Small-signal solutions of a two-dimensional doubly degenerate taxis system modeling bacterial motion in nutrient-poor environments

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Abstract

The doubly degenerate nutrient taxis model

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \nabla \cdot (u^2 v \nabla v) + \ell uv, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases}$$

is considered in smoothly bounded convex subdomains of the plane, with $\ell \geq 0$. It is shown that for any $p > 2$ and each fixed nonnegative $u_0 \in W^{1,\infty}(\Omega)$, a smallness condition exclusively involving v_0 can be identified as sufficient to ensure that an associated no-flux type initial-boundary value problem with $(u,v)|_{t=0} = (u_0, v_0)$ admits a global weak solution satisfying $\operatorname{ess\,sup}_{t>0} \|u(\cdot,t)\|_{L^p(\Omega)} < \infty$. The proof relies on the use of an apparently novel class of functional inequalities which provide estimates from below for certain Dirichlet integrals involving possibly degenerate weight functions.

Key words: chemotaxis; degenerate diffusion; a priori estimate; functional inequality
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1 Introduction

This manuscript is concerned with the parabolic system

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \chi\nabla \cdot (u^2 v\nabla v) + \ell uv, \\ v_t = \Delta v - uv, \end{cases} \tag{1.1}$$

which has been proposed as a refined model for the collective behavior in populations of *E. coli* when exposed to nutrient-poor agars. Here the pure reaction-diffusion system, as obtained on letting $\chi = 0$, was already introduced in [10] to describe experimental results revealing a strikingly prevalent tendency toward stabilization and formation of strongly structured distributions in such ensembles ([7], [6], [18]). As theoretically argued more recently in [16] and [19], and further endorsed by numerical experiments in [16], the accuracy of this description can significantly be enhanced by additionally accounting for nutrient-directed chemotactic behavior, that is, by considering (1.1) with $\chi > 0$. The hypotheses underlying this full model hence include an absorption-diffusion type evolution of a nutrient, as quantified via its concentration $v = v(x,t)$, coupled to a combination of food-supported proliferation and of random diffusive and cross-diffusive movement in the considered population, represented through its density $u = u(x,t)$; here the functional form of the corresponding migration operators particularly accounts for certain limitations in motility especially at low nutrient levels ([19]).

From a mathematical perspective, considerable challenges going along with nontrivial choices of χ already appear at the stage of questions related to global solvability and boundedness: As impressively indicated by an extensively developed theory of Keller-Segel type chemotaxis systems, attractive taxis mechanisms such as present in (1.1) may well exert considerably destabilizing influences, as becoming manifest in a number of results on the occurrence of blow-up phenomena in various particular modeling contexts ([2], [13]); in the concrete framework of (1.1), especially the superlinear growth with respect to u in the cross-diffusion rate $u^2 v$ seems to come along with a noticeable potential to substantially support such tendencies. Indications for this can be found in a rich literature on corresponding effects that prescribed asymptotics in chemotactic sensitivity functions may have on the possibility of singularity formation: In several of its versions, namely, the simple relative of (1.1) given by

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), \\ v_t = \Delta v + f(u,v), \end{cases} \tag{1.2}$$

in which especially any signal dependence of migration is disregarded, is accessible to a fairly comprehensive analysis in this direction. In the case when $f(u,v) = -v + u$, for instance, the asymptotics of the fraction $\frac{S(u)}{D(u)}$ at large values of u, relative to the spatial dimension $n \geq 2$, is known to be decisive in this respect: If, besides further technical conditions, S and D satisfy $\frac{S(u)}{D(u)} \leq Cu^\alpha$ for all $u \geq 1$ and some $C > 0$ and $\alpha < \frac{2}{n}$, then associated Neumann type initial-boundary value problems admit global bounded solutions for initial data of arbitrary size ([8]), [22]), while if $\frac{S(u)}{D(u)} \geq Cu^\alpha$ for all $u \geq 1$, some $C > 0$ and some $\alpha > \frac{2}{n}$, then unbounded solutions in balls can always be found ([29], [3], [4], [31]). Even when obviously not directly applicable to (1.1) with its more complex interactions especially in the migration parts, this may be viewed as indicating that the choices $S(u) = u^2$ and $D(u) = u$ associated with the parameter functions in (1.1), hence leading to linear growth of the quotient $\frac{S(u)}{D(u)} = u$, might go along with a nontrivial potential of destabilization. Also an inclusion of the additional dissipative mechanism of signal consumption, related to the choice $f(u,v) = -uv$ equivalent to that in (1.1), is only partially known to yield significant explosion-inhibiting effects. Beyond a scattered collection of findings on global solvability within certain regimes of S and D, possibly in frameworks of weak and hence possibly unbounded solutions only ([21], [25], [26], [27], [5], [33], [23]), a comprehensive picture about chemotaxis-absorption interplay even in simple contexts such as that in (1.2) seems yet lacking; in particular, to date it even appears to be unknown whether the simplified chemotaxis-consumption relative of (1.2) involving migration rates that exhibit growth with respect to u as those in (1.1), that is, with $D(u) = u$ and $S(u) = u^2$, in any multi-dimensional setting.

The core challenge of signal-dependent diffusion degeneracy. Main results. In light of the above discussion, a key issue in the development of any theory for (1.1) seems to consist in the problem of adequately quantifying the action of the dissipative mechanisms therein, and especially of appropriately coping with the signal-dependent degeneracy of cell diffusion. In fact, while diffusion mechanisms that exclusively involve degeneracies depending on the population density u have successfully been dealt with in a fairly effective and straightforward manner in numerous precedent studies on Keller-Segel type problems (cf. e.g. [27], [9], [12]) the inclusion of the factor v as part of the cell diffusivity in (1.1) seems to bring about quite considerable analytical demands; accordingly, the literature on (1.1) up to now seems limited to one recent contribution that concentrates on its one-dimensional version in which favorable embeddings significantly alleviate the analysis ([32]).

