p}{2}-1}e^{-\frac{\kappa c}{2}}\nabla c\qquad\text{in } L^2(\Omega\times(0,T)), \tag{5.10}$$

from which it follows that

$$\begin{aligned}&\nabla\Big\{(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\Big\}+\frac{2\kappa+p(p+1)\frac{n_\varepsilon}{n_\varepsilon+1}}{4(p+1)}(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\nabla c_\varepsilon\\ &\qquad\rightharpoonup\quad\nabla\Big\{(n+1)^{-\frac{p}{2}}e^{-\frac{\kappa c}{2}}\Big\}+\frac{2\kappa+p(p+1)\frac{n}{n+1}}{4(p+1)}(n+1)^{-\frac{p}{2}}e^{-\frac{\kappa c}{2}}\nabla c\qquad\text{in } L^2(\Omega\times(0,T))\end{aligned} \tag{5.11}$$

and that, since

$$(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\to(n+1)^{-\frac{p}{2}}e^{-\frac{\kappa c}{2}}\quad\text{in } L^2(\Omega\times(0,T))\qquad\text{as } \varepsilon=\varepsilon_j\searrow 0 \tag{5.12}$$

by the dominated convergence theorem, also

$$n_\varepsilon(n_\varepsilon+1)^{-p-1}e^{-\kappa c_\varepsilon}\nabla c_\varepsilon\quad\rightharpoonup\quad n(n+1)^{-p-1}e^{-\kappa c}\nabla c\qquad\text{in } L^1(\Omega\times(0,T)) \tag{5.13}$$

and

$$(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\nabla\Big\{(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\Big\}\rightharpoonup(n+1)^{-\frac{p}{2}}e^{-\frac{\kappa c}{2}}\nabla\Big\{(n+1)^{-\frac{p}{2}}e^{-\frac{\kappa c}{2}}\Big\}\qquad\text{in } L^1(\Omega\times(0,T)) \tag{5.14}$$

as $\varepsilon=\varepsilon_j\searrow 0$. Since $\Big((n_\varepsilon+1)^{-p-1}(\rho n_\varepsilon-\mu n_\varepsilon^\alpha)e^{-\kappa c_\varepsilon}\Big)_{\varepsilon\in(0,1)}$ can readily be seen to be uniformly integrable over $\Omega\times(0,T)$ according to (2.3), the Vitali convergence theorem ensures that furthermore

$$(n_\varepsilon+1)^{-p-1}(\rho n_\varepsilon-\mu n_\varepsilon^\alpha)e^{-\kappa c_\varepsilon}\to(n+1)^{-p-1}(\rho n-\mu n^\alpha)e^{-\kappa c}\quad\text{in } L^1(\Omega\times(0,T)), \tag{5.15}$$

while, quite similarly,

$$(n_\varepsilon+1)^{-p}c_\varepsilon e^{-\kappa c_\varepsilon}\to(n+1)^{-p}ce^{-\kappa c}\quad\text{in } L^1(\Omega\times(0,T))\qquad\text{as } \varepsilon=\varepsilon_j\searrow 0 \tag{5.16}$$

and

$$\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}(n_\varepsilon+1)^{-p}e^{-\kappa c_\varepsilon}\to n(n+1)^{-p}e^{-\kappa c}\quad\text{in } L^1(\Omega\times(0,T))\qquad\text{as } \varepsilon=\varepsilon_j\searrow 0, \tag{5.17}$$

as well as

$$(n_\varepsilon+1)^{-p}e^{-\kappa c_\varepsilon}u_\varepsilon\rightharpoonup (n+1)^{-p}e^{-\kappa c}u\quad\text{in } L^1(\Omega\times(0,T))\qquad\text{as } \varepsilon=\varepsilon_j\searrow 0 \tag{5.18}$$

due to (5.5).

Now letting ϕ,Φ and ψ be as in (4.1), we see that the properties in (2.12) and (2.13) are obvious, and that (2.14), (2.15) are immediate from (5.8)-(5.17). Moreover, given any nonnegative $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, in the corresponding identity from Corollary 4.2, upon a time integration implying that to see that

