# Boundedness in a three-dimensional Keller-Segel-Stokes system with subcritical sensitivity 

Michael Winkler*<br>Institut für Mathematik, Universität Paderborn, Warburger Str. 100, 33098 Paderborn, Germany


#### Abstract

We consider a quasilinear Keller-Segel system with density-dependent migration rates, coupled to the incompressible Stokes equations through transport and buoyancy. By means of an apparently novel approach based on certain conditional estimates for the taxis gradient and the fluid field, for diffusion rates asymptotically controllable by power-tape majorants and minorants a result on global existence and boundedness is derived under an essentially optimal condition on the strength of cross-diffusion relative to diffusion.


Key words: chemotaxis; Stokes equations; blow-up prevention
MSC (2010): 35K65 (primary); 35Q55, 92C17 (secondary)

## 1 Introduction

In a smoothly bounded domain $\Omega \subset \mathbb{R}^{n}, n \geq 2$, we consider the problem

$$
\begin{cases}n_{t}+u \cdot \nabla n=\nabla \cdot(D(n) \nabla n)-\nabla \cdot(S(n) \nabla c), & x \in \Omega, t>0  \tag{1.1}\\ c_{t}+u \cdot \nabla c=\Delta c-c+n, & x \in \Omega, t>0 \\ u_{t} & =\Delta u+\nabla P+n \nabla \Phi, \quad \nabla \cdot u=0, \\ \frac{\partial n}{\partial \nu}=\frac{\partial c}{\partial \nu}=0, \quad u=0, & x \in \Omega, t>0 \\ n(x, 0)=n_{0}(x), \quad c(x, 0)=c_{0}(x), \quad u(x, 0)=u_{0}(x), & x \in \Omega,\end{cases}
$$

which can be viewed as an extension of the classical quasilinear Keller-Segel model to frameworks in which the interaction of chemotactically moving populations with liquid environments through transport and buoyancy needs to be accounted for ([11]).
In the absence of such fluid coupling, the resulting two-component system for the unknown population density $n$ and signal concentration $c$ has been thoroughly studied in the literature, reflecting its prominent role in the refined modeling of chemotactic motion in situations when density-dependent influences on cell motility are of quantitative relevance ([6], [4]). With regard to the fundamental question whether or not the spontaneous emergence of singular structures is supported, the latter fluid-free version of (1.1) is fairly well-understood, at least in cases when $D$ is suitably smooth and positive, and such that with some $m_{0} \in \mathbb{R}, m \geq m_{0}, k_{D}>0$ and $K_{D}>0$ we have

$$
\begin{equation*}
k_{D} s^{m_{0}-1} \leq D(s) \leq K_{D} s^{m-1} \quad \text { for all } s>1 \tag{1.2}
\end{equation*}
$$

Then, namely, global bounded classical solutions can be found for all reasonably regular initial data whenever $S$ is suitably smooth and nonnegative satisfies $S(0)=0$ and complies with the subcriticality condition that $\frac{S(s)}{D(s)} \leq K_{S D} s^{\alpha}$ be valid for all $s>1$ and some $K_{S D}>0$ and $\alpha<\frac{2}{n}$ ([10], [7], [8]); additional results on the occurrence of finite-time or infinite-time explosions, available under various assumptions suitably complementing to the latter, strongly indicate that the growth of $0 \leq s \mapsto s^{\frac{2}{n}}$ indeed marks a genuinely blow-up critical relationship between $S$ and $D$ ([2], [3], [13], [16]).
In the context of the fully coupled problem (1.1), however, the picture seems much less complete in this respect so far: While said unboundedness results clearly extend in a trivial manner by simply letting $\Phi \equiv$ const. and $u_{0} \equiv 0$, in the physically most relevant space dimension $n=3$ the above condition on subcritical growth of $\frac{S}{D}$ has up to now been found to ensure boundedness in (1.1) only under the additional assumption that in (1.2) we have $m_{0}=m>-\frac{1}{3}$ ([1]; cf. also [12], [15] for precedents addressing the linear diffusion case when $m_{0}=m=1$ ); for smaller values of $m_{0}$, only a result on global existence seems available, without information on boundedness properties ([1]). In the presence of diffusion rates exhibiting fast algebraic decay at large densities, the question how far fluid interaction may influence singularity formation in (1.1) accordingly seems unsolved.
The purpose of this note is to propose an analytical approach capable of adequately coping with the challenges related to the chemotaxis-fluid coupling in (1.1) also when such strong diffusion degeneracies are involved. This