As an exemplary manifestation of this challenge which seems particularly relevant, the time evolution of the functional $\int_\Omega u^p$ for $p > 1$, constituting an apparently natural object of reasonings concerned with fundamental regularity issues in taxis-type systems ([2]), can be seen to be essentially governed by the competition between a taxis-driven contribution on the one hand, and a dissipated quantity of the form $\int_\Omega u^{p-1} v|\nabla u|^2$ on the other (cf. Lemma 3.1). To set up a cornerstone for our approach toward a basic solution theory for (1.1), in a first crucial step we shall accordingly address the problem of suitably estimating unfavorably weighted expressions of the latter type from below. Specifically, a key role in our analysis will be played by the observation that in smoothly bounded two-dimensional domains Ω and for each $p \geq 1$, a functional inequality of the form

$$\int_\Omega \varphi^{p+1}\psi \leq C(p)\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + C(p)\cdot\left\{\int_\Omega \varphi^p\right\}\cdot\int_\Omega \frac{\varphi}{\psi}|\nabla\psi|^2 + C(p)\cdot\left\{\int_\Omega \varphi\right\}^p\cdot\int_\Omega \varphi\psi \qquad (1.3)$$

holds for any reasonably regular functions $\varphi \geq 0$ and $\psi > 0$. Relying on the essentially well-known fact that, at least at a formal level, the integrals $\int_\Omega \frac{u}{v}|\nabla v|^2$ and $\int_\Omega uv$ are dissipated during the evolution of $\int_\Omega \frac{|\nabla v|^2}{v}$ and of $\int_\Omega v$ along trajectories of (1.1) (see Lemma 3.5 and (2.5)), we can utilize (1.3) to identify a certain energy-like property of the coupled quantity

$$\int_\Omega u^p + \int_\Omega \frac{|\nabla v|^2}{v}$$

in such domains and for any fixed $p > 2$, provided that the corresponding initial data satisfy a certain smallness condition in their second component (Lemma 3.7 and Lemma 3.8). Appropriate further exploitation of this will thereupon enable us to construct, for any such initial data, some global weak solutions through a limit procedure in suitably regularized approximate problems (Lemma 5.1).

To make this more precise, let us henceforth concentrate on the case $\chi = 1$ for definiteness, and accordingly consider the full initial-boundary value problem given by

$$\begin{cases} u_t = \nabla \cdot (uv\nabla u) - \nabla \cdot (u^2 v \nabla v) + \ell uv, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ (uv\nabla u - u^2 v\nabla v)\cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

in a smoothly bounded planar domain Ω, with $\ell \geq 0$, and with given nonnegative initial data u_0 and v_0. In this setting, the following solution concept to be subsequently pursued then seems fairly natural:

Definition 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a dounded domain with smooth boundary, let $\ell \geq 0$, and let $u_0 \in L^1(\Omega)$ and $v_0 \in L^1(\Omega)$ be nonnegative. Then a pair (u,v) of nonnegative functions*

$$\begin{cases} u \in L^1_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \\ v \in L^\infty_{loc}(\overline{\Omega} \times [0,\infty)) \cap L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{1.5}$$

will be called a global weak solution of (1.4) *if*

$$u^2 v \in L^1_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \text{and} \qquad u^2 \nabla v \in L^1_{loc}(\overline{\Omega} \times [0,\infty); \mathbb{R}^2), \tag{1.6}$$

and if

$$\begin{aligned} -\int_0^\infty \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot,0) &= \frac{1}{2}\int_0^\infty \int_\Omega u^2 \nabla v \cdot \nabla\varphi + \frac{1}{2}\int_0^\infty \int_\Omega u^2 v \Delta\varphi \\ &\quad + \int_0^\infty \int_\Omega u^2 v \nabla v \cdot \nabla \varphi + \ell \int_0^\infty \int_\Omega uv\varphi \end{aligned} \tag{1.7}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$, as well as

$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega uv\varphi \tag{1.8}$$

for each $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$.

Our main result now reveals that in the presence of suitably small initial data v_0, such a global weak solution can indeed always be found.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, and let $\ell \geq 0$. Then for any choice of $p > 2$ and $K > 0$, one can find $\delta(p,K) > 0$ and $C(p,K) > 0$ with the property that whenever*

$$u_0 \in W^{1,\infty}(\Omega) \quad \text{and} \quad v_0 \in W^{1,\infty}(\Omega) \qquad \text{are such that } u_0 \geq 0 \text{ and } v_0 > 0 \text{ in } \overline{\Omega} \tag{1.9}$$

with

$$\|u_0\|_{L^p(\Omega)} \leq K \tag{1.10}$$

and

$$\|v_0\|_{L^\infty(\Omega)} \le \delta(p,K) \qquad \textit{and} \qquad \left\|\nabla\sqrt{v_0}\right\|_{L^\infty(\Omega)} \le \delta(p,K), \tag{1.11}$$

the problem (1.4) admits a global weak solution (u,v) in the sense of Definition 1.1. Moreover, $v>0$ a.e. in $\Omega\times(0,\infty)$, and we have

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C(p,K) \qquad \textit{for a.e.} \quad t>0$$

as well as

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \delta(p,K) \quad \textit{and} \quad \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le 1 \qquad \textit{for a.e.} \quad t>0.$$

We emphasize that since no condition on the size of the initial population density u_0 is required, the requirements in Theorem 1.2 seem well compatible with the underlying intention to use (1.1) in application contexts determined by small nutrient concentrations ([16]).

2 Local existence and basic properties in an approximate problem

In order to construct a solution to (1.4) through approximation by solutions to conveniently regularized problems, for $\varepsilon\in(0,1)$ let us consider

$$\begin{cases} u_{\varepsilon t} = \nabla\cdot(u_\varepsilon v_\varepsilon\nabla u_\varepsilon) - \nabla\cdot(u_\varepsilon^2 v_\varepsilon\nabla v_\varepsilon) + \ell u_\varepsilon v_\varepsilon, & x\in\Omega,\ t>0,\\ v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon v_\varepsilon, & x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0) = u_0(x)+\varepsilon, \quad v_\varepsilon(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{2.1}$$

which according to well-established approaches allow for the following statement on existence, extensibility and basic properties.