$$\begin{aligned}
&\frac{4(p+1)}{p}\int_0^\infty\int_\Omega\left|\nabla\Big\{(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\Big\}+\frac{2\kappa+p(p+1)\frac{n_\varepsilon}{n_\varepsilon+1}}{4(p+1)}(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\nabla c_\varepsilon\right|^2\varphi\\
&+\int_0^\infty\int_\Omega\frac{4\kappa^2-p(p+1)^2\frac{n_\varepsilon^2}{(n_\varepsilon+1)^2}}{4(p+1)}(n_\varepsilon+1)^{-p}e^{-\kappa c_\varepsilon}|\nabla c_\varepsilon|^2\varphi\\
&\quad=\int_0^\infty\int_\Omega(n_\varepsilon+1)^{-p}e^{-\kappa c_\varepsilon}\varphi_t+\int_\Omega(n_0+1)^{-p}e^{-\kappa c_0}\varphi(\cdot,0)\\
&\qquad-2\int_0^\infty\int_\Omega(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\nabla\Big\{(n_\varepsilon+1)^{-\frac{p}{2}}e^{-\frac{\kappa c_\varepsilon}{2}}\Big\}\cdot\nabla\varphi\\
&\qquad-p\int_0^\infty\int_\Omega n_\varepsilon(n_\varepsilon+1)^{-p-1}e^{-\kappa c_\varepsilon}\nabla c_\varepsilon\cdot\nabla\varphi\\
&\qquad+\int_0^\infty\int_\Omega(n_\varepsilon+1)^{-p}e^{-\kappa c_\varepsilon}u_\varepsilon\cdot\nabla\varphi-p\int_0^\infty\int_\Omega(n_\varepsilon+1)^{-p-1}(\rho n_\varepsilon-\mu n_\varepsilon^\alpha)e^{-\kappa c_\varepsilon}\varphi\\
&\qquad+\kappa\int_0^\infty\int_\Omega(n_\varepsilon+1)^{-p}c_\varepsilon e^{-\kappa c_\varepsilon}\varphi+\kappa\int_0^\infty\int_\Omega\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}(n_\varepsilon+1)^{-p}e^{-\kappa c_\varepsilon}\varphi\qquad\text{for all } \varepsilon\in(0,1),
\end{aligned}$$

we may once again rely on our hypothesis $\kappa>\frac{\sqrt{p}(p+1)}{2}$ to infer that since thus $4\kappa^2-p(p+1)^2\frac{n_\varepsilon^2}{(n_\varepsilon+1)^2}$ is nonnegative for all $\varepsilon\in(0,1)$, from Lemma 5.1 and Fatou's lemma we obtain that

$$\begin{aligned}
&\int_0^\infty\int_\Omega\frac{4\kappa^2-p(p+1)^2\frac{n^2}{(n+1)^2}}{4(p+1)}(n+1)^{-p}e^{-\kappa c}|\nabla c|^2\varphi\\
&\qquad\le\liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^\infty\int_\Omega\frac{4\kappa^2-p(p+1)^2\frac{n_\varepsilon^2}{(n_\varepsilon+1)^2}}{4(p+1)}(n_\varepsilon+1)^{-p}e^{-\kappa c_\varepsilon}|\nabla c_\varepsilon|^2\varphi.
\end{aligned}$$

Along with (5.11) and a standard argument based on lower semicontinuity property of L^2 norms with respect to weak convergence, due to (5.13)-(5.18) this can readily be verified to entail (2.16). □

We can thereby complete the derivation of our main results:

Proof of Theorem 1.1. The claim follows by combining Lemma 5.1 with Lemma 5.2. □

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References

[1] Espejo E. E., Suzuki T.: *Reaction terms avoiding aggregation in slow fluids.* Nonlinear Anal. Real World Appl. **21**, 110-126 (2015)

[2] Friedman, A.: *Partial Differential Equations.* Holt, Rinehart & Winston, New York, 1969

[3] Galakhov, E., Salieva, O., Tello, J.I.: *On a parabolic-elliptic system with chemotaxis and logistic type growth.* Journal of Differential Equations, **261**, 4631-4647 (2016)

[4] Giga, Y., Sohr, H.: *Abstract L^p Estimates for the Cauchy Problem with Applications to the Navier-Stokes Equations in Exterior Domains.* J. Functional Analysis **102**, 72-94 (1991)

[5] Herrero, M.A., Velázquez, J.J.L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **24**, 633-683 (1997)

[6] Henry, D.: *Geometric Theory of Semilinear Parabolic Equations.* Springer-Verlag, Berlin Heidelberg, 1981

[7] Jin, H., Xiang, T.: *Chemotaxis effect vs. logistic damping on boundedness in the 2-D minimal Keller-Segel model.* C. R. Math. Acad. Sci. Paris **356**, 875-885 (2018)

[8] Keller, E.F., Segel, L.A.: *Initiation of slime mold aggregation viewed as an instability.* J. Theor. Biol. **26**, 399-415 (1970)

[9] Kiselev, A., Ryzhik, L.: *Biomixing by chemotaxis and enhancement of biological reactions.* Comm Partial Differential Equations **37**, 298-318 (2012)