[^0]method, in contrast to previously pursued strategies relying on certain conditional estimates for $\nabla c$ and $u$ recently obtained for solutions $(c, u)$ to the respective sub-problem of (1.1) in a more general setting ([17]), will enable us to derive the following result which essentially provides a complete answer to the above question, namely asserting that fluid interaction does not affect the potential of (1.1) to generate singular behavior also when diffusion of arbitrary algebraic strength near $n=\infty$ are present. Here and below, we let $A$ denote the realization of the Stokes operator in $L^{2}\left(\Omega ; \mathbb{R}^{3}\right)$, with its domain of definition given by $D(A)=W^{2,2}\left(\Omega ; \mathbb{R}^{3}\right) \cap W_{0}^{1,2}\left(\Omega ; \mathbb{R}^{3}\right) \cap L_{\sigma}^{2}(\Omega)$, where $L_{\sigma}^{2}(\Omega):=\{\varphi \in$ $\left.L^{2}\left(\Omega ; \mathbb{R}^{3}\right) \mid \nabla \cdot \varphi=0\right\}$, and for $\vartheta>0$ we let $A^{\vartheta}$ denote the corresponding fractional power ([9]).

Theorem 1.1 Suppose that $\Omega \subset \mathbb{R}^{3}$ is a bounded domain with smooth boundary, that $\Phi \in W^{2, \infty}(\Omega)$, and that $D \in \bigcup_{\iota>0} C^{1+\iota}([0, \infty))$ and $S \in \bigcup_{\iota>0} C^{1+\iota}([0, \infty))$ are such that $D>0$ in $[0, \infty)$, that $S(0)=0$, and that (1.2) as well as

$$
\begin{equation*}
\frac{|S(s)|}{D(s)} \leq K_{S D} s^{\alpha} \quad \text { for all } s>1 \tag{1.3}
\end{equation*}
$$

with some $m_{0} \in \mathbb{R}, m \geq m_{0}, k_{D}>0, K_{D}>0, K_{S D}>0$ and $\alpha<\frac{2}{3}$. Then for any choice of $0 \leq n_{0} \in C^{0}(\bar{\Omega})$, $0 \leq c_{0} \in W^{1, \infty}(\Omega)$ and $u_{0} \in \bigcup_{\vartheta \in\left(\frac{3}{4}, 1\right)} D\left(A^{\vartheta}\right)$, one can find functions

$$
\left\{\begin{array}{l}
n \in C^{0}(\bar{\Omega} \times[0, \infty)) \cap C^{2,1}(\bar{\Omega} \times(0, \infty)), \\
c \in \bigcap_{q>3} C^{0}\left([0, \infty) ; W^{1, q}(\Omega)\right) \cap C^{2,1}(\bar{\Omega} \times(0, \infty)), \\
u \in \bigcup_{\vartheta \in\left(\frac{3}{4}, 1\right)} C^{0}\left([0, \infty) ; D\left(A^{\vartheta}\right)\right) \cap C^{2,1}\left(\bar{\Omega} \times(0, \infty) ; \mathbb{R}^{3}\right) \quad \text { and } \\
P \in C^{1,0}(\Omega \times(0, \infty))
\end{array}\right.
$$

such that $n \geq 0$ and $c \geq 0$ in $\Omega \times(0, \infty)$, and that ( $n, c, u, P)$ forms a classical solution of (1.1). Moreover,

$$
\begin{equation*}
\sup _{t>0}\left\{\|n(\cdot, t)\|_{L^{\infty}(\Omega)}+\|c(\cdot, t)\|_{W^{1, \infty}(\Omega)}+\left\|A^{\vartheta} u(\cdot, t)\right\|_{L^{2}(\Omega)}\right\}<\infty \quad \text { for all } \vartheta \in\left(\frac{3}{4}, 1\right) \tag{1.4}
\end{equation*}
$$

## 2 Preliminaries. Conditional bounds for $c$ and $u$

Lemma 2.1 Under the assumption of Theorem 1.1, there exist $T_{\max } \in(0, \infty]$ and functions

$$
\left\{\begin{array}{l}
n \in C^{0}\left(\bar{\Omega} \times\left[0, T_{\max }\right)\right) \cap C^{2,1}\left(\bar{\Omega} \times\left(0, T_{\max }\right)\right), \\
c \in \bigcap_{q>3} C^{0}\left(\left[0, T_{\max }\right) ; W^{1, q}(\Omega)\right) \cap C^{2,1}\left(\bar{\Omega} \times\left(0, T_{\max }\right)\right), \\
u \in \bigcup_{\vartheta \in\left(\frac{1}{2}, 1\right)} C^{0}\left(\left[0, T_{\max }\right) ; D\left(A^{\vartheta}\right)\right) \cap C^{2,1}\left(\bar{\Omega} \times\left(0, T_{\max }\right) ; \mathbb{R}^{3}\right) \quad \text { and } \\
P \in C^{1,0}\left(\Omega \times\left(0, T_{\max }\right)\right)
\end{array}\right.
$$