Lemma 2.1 *Suppose that $n\ge1$ and that $\Omega\subset\mathbb{R}^n$ is a bounded domain with smooth boundary, and assume that (1.9) holds. Then for each $\varepsilon\in(0,1)$ one can find $T_{max,\varepsilon}\in(0,\infty]$ and functions*

$$\begin{cases} u_\varepsilon \in C^0(\overline{\Omega}\times[0,T_{max,\varepsilon}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max,\varepsilon})) \qquad \textit{and}\\ v_\varepsilon \in \bigcap_{q\ge1} C^0([0,T_{max,\varepsilon});W^{1,q}(\Omega))\cap C^{2,1}(\overline{\Omega}\times(0,T_{max,\varepsilon})) \end{cases} \tag{2.2}$$

such that $u_\varepsilon>0$ and $v_\varepsilon>0$ in $\overline{\Omega}\times[0,\infty)$, that $(u_\varepsilon,v_\varepsilon)$ solves (2.1) in the classical sense in $\Omega\times(0,T_{max,\varepsilon})$, and that

$$\textit{if } T_{max,\varepsilon}<\infty, \qquad \textit{then} \qquad \limsup_{t\nearrow T_{max,\varepsilon}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.3}$$

This solution has the additional properties that

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t\in(0,T_{max,\varepsilon}), \tag{2.4}$$

and that

$$\int_0^{T_{max,\varepsilon}}\int_\Omega u_\varepsilon v_\varepsilon \le \int_\Omega v_0. \tag{2.5}$$

PROOF. Standard arguments from local existence theories of taxis-type parabolic problems involving nonlinear diffusion ([1], [12]) readily provide $T_{max,\varepsilon} \in (0,\infty]$ and positive functions u_ε and v_ε which are such that (2.2) holds, that $(u_\varepsilon, v_\varepsilon)$ forms a classical solution of (2.1) in $\Omega \times (0, T_{max,\varepsilon})$, and that

$$\text{if } T_{max,\varepsilon} < \infty, \qquad \text{then}$$
$$\limsup_{t \nearrow T_{max,\varepsilon}} \left\{ \|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \left\| \frac{1}{u_\varepsilon(\cdot,t)} \right\|_{L^\infty(\Omega)} + \left\| \frac{1}{v_\varepsilon(\cdot,t)} \right\|_{L^\infty(\Omega)} \right\} = \infty; \tag{2.6}$$

the monotonicity property in (2.4) then immediately follows from the maximum principle, while (2.5) directly results from an integration in the second equation from (2.1).

To see that we actually must have (2.3), let us assume that for some $\varepsilon \in (0,1)$ we had $T_{max,\varepsilon} < \infty$ but

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_1(\varepsilon) \qquad \text{for all } t \in (0, T_{max,\varepsilon}) \tag{2.7}$$

with some $c_1(\varepsilon) > 0$. Then a straightforward application of known semigroup estimates ([28]) to the second equation in (2.1) would yield $c_2(\varepsilon) > 0$ fulfilling

$$\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_2(\varepsilon) \qquad \text{for all } t \in (0, T_{max,\varepsilon}), \tag{2.8}$$

while standard parabolic Hölder regularity theory ([20]) would ensure the existence of $\theta_1 = \theta_1(\varepsilon) \in (0,1)$ such that

$$v_\varepsilon \in C^{\theta_1, \frac{\theta_1}{2}}(\overline{\Omega} \times [0, T_{max,\varepsilon}]). \tag{2.9}$$

Apart from that, (2.7) would imply that $v_{\varepsilon t} \ge \Delta v_\varepsilon - c_1(\varepsilon) v_\varepsilon$ in $\Omega \times (0, T_{max,\varepsilon})$, and that hence writing $c_3 := \inf_{x\in\Omega} v_0(x) > 0$ we would have $v_\varepsilon(x,t) \ge \underline{v}(x,t) := c_3 e^{-c_1(\varepsilon)t}$ for all $x \in \Omega$ and $t \in (0, T_{max,\varepsilon})$ by the comparison principle, because $\underline{v}(x,0) = c_3 \le v_0(x) = v(x,0)$ for all $x \in \Omega$, and because $\underline{v}_t - \Delta\underline{v} + c_1(\varepsilon)\underline{v} = 0$ in $\Omega \times (0, T_{max,\varepsilon})$. In consequence, this would particularly show that

$$v_\varepsilon \ge c_4(\varepsilon) := c_3 e^{-c_1(\varepsilon) T_{max,\varepsilon}} \qquad \text{in } \Omega \times (0, T_{max,\varepsilon}), \tag{2.10}$$

whence in the identity

$$u_{\varepsilon t} = \nabla \cdot a_\varepsilon(x,t,u_\varepsilon,\nabla u_\varepsilon) + b_\varepsilon(x,t,u_\varepsilon), \qquad x \in \Omega,\ t \in (0, T_{max,\varepsilon}),$$

due to (2.1) valid with

$$\begin{aligned} &a_\varepsilon(x,t,z,\xi) := v_\varepsilon(x,t)|z|\xi - u_\varepsilon^{\frac{3}{2}}(x,t) v_\varepsilon(x,t)|z|^{\frac12} \nabla v_\varepsilon(x,t) \qquad \text{and} \\ &b_\varepsilon(x,t) := \ell u_\varepsilon(x,t) v_\varepsilon(x,t), \qquad (x,t,z,\xi) \in \Omega \times (0, T_{max,\varepsilon}) \times \mathbb{R} \times \mathbb{R}^2, \end{aligned}$$

a combination of (2.10) with (2.7), (2.8) and Young's inequality would entail that with some $c_5(\varepsilon) > 0$ and $c_6(\varepsilon) > 0$, the key estimate