[10] Kiselev, A., Ryzhik, L.: *Biomixing by chemotaxis and efficiency of biological reactions: the critical reaction case.* J. Math. Phys. **53**, 115609, 9pp. (2012)

[11] Lankeit, J.: *Eventual smoothness and asymptotics in a three-dimensional chemotaxis system with logistic source.* J. Differential Equations **258**, 1158-1191 (2015)

[12] Lankeit, J., Winkler M.: *A generalized solution concept for the Keller-Segel system with logarithmic sensitivity: global solvability for large nonradial data.* Nonlinear Differ. Equ. Appl. **24**, Paper 49 (2017)

[13] Osaki, K., Tsujikawa, T., Yagi, A., Mimura, M.: *Exponential attractor for a chemotaxis-growth system of equations.* Nonlinear Analysis **51**, 119-144 (2002)

[14] Sohr, H.: *The Navier-Stokes Equations. An Elementary Functional Analytic Approach.* Birkhäuser, Basel, 2001

[15] Tao, Y., Winkler, M.: *Blow-up prevetion by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system.* Z. Angew. Math. Phys. **12**, Art.138 (2016)

[16] Tao, Y., Winkler, M.: *Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system.* Z. Angew. Math. Phys. **66**, 2555-2573 (2015)

[17] Tello, J.I., Winkler, M.: *A chemotaxis system with logistic source.* Comm Partial Differential Equations **32**, 849-877 (2007)

[18] Temam, R.: *Navier-Stokes equations. Theory and numerical analysis. Studies in Mathematics and its Applications. Vol. 2.* North-Holland, Amsterdam, 1977

[19] Tuval, I., Cisneros, L., Dombrowski, C., Wolgemuth, C.W., Kessler, J.O., Goldstein, R.E.: *Bacterial swimming and oxygen transport near contact lines.* Proc. Nat. Acad. Sci. USA **102**, 2277-2282 (2005)

[20] Viglialoro, G: *Very weak global solutions to a parabolic-parabolic chemotaxis-system with logistic source.* J. Math. Anal. Appl. **439**, 197-212 (2016)

[21] Viglialoro, G: *Boundedness properties of very weak solutions to a fully parabolic chemotaxis system with logistic source.* Nonlinear Anal. Real World Appl. **34**, 520-535 (2017)

[22] Wang, Y., Xiang, Z.: *Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation: The 3D case.* J. Differential Eq. **261**, 4944-4973 (2016)

[23] Wang, Y., Winkler, M., Xiang, Z.: *Global classical solutions in a two-dimensional chemotaxis-Navier-Stokes system with subcritical sensitivity,* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **18**, 421-466 (2018)

[24] Wang, Y., Winkler M., Xiang, Z.: *Local energy estimates and global solvability in a three-dimensional chemotaxis-fluid system with prescribed signal on the boundary.* Preprint

[25] Winkler, M.: *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source.* Comm. Part. Differential Eq. **35**, 1516-1537 (2010)

[26] Winkler, M.: *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction.* J. Math. Anal. Appl. **384**, 261-272 (2011)

[27] Winkler, M.: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013)

[28] Winkler, M.: *Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities.* SIAM J. Math. Anal. **47**, 3092-3115 (2015)

[29] Winkler, M.: *Boundedness and large time behavior in a three-dimensional chemotaxis-Stokes system with nonlinear diffusion and general sensitivity.* Cal. Var. Partial Differential Equations, **54**, 3789-3828 (2016).

[30] Winkler, M.: *Finite-time blow-up in low-dimensional Keller-Segel systems with logistic-type superlinear degradation.* Z. Angew. Math. Phys. **69**, Art.40 (2018)

[31] Winkler, M.: *A three-dimensional Keller-Segel-Navier-Stokes system with logistic source: Global weak solutions and asymptotic stabilization.* J. Functional Analysis **276**, 1339-1401 (2019)

[32] Winkler, M.: *The role of superlinear damping in the construction of solutions to drift-diffusion problems with initial data in L^1.* Adv. Nonlin. Anal. **9**, 526-566 (2020)

[33] Winkler, M.: *L^1 solutions to parabolic Keller-Segel systems involving arbitrary superlinear degradation.* Preprint

[34] Xiang, T.: *How strong a logistic damping can prevent blow-up for the minimal Keller-Segel chemotaxis system?* Non. Anal. Real World Appl. **459**, 1172-1200 (2018)

[35] Xiang, T.: *Chemotactic aggregation versus logistic damping on boundedness in the 3D minimal Keller-Segel model?* SIAM J. Appl. Math. **78**, 2420-2438 (2018)

[36] Yan, J., Fuest, M: *When do Keller-Segel systems with heterogeneous logistic sources admit generalized solutions?* arXiv:2004.02153