such that $n \geq 0$ and $c \geq 0$ in $\Omega \times\left(0, T_{\max }\right)$, that $(n, c, u, P)$ solves (1.1) classically in $\Omega \times\left(0, T_{\max }\right)$, and that if $T_{\text {max }}<\infty$, then for all $\vartheta \in\left(\frac{3}{4}, 1\right)$ we have

$$
\begin{equation*}
\limsup _{t \nearrow T_{\max }}\left\{\|n(\cdot, t)\|_{L^{\infty}(\Omega)}+\|c(\cdot, t)\|_{W^{1, \infty}(\Omega)}+\left\|A^{\vartheta} u(\cdot, t)\right\|_{L^{2}(\Omega)}\right\}=\infty \tag{2.1}
\end{equation*}
$$

In addition, this solution satisfies

$$
\begin{equation*}
\int_{\Omega} n(\cdot, t)=\int_{\Omega} n_{0} \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{2.2}
\end{equation*}
$$

Proof. This can be seen by straightforward adaptation of standard arguments, a detailed application of which to some closely related contexts can be found in [5, Section 2] and [14, Lemma 2.1], for instance.

The following conditional bounds for the second and third components of this solution are immediate consequences of [17, Theorem 1.2 and Proposition 1.1].
Lemma 2.2 Under the assumptions of Theorem 1.1, given any $\theta \in\left(\frac{3}{4}, 1\right), p>3$ and $\eta>0$ one can find $C=$ $C(\theta, p, \eta)>0$ such that

$$
\begin{equation*}
\|c(\cdot, t)\|_{W^{1, \infty}(\Omega)} \leq C \cdot\left\{1+\sup _{s \in(0, t)}\|n(\cdot, s)\|_{L^{p}(\Omega)}\right\}^{\frac{p}{p-1} \cdot\left(\frac{2}{3}+\eta\right)} \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{2.3}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|A^{\theta} u(\cdot, t)\right\|_{L^{2}(\Omega)} \leq C \cdot\left\{1+\sup _{s \in(0, t)}\|n(\cdot, s)\|_{L^{p}(\Omega)}\right\}^{\frac{p}{p-1} \cdot\left(\frac{4 \theta-1}{6}+\eta\right)} \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{2.4}
\end{equation*}
$$

## $3 \quad L^{p}$ estimates for $n$. Proof of Theorem 1.1

In order to appropriately prepare our testing procedure to be detailed in Lemma 3.2, let us note two basic properties of a function to be utilized therein.
Lemma 3.1 Under the assumptions from Theorem 1.1, for each $p>1$ there exists $C=C(p)>0$ such that

$$
\begin{equation*}
\psi_{p}(s):=\int_{0}^{s} \int_{0}^{\sigma} \frac{\tau^{m+p-3}}{D(\tau)} d \tau d \sigma, \quad s \geq 0 \tag{3.1}
\end{equation*}
$$

satisfies

$$
\frac{1}{C} s^{p}-1 \leq \psi_{p}(s) \leq C s^{p+m-m_{0}}+C \quad \text { for all } s \geq 0
$$

Proof. As this can be derived from (1.2) and (1.3) by quite elementary arguments, we may refrain from giving details here.
The core of our reasoning can now be found in the following outcome of a testing procedure which in an essential manner relies on the outcome of Lemma 2.2.

Lemma 3.2 Suppose that the assumptions from Theorem 1.1 and satisfied, and let $T_{\max }$ and $(n, c, u, P)$ be as in Lemma 2.1. Then for all $p_{0}>1$ there exist $p \geq p_{0}$ and $C>0$ such that

$$
\begin{equation*}
\|n(\cdot, t)\|_{L^{p}(\Omega)} \leq C \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{3.2}
\end{equation*}
$$