$$\begin{aligned} a_\varepsilon(x,t,z,\xi)\cdot\xi &= v_\varepsilon(x,t)|z|\cdot|\xi|^2 - u_\varepsilon^{\frac32}(x,t) v_\varepsilon(x,t)|z|^{\frac12} \nabla v_\varepsilon(x,t)\cdot\xi \\ &\ge c_4(\varepsilon)|z|\cdot|\xi|^2 - c_5(\varepsilon)|z|^{\frac12}|\xi| \\ &\ge \frac{c_4(\varepsilon)}{2}|z|\cdot|\xi|^2 - c_6(\varepsilon) \end{aligned}$$

would hold for all $(x,t,z,\xi) \in \Omega \times (0,T_{max,\varepsilon}) \times \mathbb{R} \times \mathbb{R}^2$. As (2.7) and (2.8) would furthermore show that there exist $c_7(\varepsilon) > 0$ and $c_8(\varepsilon) > 0$ satisfying

$$\begin{aligned} &\big|a_\varepsilon(x,t,z,\xi)\big| \le c_7(\varepsilon)|z|\cdot|\xi| + c_7(\varepsilon)|z|^{\frac{1}{2}} \qquad \text{and} \\ &\big|b_\varepsilon(x,t)\big| \le c_8(\varepsilon) \qquad \text{for all } (x,t,z,\xi) \in \Omega \times (0,T_{max,\varepsilon}) \times \mathbb{R} \times \mathbb{R}^2, \end{aligned}$$

again relying on the Hölder estimates provided by [20] we could find $\theta_2 = \theta_2(\varepsilon) \in (0,1)$ such that

$$u_\varepsilon \in C^{\theta_2,\frac{\theta_2}{2}}(\overline{\Omega} \times [0,T_{max,\varepsilon}]). \tag{2.11}$$

This in turn would enable us to apply standard parabolic Schauder theory ([11]) to see that with some $\theta_3 = \theta_3(\varepsilon) \in (0,1)$ we would have $v_\varepsilon \in C^{2+\theta_3,1+\frac{\theta_3}{2}}(\overline{\Omega} \times [\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon}])$, whence we could especially pick $c_9(\varepsilon) > 0$ such that

$$|\Delta v_\varepsilon| \le c_9(\varepsilon) \qquad \text{in } \Omega \times \Big(\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon}\Big). \tag{2.12}$$

In the first equation from (2.1), now rearranged so as to become

$$u_{\varepsilon t} = A_\varepsilon(x,t)\Delta u_\varepsilon + B_\varepsilon(x,t)\cdot\nabla u_\varepsilon + D_\varepsilon(x,t)u_\varepsilon, \qquad x \in \Omega,\ t \in (0,T_{max,\varepsilon}),$$

with

$$\begin{aligned} &A_\varepsilon(x,t) := u_\varepsilon(x,t)v_\varepsilon(x,t), \\ &B_\varepsilon(x,t) := v_\varepsilon(x,t)\nabla u_\varepsilon(x,t) + u_\varepsilon(x,t)\nabla v_\varepsilon(x,t) - 2u_\varepsilon(x,t)v_\varepsilon(x,t)\nabla v_\varepsilon(x,t) \qquad \text{and} \\ &D_\varepsilon(x,t) := -u_\varepsilon(x,t)v_\varepsilon(x,t)\Delta v_\varepsilon(x,t) - 2u_\varepsilon(x,t)|\nabla v_\varepsilon(x,t)|^2 + \ell v_\varepsilon(x,t), \qquad (x,t) \in \Omega \times (0,T_{max,\varepsilon}), \end{aligned}$$

we could then identify a positive constant $c_{10}(\varepsilon)$ such that

$$D_\varepsilon \ge -c_{10}(\varepsilon) \qquad \text{in } \Omega \times \Big(\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon}\Big),$$

so that abbreviating $c_{11}(\varepsilon) := \inf_{x\in\Omega} u_\varepsilon(x,\frac{1}{4}T_{max,\varepsilon}) > 0$, parabolic comparison of u_ε with $\underline{u}(x,t) := c_{11}(\varepsilon)e^{-c_{10}(\varepsilon)\cdot(t-\frac{1}{4}T_{max,\varepsilon})}$, $(x,t) \in \overline{\Omega} \times [\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon})$, would show that since

$$\underline{u}_t - A_\varepsilon\Delta\underline{u} - B_\varepsilon\cdot\nabla\underline{u} - D_\varepsilon\underline{u} = -c_{10}(\varepsilon)\underline{u} - D_\varepsilon\underline{u} \le 0 \qquad \text{in } \Omega \times \Big(\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon}\Big)$$

and $\underline{u}(\cdot,\frac{1}{4}T_{max,\varepsilon}) \le u_\varepsilon(\cdot,\frac{1}{4}T_{max,\varepsilon})$, we would have $u_\varepsilon \ge \underline{u}$ in $\Omega \times (\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon})$ and hence, in particular,

$$u_\varepsilon \ge c_{11}(\varepsilon)e^{-c_{10}(\varepsilon)\cdot\frac{3}{4}T_{max,\varepsilon}} \qquad \text{in } \Omega \times \Big(\frac{1}{4}T_{max,\varepsilon}, T_{max,\varepsilon}\Big). \tag{2.13}$$

Thereupon, first order parabolic Hölder regularity theory ([14]) would become applicable so as to yield $\theta_4 = \theta_4(\varepsilon) \in (0,1)$ such that $u_\varepsilon \in C^{1+\theta_4,\frac{1+\theta_4}{2}}(\overline{\Omega} \times [\frac{1}{2}T_{max,\varepsilon}, T_{max,\varepsilon}])$, inter alia meaning that with some $c_{12}(\varepsilon) > 0$ we would have

$$\|u_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_{12}(\varepsilon) \qquad \text{for all } t \in \Big(\frac{1}{2}T_{max,\varepsilon}, T_{max,\varepsilon}\Big).$$

Together with (2.8), (2.13) and (2.10), however, this would contradict (2.6), so that indeed (2.3) must hold. $\square$

In our subsequent reasoning, we consider the convex domain $\Omega \subset \mathbb{R}^2$ and the parameter $\ell \geq 0$ as fixed, and whenever functions $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ with $u_0 \geq 0$ and $v_0 > 0$ in $\overline{\Omega}$ have been selected, without any further explicit mentioning we shall let $(u_\varepsilon, v_\varepsilon)$ and $T_{max,\varepsilon}$ be as accordingly provided by Lemma 2.1.

As an elementary but crucial preparation for our loop-type argument in the next section, let us note the following consequence of the regularizing action of parabolicity on a certain preservation of smallness in the second solution components, presupposing the presence of an L^p bound for the first solution component. The restriction on p in Theorem 1.2 is precisely due to our argument in this direction:

Lemma 2.2 *For all $p > 2, L > 0$ and $\eta > 0$, there exists $\delta_0(p, L, \eta) > 0$ with the property that if (1.9) holds with*

$$\|v_0\|_{L^\infty(\Omega)} \leq \delta_0(p, L, \eta) \qquad \textit{and} \qquad \|\nabla v_0\|_{L^\infty(\Omega)} \leq \delta_0(p, L, \eta), \tag{2.14}$$

and if for some $\varepsilon \in (0, 1)$ and $T \in (0, T_{max,\varepsilon})$ we have

$$\int_\Omega u_\varepsilon^p(\cdot, t) \leq L \qquad \textit{for all } t \in (0, T), \tag{2.15}$$

then

$$\|\nabla v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \eta \qquad \textit{for all } t \in (0, T), \tag{2.16}$$

Proof. We recall known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ on Ω ([28]) to fix $c_1 > 0, \lambda > 0$ and $c_2(p) > 0$ such that

$$\|\nabla e^{t\Delta}\varphi\|_{L^\infty(\Omega)} \leq c_1 \|\nabla \varphi\|_{L^\infty(\Omega)} \qquad \text{for all } t > 0 \text{ and any } \varphi \in W^{1,\infty}(\Omega), \tag{2.17}$$

and that

$$\|\nabla e^{t\Delta}\varphi\|_{L^\infty(\Omega)} \leq c_2(p) \cdot \big(1 + t^{-\frac{1}{2}-\frac{1}{p}}\big) e^{-\lambda t} \|\varphi\|_{L^p(\Omega)} \qquad \text{for all } t > 0 \text{ and each } \varphi \in C^0(\overline{\Omega}). \tag{2.18}$$

Since our assumption $p > 2$ implies that $c_3(p) := \int_0^\infty (1 + \sigma^{-\frac{1}{2}-\frac{1}{p}}) e^{-\lambda\sigma} d\sigma$ is finite, given $L > 0$ and $\eta > 0$ we can thereafter choose $\delta_0(p, L, \eta) > 0$ small enough such that

$$\big(c_1 + c_2(p) c_3(p) L\big) \cdot \delta_0(p, L, \eta) \leq \eta, \tag{2.19}$$

and assuming (1.9), (2.14) and (2.15) to be satisfied for some $\varepsilon \in (0, 1)$ and $T \in (0, T_{max,\varepsilon})$, thanks to the second equation in (2.1) and (2.4) we then obtain from (2.17)-(2.19) that, indeed,

$$\begin{aligned}
\|\nabla v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} &= \bigg\|\nabla e^{t\Delta} v_0 - \int_0^t \nabla e^{(t-s)\Delta}\big\{u_\varepsilon(\cdot, s) v_\varepsilon(\cdot, s)\big\} ds\bigg\|_{L^\infty(\Omega)} \\
&\leq c_1 \|\nabla v_0\|_{L^\infty(\Omega)} + c_2(p) \int_0^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{1}{p}}\Big) e^{-\lambda(t-s)} \big\|u_\varepsilon(\cdot, s) v_\varepsilon(\cdot, s)\big\|_{L^p(\Omega)} ds \\
&\leq c_1 \|\nabla v_0\|_{L^\infty(\Omega)} + c_2(p) \|v_0\|_{L^\infty(\Omega)} \int_0^t \Big(1 + (t-s)^{-\frac{1}{2}-\frac{1}{p}}\Big) e^{-\lambda(t-s)} \|u_\varepsilon(\cdot, s)\|_{L^p(\Omega)} ds \\
&\leq c_1 \delta_0(p, L, \eta) + c_2(p) \delta_0(p, L, \eta) c_3(p) L \\
&\leq \eta
\end{aligned}$$

for all $t \in (0,T)$. $\square$

3 A self-map type argument controlling u_ε in L^p for small v_0

We next enter the key stage of our argument by carefully documenting the outcome of a standard testing procedure associated with the first sub-problem of (2.1).

Lemma 3.1 *Let $p>2, L>0$ and $\eta>0$, and with $\delta_0(p,L,\eta)$ taken from Lemma 2.2, suppose that (1.9) as well as (2.14) and (2.15) hold for some $\varepsilon\in(0,1)$ and $T\in(0,T_{max,\varepsilon})$. Then for any choice of $a>0$,*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + a\int_\Omega u_\varepsilon^2 v_\varepsilon \\
&\quad\le p^2\eta^2\int_\Omega u_\varepsilon^{p+1}v_\varepsilon + \left\{p\ell^p\cdot\Big(\frac{2}{\eta^2}\Big)^{p-1} + a^{\frac{p}{p-1}}\cdot\Big(\frac{2}{p^2\eta^2}\Big)^{\frac{1}{p-1}}\right\}\int_\Omega u_\varepsilon v_\varepsilon \qquad \textit{for all } t\in(0,T). \qquad (3.1)
\end{aligned}$$

Proof. We use the first equation in (2.1) and integrate by parts to see that due to Young's inequality and (2.16),

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + p(p-1)\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \\
&\quad= p(p-1)\int_\Omega u_\varepsilon^p v_\varepsilon\nabla u_\varepsilon\cdot\nabla v_\varepsilon + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon \\
&\quad\le \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon|\nabla v_\varepsilon|^2 + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon \\
&\quad\le \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{p(p-1)\eta^2}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon + p\ell\int_\Omega u_\varepsilon^p v_\varepsilon \qquad \text{for all } t\in(0,T).
\end{aligned}$$