Proof. Given $p_{0}>1$, we use that $2-3 \alpha>0$ in choosing $p \geq p_{0}$ such that abbreviating $\gamma:=m-m_{0} \geq 0$ we have

$$
\begin{equation*}
p>\max \left\{\frac{2-3 \alpha+2 \gamma}{2-3 \alpha}, 4-2 \alpha-m, 3-m, \frac{3+\gamma-3 m}{2}\right\} \tag{3.3}
\end{equation*}
$$

and in observing that the first restriction contained herein ensures that

$$
(p-1)(4-3 \alpha)-3(p+\gamma-1) \cdot \frac{2}{3}=p \cdot(2-3 \alpha)-(2-3 \alpha+2 \gamma)>0
$$

and that hence $\frac{3(p+\gamma-1)}{(p-1)(4-3 \alpha)} \cdot \frac{2}{3}<1$, so that we can pick $\eta>0$ such that

$$
\begin{equation*}
\kappa:=\frac{3(p+\gamma-1)}{(p-1)(4-3 \alpha)} \cdot\left(\frac{2}{3}+\eta\right) \quad \text { satisfies } \quad \kappa<1 \tag{3.4}
\end{equation*}
$$

With these values of $p$ and $\eta$ fixed henceforth, we next draw on Lemma 2.2 to obtain $C_{1}>0$ such that writing

$$
M_{p}(t):=1+\sup _{s \in(0, t)}\|n(\cdot, s)\|_{L^{p}(\Omega)}, \quad t \in\left(0, T_{\max }\right)
$$

we have

$$
\begin{equation*}
\|\nabla c(\cdot, t)\|_{L^{\infty}(\Omega)} \leq C_{1} M_{p}^{\frac{p}{p-1} \cdot\left(\frac{2}{3}+\eta\right)}(t) \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{3.5}
\end{equation*}
$$

In order to estimate $M_{p}$ on the basis thereof, we first follow the classical approach from [10] by taking $\psi_{p}$ as accordingly defined in (3.1) and using (1.1) to see that for all $t \in\left(0, T_{\max }\right)$, since $\psi_{p}^{\prime \prime}(s)=\frac{s^{m+p-3}}{D(s)}$ for all $s \geq 0$ and $\nabla \cdot u=0$ in $\Omega \times\left(0, T_{\max }\right)$,

$$
\begin{align*}
\frac{d}{d t} \int_{\Omega} \psi_{p}(n) & =-\int_{\Omega} n^{m+p-3}|\nabla n|^{2}+\int_{\Omega} n^{m+p-3} \frac{S(n)}{D(n)} \nabla n \cdot \nabla c \\
& \leq-\frac{1}{2} \int_{\Omega} n^{m+p-3}|\nabla n|^{2}+\frac{1}{2} \int_{\Omega} n^{m+p-3} \frac{S^{2}(n)}{D^{2}(n)}|\nabla c|^{2} \\
& =-\frac{2}{(m+p-1)^{2}} \int_{\Omega}\left|\nabla n^{\frac{m+p-1}{2}}\right|^{2}+\frac{1}{2} \int_{\Omega} n^{m+p-3} \frac{S^{2}(n)}{D^{2}(n)}|\nabla c|^{2} \tag{3.6}
\end{align*}
$$

according to Young's inequality, where thanks to (1.3),

$$
\begin{align*}
\frac{1}{2} \int_{\Omega} n^{m+p-3} \frac{S^{2}(n)}{D^{2}(n)}|\nabla c|^{2} & =\frac{1}{2} \int_{\{n \leq 1\}} n^{m+p-3} \frac{S^{2}(n)}{D^{2}(n)}|\nabla c|^{2}+\frac{1}{2} \int_{\{n>1\}} n^{m+p-3} \frac{S^{2}(n)}{D^{2}(n)}|\nabla c|^{2} \\
& \leq C_{2} \int_{\Omega}|\nabla c|^{2}+\frac{K_{S D}^{2}}{2} \int_{\Omega} n^{m+p-3+2 \alpha}|\nabla c|^{2} \quad \text { for all } t \in\left(0, T_{\text {max }}\right) \tag{3.7}
\end{align*}
$$

with $C_{2}:=\frac{1}{2} \max _{s \in[0,1]}\left\{s^{m+p-3} \frac{S^{2}(s)}{D^{2}(s)}\right\}$ being finite by continuity of $S$ and $D$ and by positivity of $D$, and by the fact that $m-p-3 \geq 0$ due to (3.3).
From now on, however, our strategy apparently deviates from all those previously pursued in that instead of explicitly involving the second equation in (1.1), our method of controlling $\nabla c$ exclusively relies on (3.5), according to which, namely, (3.7) implies that for all $t \in\left(0, T_{\max }\right)$,

$$
\begin{equation*}
\frac{1}{2} \int_{\Omega} n^{m+p-3} \frac{S^{2}(n)}{D^{2}(n)}|\nabla c|^{2} \leq C_{1}^{2} C_{2}|\Omega| M_{p}^{\frac{2 p}{p-1} \cdot\left(\frac{2}{3}+\eta\right)}(t)+\frac{K_{S D}^{2} C_{1}^{2}}{2} M_{p}^{\frac{2 p}{p-1} \cdot\left(\frac{2}{3}+\eta\right)}(t) \int_{\Omega} n^{m+p-3+2 \alpha} \tag{3.8}
\end{equation*}
$$

To appropriately estimate the latter integral by interpolation, we now use that the restrictions $p>4-2 \alpha-m$ and $p>3-m$ in (3.3) imply that $m+p-3+2 \alpha>1$ and that, again since $\alpha<\frac{2}{3}$,

$$
3(m+p-1)-(m+p-3+2 \alpha)=2 p-2 \alpha+2 m>2 \cdot(3-m)-2 \alpha+2 m=6-2 \alpha>0
$$

meaning that $\frac{2}{m+p-1}<\frac{2(m+p-3+2 \alpha)}{m+p-1}<6$ and that hence setting $a:=\frac{3(m+p-1)(m+p-4+2 \alpha)}{(3 m+3 p-4)(m+p-3+2 \alpha)}$ defines a number $a$ belonging to $(0,1)$. As $\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2}{m+1}}=\int_{\Omega} n=\int_{\Omega} n_{0}$ for all $t \in\left(0, T_{\max }\right)$ by (2.2), we may thus employ the Gagliardo-Nirenberg inequality to see that with some $C_{3}>0$ and $C_{4}>0$ we have

$$
\begin{align*}
& \int_{\Omega} n^{m+p-3+2 \alpha}=\left\|n^{\frac{m+p-1}{2}}\right\|_{L}^{\frac{2(m+p-3+2 \alpha)}{m+p-1}} \begin{array}{l}
\frac{2(m+p+2 \alpha)}{m+p-1}
\end{array}(\Omega) \\
& \leq \quad C_{3}\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^{2}(\Omega)}^{\frac{2(m+p-3+2 \alpha)}{m+p-1} \cdot a}\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2(m+p-3+2 \alpha)}{m+p-1}} \cdot \frac{2(1-a)}{\frac{2}{m+p}(\Omega)}}^{\frac{2}{}}+C_{3}\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2(m+p-3+2 \alpha)}{m+p-1}} \frac{\frac{2}{m+p-1}(\Omega)}{\frac{2}{m}}} \\
& \leq \quad C_{4}\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^{2}(\Omega)}^{2 \cdot \frac{3(m+p-4+2 \alpha)}{3 m+3 p-4}}+C_{4} \quad \text { for all } t \in\left(0, T_{\max }\right) . \tag{3.9}
\end{align*}
$$

Since here $\frac{3(m+p-4+2 \alpha)}{3 m+3 p-4}-1=-\frac{8-6 \alpha}{3 m+3 p-4}<0$ due to the inequality $\alpha<\frac{4}{3}$, it follows that $\lambda:=\frac{3 m+3 p-4}{3(m+p-4+2 \alpha)}$ satisfies $\lambda>1$, so that Young's inequality provides $C_{5}>0$ fulfilling

$$
\begin{aligned}
\frac{K_{S D}^{2} C_{1}^{2}}{2} M^{\frac{2 p}{p-1} \cdot\left(\frac{2}{3}+\eta\right)}(t) \cdot C_{4}\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^{2}(\Omega)}^{2 \cdot \frac{3(m+p-4+2 \alpha)}{3 m+3 p-4}} \leq & \frac{1}{(m+p-1)^{2}}\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^{2}(\Omega)}^{2} \\
& +C_{5} M^{\frac{2 p}{p-1} \cdot\left(\frac{2}{3}+\eta\right) \cdot \frac{\lambda}{\lambda-1}}(t) \quad \text { for all } t \in\left(0, T_{\text {max }}\right)
\end{aligned}
$$

which together with (3.9), (3.8), (3.7) and the fact that $\frac{\lambda}{\lambda-1}=\frac{3 m+3 p-4}{8-6 \alpha}>1$ implies that since $M_{p} \geq 1$, writing $C_{6}:=C_{5}+C_{1}^{2} C_{2}|\Omega|+\frac{K_{S D}^{2} C_{1}^{2} C_{4}}{2}$ we have