Since two further applications of Young's inequality show that

$$\begin{aligned}
p\ell\int_\Omega u_\varepsilon^p v_\varepsilon &= \int_\Omega\Big(\frac{p\eta^2}{2}u_\varepsilon^{p+1}v_\varepsilon\Big)^{\frac{p-1}{p}}\cdot p\ell\Big(\frac{2}{p\eta^2}\Big)^{\frac{p-1}{p}}u_\varepsilon^{\frac1p}v_\varepsilon^{\frac1p} \\
&\le \frac{p\eta^2}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon + \left\{p\ell\cdot\Big(\frac{2}{p\eta^2}\Big)^{\frac{p-1}{p}}\right\}^p\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon})
\end{aligned}$$

and that

$$\begin{aligned}
a\int_\Omega u_\varepsilon^2 v_\varepsilon &= \int_\Omega\Big(\frac{p^2\eta^2}{2}u_\varepsilon^{p+1}v_\varepsilon\Big)^{\frac1p}\cdot a\Big(\frac{2}{p^2\eta^2}\Big)^{\frac1p}u_\varepsilon^{\frac{p-1}{p}}v_\varepsilon^{\frac{p-1}{p}} \\
&\le \frac{p^2\eta^2}{2}\int_\Omega u_\varepsilon^{p+1}v_\varepsilon + \left\{a\cdot\Big(\frac{2}{p^2\eta^2}\Big)^{\frac1p}\right\}^{\frac{p}{p-1}}\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}),
\end{aligned}$$

this implies (3.1), because $\left\{p\ell\cdot\big(\frac{2}{p\eta^2}\big)^{\frac{p-1}{p}}\right\}^p = p\ell^p\cdot\big(\frac{2}{\eta^2}\big)^{p-1}$. $\square$

Now a core task will consist in appropriately estimating the first summand on the right-hand side of (3.1) against dissipated quantities, where a fundamental obstacle is linked to the circumstance that the diffusion-related contribution to (3.1) contains the expectedly small weight v_ε. In order to prepare our approach to adequately overcome this, let us separately derive the following functional inequality which does rely on our overall restriction on planarity of the spatial setting, but which actually does not require Ω to be convex.

Lemma 3.2 *For each $p \ge 1$, one can find $C(p) > 0$ such that if $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$ are such that $\varphi \ge 0$ and $\psi > 0$ in $\overline{\Omega}$, then*

$$\int_\Omega \varphi^{p+1}\psi \le C(p)\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + C(p)\cdot\Big\{\int_\Omega \varphi^p\Big\}\cdot\int_\Omega \frac{\varphi}{\psi}|\nabla\psi|^2 + C(p)\cdot\Big\{\int_\Omega\varphi\Big\}^p\cdot\int_\Omega\varphi\psi. \quad (3.2)$$

PROOF. According to the Sobolev inequality in the two-dimensional domain Ω, given $p \ge 1$ we can find $c_1(p) > 0$ such that

$$\int_\Omega \rho^2 \le c_1(p)\|\nabla\rho\|^2_{L^1(\Omega)} + c_1(p)\|\rho\|^2_{L^{\frac{2}{p+1}}(\Omega)} \qquad \text{for all } \rho \in C^1(\overline{\Omega}),$$

which for fixed nonnegative $\varphi \in C^1(\overline{\Omega})$ and positive $\psi \in C^1(\overline{\Omega})$ we apply to $\rho := \varphi^{\frac{p+1}{2}}\psi^{\frac12}$ to infer that

$$\begin{aligned}\int_\Omega \varphi^{p+1}\psi &\le c_1(p)\cdot\Big\{\int_\Omega\Big|\frac{p+1}{2}\varphi^{\frac{p-1}{2}}\psi^{\frac12}\nabla\varphi + \frac12\varphi^{\frac{p+1}{2}}\psi^{-\frac12}\nabla\psi\Big|\Big\}^2 \\ &\quad + c_1(p)\cdot\Big\{\int_\Omega\varphi\psi^{\frac{1}{p+1}}\Big\}^{p+1} \\ &\le \frac{(p+1)^2c_1(p)}{2}\cdot\Big\{\int_\Omega\varphi^{\frac{p-1}{2}}\psi^{\frac12}|\nabla\varphi|\Big\}^2 + \frac{c_1(p)}{2}\cdot\Big\{\int_\Omega\varphi^{\frac{p+1}{2}}\psi^{-\frac12}|\nabla\psi|\Big\}^2 \\ &\quad + c_1(p)\cdot\Big\{\int_\Omega\varphi\psi^{\frac{1}{p+1}}\Big\}^{p+1}. \end{aligned} \quad (3.3)$$

Here by the Cauchy-Schwarz inequality,

$$\Big\{\int_\Omega\varphi^{\frac{p-1}{2}}\psi^{\frac12}|\nabla\varphi|\Big\}^2 \le |\Omega|\int_\Omega\varphi^{p-1}\psi|\nabla\varphi|^2$$

and

$$\Big\{\int_\Omega\varphi^{\frac{p+1}{2}}\psi^{-\frac12}|\nabla\psi|\Big\}^2 \le \Big\{\int_\Omega\varphi^p\Big\}\cdot\int_\Omega\frac{\varphi}{\psi}|\nabla\psi|^2,$$

while using the Hölder inequality we see that

$$\Big\{\int_\Omega\varphi\psi^{\frac{1}{p+1}}\Big\}^{p+1} \le \Big\{\int_\Omega\varphi\Big\}^p\cdot\int_\Omega\varphi\psi.$$

The claim therefore results from (3.3) if we let $C(p) := \max\big\{\frac{(p+1)^2c_1(p)|\Omega|}{2}, c_1(p)\big\}$. □

Indeed, employing the latter with $(\varphi,\psi) = (u_\varepsilon, v_\varepsilon)$ yields the following as a consequence of Lemma 3.1 in which, for definiteness, we already specify a selection of the parameter a that will turn out to be convenient later on.