$$
\begin{align*}
\frac{1}{2} \int_{\Omega} n^{m+p-3} \frac{S^{2}(n)}{D^{2}(n)}|\nabla c|^{2} \leq & \frac{K_{S D}^{2} C_{1}^{2}}{2} M_{p}^{\frac{2 p}{p-1} \cdot\left(\frac{2}{3}+\eta\right)}(t) \cdot\left\{C_{4}\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^{2}(\Omega)}^{2 \cdot \frac{3(m+p-4+2 \alpha)}{3 m+3 p-4}}+C_{4}\right\} \\
& +C_{1}^{2} C_{2}|\Omega| M_{p}^{\frac{2 p}{p-1} \cdot\left(\frac{2}{3}+\eta\right)}(t) \\
\leq & \frac{1}{(m+p-1)^{2}} \int_{\Omega}\left|\nabla n^{\frac{m+p-1}{2}}\right|^{2}+C_{5} M_{p}^{\frac{2 p}{p-1} \cdot\left(\frac{2}{3}+\eta\right) \cdot \frac{\lambda}{\lambda-1}}(t) \\
& +\frac{K_{S D}^{2} C_{1}^{2} C_{4}}{2} M_{p}^{\frac{2 p}{p-1} \cdot\left(\frac{2}{3}+\eta\right)}(t)+C_{1}^{2} C_{2}|\Omega| M_{p}^{\frac{p}{p-1} \cdot\left(\frac{2}{3}+\eta\right)}(t) \\
\leq & \frac{1}{(m+p-1)^{2}} \int_{\Omega}\left|\nabla n^{\frac{m+p-1}{2}}\right|^{2} \\
& +C_{6} M_{p}^{\frac{p}{p-1} \cdot\left(\frac{2}{3}+\eta\right) \cdot \frac{3 m+3 p-4}{4-3 \alpha}}(t) \quad \text { for all } t \in\left(0, T_{\max }\right) . \tag{3.10}
\end{align*}
$$

We next turn part of the dissipative contribution to (3.6) into a superlinear zero-order absorptive term by recalling that due to Lemma 3.1 and our definition of $\gamma$, there exists $C_{7}>0$ such that

$$
\begin{equation*}
\int_{\Omega} \psi_{p}(n) \leq C_{7} \int_{\Omega} n^{p+\gamma}+C_{7} \quad \text { for all } t \in\left(0, T_{\max }\right) \tag{3.11}
\end{equation*}
$$

and that the rightmost restriction expressed in (3.3) guarantees that $6(m+p-1)-2(p+\gamma)=4 p-2(3+\gamma-3 m)>0$ and that thus $\frac{2}{m+p-1}<\frac{2(p-\gamma)}{m+p-1}<6$, because clearly $p+\gamma>1$, Therefore, namely, the number $b:=\frac{3(m+p-1)(p+\gamma-1)}{(3 m+3 p-4)(p+\gamma)}$ satisfies $b \in(0,1)$, so that we may once more employ the Gagliardo-Nirenberg inequality along with (2.2) to see that with some $C_{8}>0$ and $C_{9}>0$, due to (3.11) we have

$$
\begin{aligned}
& \left\{\int_{\Omega} \psi_{p}(n)\right\}^{\frac{3 m+3 p-4}{3(p+\gamma-1)}} \leq\left(2 C_{7}\right)^{\frac{3 m+3 p-4}{3(p+\gamma-1)}}\left\{\int_{\Omega} n^{p+\gamma}\right\}^{\frac{3 m+3 p-4}{3(p+\gamma-1)}}+\left(2 C_{7}\right)^{\frac{3 m+3 p-4}{3(p+\gamma-1)}} \\
& =\left(2 C_{7}\right)^{\frac{3 m+3 p-4}{3(p+\gamma-1)}}\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2(3 m+3 p-4)(p+\gamma)}{3(m+p-1)(p+\gamma-1)}}(\Omega)}^{\frac{2(1)}{m+p-1}}+\left(2 C_{7}\right)^{\frac{3 m+3 p-4}{3(p+\gamma-1)}} \\
& \leq \quad C_{8}\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^{2}(\Omega)}^{\frac{2(3 m+3 p-4)(p+\gamma)(p+\gamma-1)}{3(m)} \cdot b}\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2(3 m+3 p-4)(p+\gamma)}{3(m+1)(p+1)}(\Omega-1)} \cdot(1-b)}^{\substack{m+p-1}} \\
& +C_{8}\left\|n^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2(3 m+3 p-4)(p+\gamma)}{3(m+p-1)(p+\gamma-1)}}(\Omega)}^{\frac{2}{m+p-1}\left(2 C_{7}\right)^{\frac{3 m+3 p-4}{3(p+\gamma-1)}}} \\
& \leq \quad C_{9}\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^{2}(\Omega)}^{\frac{2(3 m+3 p-4)(p+\gamma)}{3(2+1)(p+\gamma-1)} \cdot b}+C_{9} \\
& =C_{9}\left\|\nabla n^{\frac{m+p-1}{2}}\right\|_{L^{2}(\Omega)}^{2}+C_{9} \quad \text { for all } t \in\left(0, T_{\max }\right) \text {. }
\end{aligned}
$$