Lemma 3.3 *Let $p>2$. Then there exist $\eta_0(p)>0$ and $C(p)>0$ with the property that if $L>0$, and if u_0 and v_0 are such that (1.9) and (2.14) as well as (2.15) hold for some $\eta\in(0,\eta_0(p)]$, $\varepsilon\in(0,1)$ and $T\in(0,T_{max,\varepsilon})$, with $\delta_0(p,L,\eta)>0$ as given by Lemma 2.2, then*

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + 2(7+4\sqrt{2})\int_\Omega u_\varepsilon^2 v_\varepsilon + \eta^2\int_\Omega u_\varepsilon^{p+1}v_\varepsilon \\
&\qquad \le\ C(p)\cdot L\eta^2\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + C(p)\cdot\frac{1+L}{\eta^{2(p-1)}}\int_\Omega u_\varepsilon v_\varepsilon \qquad \textit{for all } t\in(0,T).
\end{aligned} \tag{3.4}$$

PROOF. We first invoke Lemma 3.2 to fix $c_1(p)>0$ in such a way that whenever $0\le\varphi\in C^1(\overline{\Omega})$ and $0<\psi\in C^1(\overline{\Omega})$,

$$\int_\Omega \varphi^{p+1}\psi \le c_1(p)\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + c_1(p)\cdot\Big\{\int_\Omega\varphi^p\Big\}\cdot\int_\Omega\frac{\varphi}{\psi}|\nabla\psi|^2 + c_1(p)\cdot\Big\{\int_\Omega\varphi\Big\}^p\cdot\int_\Omega\varphi\psi, \tag{3.5}$$

and we then pick $\eta_0(p)>0$ small enough such that

$$\eta_0(p)\le\min\Big\{1\,,\,\sqrt{\frac{p(p-1)}{4(p^2+1)c_1(p)}}\Big\}. \tag{3.6}$$

Then assuming that $0\le u_0\in W^{1,\infty}(\Omega)$, and that $v_0\in W^{1,\infty}(\Omega)$, $\varepsilon\in(0,1)$ and $T\in(0,T_{max,\varepsilon})$ are such that (2.14) and (2.15) hold for some $L>0$ and $\eta\in(0,\eta_0(p)]$, we note that since $\eta\le1$ we have $\eta^{-\frac{2}{p-1}}\le\eta^{-2(p-1)}$ and hence

$$p\ell^p\cdot\Big(\frac{2}{\eta^2}\Big)^{p-1} + \big\{2(7+4\sqrt{2})\big\}^p\cdot\Big(\frac{2}{p^2\eta^2}\Big)^{\frac{1}{p-1}} \le c_2(p)\eta^{2(p-1)}$$

with $c_2(p):=2^{p-1}p\ell^p+\big\{2(7+4\sqrt{2})\big\}^p\cdot\big(\frac{2}{p^2}\big)^{\frac{1}{p-1}}$. Therefore, Lemma 3.1 together with (3.5) implies that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + 2(7+4\sqrt{2})\int_\Omega u_\varepsilon^2 v_\varepsilon + \eta^2\int_\Omega u_\varepsilon^{p+1}v_\varepsilon \\
&\quad\le\ (p^2+1)\eta^2\int_\Omega u_\varepsilon^{p+1}v_\varepsilon + c_2(p)\eta^{-2(p-1)}\int_\Omega u_\varepsilon v_\varepsilon \\
&\quad\le\ (p^2+1)c_1(p)\eta^2\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + (p^2+1)c_1(p)\eta^2\cdot\Big\{\int_\Omega u_\varepsilon^p\Big\}\cdot\int_\Omega\frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\
&\qquad +(p^2+1)c_1(p)\eta^2\cdot\Big\{\int_\Omega u_\varepsilon\Big\}^p\cdot\int_\Omega u_\varepsilon v_\varepsilon + c_2(p)\eta^{-2(p-1)}\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t\in(0,T),
\end{aligned} \tag{3.7}$$

where according to (3.6),

$$(p^2+1)c_1(p)\eta^2\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \le \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \qquad \text{for all } t\in(0,T),$$

and where thanks to (2.15) we know that

$$(p^2+1)c_1(p)\eta^2\cdot\Big\{\int_\Omega u_\varepsilon^p\Big\}\cdot\int_\Omega\frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \le (p^2+1)c_1(p)L\eta^2\int_\Omega\frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \qquad \text{for all } t\in(0,T).$$

As due to the Hölder inequality, (2.15) moreover entails that again since $\eta \le 1$ we can estimate

$$
\begin{aligned}
(p^2+1)c_1(p)\eta^2 \cdot \left\{ \int_\Omega u_\varepsilon \right\}^p \cdot \int_\Omega u_\varepsilon v_\varepsilon &\le (p^2+1)c_1(p)\eta^2|\Omega|^{p-1} \cdot \left\{ \int_\Omega u_\varepsilon^p \right\} \cdot \int_\Omega u_\varepsilon v_\varepsilon \\
&\le (p^2+1)c_1(p)|\Omega|^{p-1}L\eta^2 \int_\Omega u_\varepsilon v_\varepsilon \\
&\le (p^2+1)c_1(p)|\Omega|^{p-1}L\eta^{-2(p-1)} \int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t \in (0,T),
\end{aligned}
$$

from (3.7) we infer that, in fact, (3.4) holds with $C(p) := \max\big\{(p^2+1)c_1(p)\,,\, c_2(p)\,,\, (p^2+1)c_1(p)|\Omega|^{p-1}\big\}$.
$\square$

While the second summand on the right of (3.4) allows for a suitable control via the global disspiation property in (2.5), the corresponding first integral seems to require a separate treatment especially due to the singular factor $\frac{1}{v_\varepsilon}$ appearing therein. This will be faced in the course of a second, again fairly well-established, testing-based argument which we prepare by expatiating an elementary functional inequality which in its principal form and with appropriately adapted constants, as we may note without explicit proof, actually continues to hold in any smoothly bounded n-dimensional domain with $n \ge 1$.