As thus

$$
\int_{\Omega}\left|\nabla n^{\frac{m+p-1}{2}}\right|^{2} \geq \frac{1}{C_{9}} \cdot\left\{\int_{\Omega} \psi_{p}(n)\right\}^{\frac{3 m+3 p-4}{3(p+\gamma-1)}}-1 \quad \text { for all } t \in\left(0, T_{\max }\right)
$$

from (3.6) and (3.10) we accordingly obtain that for any fixed $t_{0} \in\left(0, T_{\max }\right)$, again since $M_{p} \geq 1$, and since $M_{p}$ is nondecreasing,

$$
\frac{d}{d t} \int_{\Omega} \psi_{p}(n)+C_{10} \cdot\left\{\int_{\Omega} \psi_{p}(n)\right\}^{\frac{3 m+3 p-4}{3(p+\gamma-1)}} \leq c_{11} M_{p}^{\frac{p}{p-1} \cdot\left(\frac{2}{3}+\eta\right) \cdot \frac{3 m+3 p-4}{4-3 \alpha}}\left(t_{0}\right) \quad \text { for all } t \in\left(0, t_{0}\right)
$$

with $C_{10}:=\frac{1}{(m+p-1)^{2} C_{9}}$ and $C_{11}:=C_{6}+\frac{1}{(m+p-1)^{2}}$. A simple ODE comparison argument reveals that therefore

$$
\int_{\Omega} \psi_{p}(n(\cdot, t)) \leq \max \left\{\int_{\Omega} \psi_{p}\left(n_{0}\right),\left\{\frac{C_{11}}{C_{10}} M_{p}^{\frac{p}{p-1} \cdot\left(\frac{2}{3}+\eta\right) \cdot \frac{3 m+3 p-4}{4-3 \alpha}}\left(t_{0}\right)\right\}^{\frac{3(p+\gamma-1)}{3 m+3 p-4}}\right\} \quad \text { for all } t \in\left(0, t_{0}\right)
$$

and that consequently, according to our definition of $\kappa$ in (3.4) we have

$$
\int_{\Omega} \psi_{p}(n(\cdot, t)) \leq C_{12} M_{p}^{p \cdot \frac{3(p+\gamma-1)}{(p-1)(4-3 \alpha)} \cdot\left(\frac{2}{3}+\eta\right)}\left(t_{0}\right)=C_{12} M_{p}^{p \kappa}\left(t_{0}\right) \quad \text { for all } t \in\left(0, t_{0}\right)
$$

with $C_{12}:=\max \left\{\int_{\Omega} \psi_{p}\left(n_{0}\right),\left(\frac{C_{11}}{C_{10}}\right)^{\frac{3(p+\gamma-1)}{3 m+3 p-4}}\right\}$. Since, on the other hand, Lemma 3.1 furthermore provides $C_{13}>0$ such that $\int_{\Omega} n^{p} \leq C_{13} \int_{\Omega} \psi_{p}(n)+C_{13}$ for all $t \in\left(0, T_{\max }\right)$, this implies that

$$
\int_{\Omega} n^{p}(\cdot, t) \leq C_{12} C_{13} M_{p}^{p \kappa}\left(t_{0}\right)+C_{13} \leq\left(C_{12}+1\right) C_{13} M_{p}^{p \kappa}\left(t_{0}\right) \quad \text { for all } t_{0} \in\left(0, T_{\max }\right) \text { and } t \in\left(0, t_{0}\right)
$$

and that according to the definition of $M_{p}$ we therefore have

$$
M_{p}\left(t_{0}\right) \leq 1+\left\{\left(C_{12}+1\right) C_{13}\right\}^{\frac{1}{p}} M_{p}^{\kappa}\left(t_{0}\right) \leq C_{14} M_{p}^{\kappa}\left(t_{0}\right) \quad \text { for all } t_{0} \in\left(0, T_{\max }\right)
$$

where $C_{14}:=1+\left\{\left(C_{12}+1\right) C_{13}\right\}^{\frac{1}{p}}$. We now only need to use that $\kappa<1$, as asserted by (3.4), to infer that $M_{p}\left(t_{0}\right) \leq C_{14}^{\frac{1}{1-\kappa}}$ for all $t_{0} \in\left(0, T_{\max }\right)$, and to thereupon conclude as intended.
With this lemma at hand, we can readily employ standard arguments to derive our main results:
Proof of Theorem 1.1. When combined with (1.3) and the upper estimate for $D$ in (1.2), in view of Lemma 2.2 an application of Lemma 3.2 readily shows that both $n$ and $h:=S(n) \nabla c+n u$ belong to $L^{\infty}\left(\left(0, T_{\max }\right) ; L^{p}(\Omega)\right)$ for any $p>1$. A Moser-type iterative argument on the basis of the identity $n_{t}=\nabla \cdot(D(n) \nabla n)-\nabla \cdot h$ ([10, Lemma A.1] therefore shows that actually $n \in L^{\infty}\left(\Omega \times\left(0, T_{\max }\right)\right)$, so that, again due to Lemma 2.2, the claim results from Lemma 2.1.

Acknowledgement. The author acknowledges support of the Deutsche Forschungsgemeinschaft in the context of the project Emergence of structures and advantages in cross-diffusion systems (No. 411007140, GZ: WI 3707/5-1).

## References

[1] CAO, X.: Fluid interaction does not affect the critical exponent in a three-dimensional Keller-Segel-Stokes model. Z. Angew. Math. Phys. 71, 61 (2020)
[2] Cieślak, T., Stinner, C.: Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions. J. Differ. Eq. 252 (10), 5832-5851 (2012)
[3] Cieślak, T., Stinner, C.: New critical exponents in a fully parabolic quasilinear Keller-Segel system and applications to volume filling models. J. Differ. Eq. 258 (6), 2080-2113 (2015)
[4] Hillen, T., Painter, K.J.: A user's guide to PDE models for chemotaxis. J. Math. Biol. 58, 183-217 (2009)
[5] Lankeit, J.: Locally bounded global solutions to a chemotaxis consumption model with singular sensitivity and nonlinear diffusion. J. Differential Eq. 262, 4052-4084 (2017)
[6] Painter, K.J., Hillen, T.: Volume-filling and quorum-sensing in models for chemosensitive movement. Can. Appl. Math. Q. 10, 501-543 (2002)
[7] Ishida, S., Seki, K., Yokota, T.: Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on nonconvex bounded domains. J. Differential Eq. 256, 2993-3010 (2014)
[8] Senba, T., Suzuki, T.: A quasi-linear system of chemotaxis. Abstr. Appl. Anal. 2006, 1-21 (2006)
[9] Sohr, H.: The Navier-Stokes Equations. An Elementary Functional Analytic Approach. Birkhäuser, Basel, 2001
[10] TaO, Y., Winkler, M.: Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity. J. Differential Eq. 252 (1), 692-715 (2012)
[11] Tuval, I., Cisneros, L., Dombrowski, C., Wolqemuth, C.W., Kessler, J.O., Goldstein, R.E.: Bacterial swimming and oxygen transport near contact lines. Proc. Nat. Acad. Sci. USA 102, 2277-2282 (2005)
[12] Wang, Y., Xiang, Z.: Global existence and boundedness in a Keller-Segel-Stokes system involving a tensor-valued sensitivity with saturation: The 3D case. J. Differential Eq. 261, 4944-4973 (2016)
[13] Winkler, M.: Does a 'volume-filling effect' always prevent chemotactic collapse? Math. Meth. Appl. Sci. 33, 12-24 (2010)
[14] Winkler, M.: Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops. Comm. Partial Differ. Eq. 37, 319-351 (2012)
[15] Winkler, M.: Does Fluid Interaction Affect Regularity in the Three-Dimensional Keller-Segel System with Saturated Sensitivity? J. Math. Fluid Mech. 20, 1889-1909 (2018)
[16] Winkler, M: Global classical solvability and generic infinite-time blow-up in quasilinear Keller-Segel systems with bounded sensitivities. J. Differential Eq. 266, 8034-8066 (2019)
[17] Winkler, M.: Conditional estimates in three-dimensional chemotaxis-Stokes systems and application to a Keller-Segelfluid model accounting for gradient-dependent flux limitation. arXiv:2009.07074


[^0]:    *michael.winkler@math.uni-paderborn.de