Lemma 3.4 *Let $\varphi \in C^2(\overline{\Omega})$ be such that $\varphi > 0$ in $\overline{\Omega}$. Then*

$$
\int_\Omega \varphi |D^2 \ln \varphi|^2 \ge \frac{1}{2(7+4\sqrt{2})} \int_\Omega \frac{|D^2\varphi|^2}{\varphi}. \tag{3.8}
$$

Proof. According to [30, Lemma 3.3], we have

$$
\int_\Omega \frac{|\nabla\varphi|^4}{\varphi^3} \le (2+\sqrt{2})^2 \int_\Omega \varphi |D^2 \ln\varphi|^2,
$$

which we combine with the observation that due to Young's inequality,

$$
\begin{aligned}
\int_\Omega \varphi|D^2\ln\varphi|^2 &= \int_\Omega \frac{|D^2\varphi|^2}{\varphi} - 2\int_\Omega \frac{1}{\varphi^2}\nabla\varphi\cdot(D^2\varphi\cdot\nabla\varphi) + \int_\Omega \frac{|\nabla\varphi|^4}{\varphi^3} \\
&\ge \frac{1}{2}\int_\Omega \frac{|D^2\varphi|^2}{\varphi} - \int_\Omega \frac{|\nabla\varphi|^4}{\varphi^3}.
\end{aligned}
$$

Therefore, namely, writing $\theta := \frac{1}{1+(2+\sqrt{2})^2} \equiv \frac{1}{7+4\sqrt{2}}$ we see that, indeed,

$$
\begin{aligned}
\int_\Omega \varphi|D^2\ln\varphi|^2 &\ge (1-\theta)\cdot\frac{1}{(2+\sqrt{2})^2}\int_\Omega \frac{|\nabla\varphi|^4}{\varphi^3} + \theta\cdot\left\{\frac{1}{2}\int_\Omega\frac{|D^2\varphi|^2}{\varphi} - \int_\Omega\frac{|\nabla\varphi|^4}{\varphi^3}\right\} \\
&= \frac{\theta}{2}\int_\Omega \frac{|D^2\varphi|^2}{\varphi},
\end{aligned}
$$

because $(1-\theta)\cdot\frac{1}{(2+\sqrt{2})^2} - \theta = \frac{1}{(2\sqrt{2})^2} - \left(\frac{1}{(2+\sqrt{2})^2}+1\right)\cdot\frac{1}{1+(2+\sqrt{2})^2} = 0.$ $\square$

We can now proceed to identify the integral $\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2$ as part of the dissipation-induced contribution to the outcome of a testing procedure applied to the second equation from (2.1). Our derivation of this is the only place in this manuscript where convexity of Ω is explicitly needed, and we remark without detailing a corresponding argument here that at the cost of moderately extended technical efforts, this assumption could actually be removed.

Lemma 3.5 *Assume (1.9), and let $\varepsilon \in (0,1)$. Then*

$$\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \le 2(7+4\sqrt{2})\int_\Omega u_\varepsilon^2 v_\varepsilon \qquad \textit{for all } t\in(0,T_{max,\varepsilon}). \tag{3.9}$$

Proof. Using integration by parts in a straightforward manner (cf. [30, Lemma 3.2] for details), from the second equation in (2.1) we obtain the identity

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + 2\int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 \\ &\qquad = -2\int_\Omega \nabla u_\varepsilon\cdot\nabla v_\varepsilon + \int_{\partial\Omega}\frac{1}{v_\varepsilon}\cdot\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \qquad \text{for all } t\in(0,T_{max,\varepsilon}), \end{aligned} \tag{3.10}$$

where as a consequence of another integration by parts and Young's inequality,

$$\begin{aligned} -2\int_\Omega \nabla u_\varepsilon\cdot\nabla v_\varepsilon &= 2\int_\Omega u_\varepsilon \Delta v_\varepsilon \\ &\le \frac{1}{2(7+4\sqrt{2})}\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} + 2(7+4\sqrt{2})\int_\Omega u_\varepsilon^2 v_\varepsilon \qquad \text{for all } t\in(0,T_{max,\varepsilon}). \end{aligned}$$

Since $|\Delta v_\varepsilon|^2 \le 2|D^2 v_\varepsilon|^2$ by the Cauchy-Schwarz inequality, and since thus

$$\frac{1}{2(7+4\sqrt{2})}\int_\Omega \frac{|\Delta v_\varepsilon|^2}{v_\varepsilon} \le \frac{1}{7+4\sqrt{2}}\int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} \le 2\int_\Omega v_\varepsilon |D^2\ln v_\varepsilon|^2 \qquad \text{for all } t\in(0,T_{max,\varepsilon})$$

thanks to Lemma 3.4, from (3.10) we thus infer (3.9) upon noting that $\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \le 0$ on $\partial\Omega\times(0,T_{max,\varepsilon})$ by convexity of Ω ([17]). $\square$

As, fortunately, the expression on the right-hand side of (3.9) can be absorbed by the third summand on the left of (3.4), we are now in the position to close the loop in our argument by combining Lemma 3.3 with Lemma 3.5. In order to be able to thereby draw a conclusion that simultaneosly includes some bound for the gradient of a suitably chosen quantity, and that hence serves as a preparation for an Aubin-Lions type compactness reasoning in Lemma 5.1 below, let us briefly add the following basic observation.

Lemma 3.6 *Let $p\ge 1$, and let $\varphi\in C^1(\overline{\Omega})$ and $\psi\in C^1(\overline{\Omega})$ be such that $\varphi\ge 0$ and $\psi>0$ in $\overline{\Omega}$. Then*

$$\bigg\{\int_\Omega \Big|\nabla\big(\varphi^{\frac{p+1}{2}}\psi\big)\Big|\bigg\}^2 \le \frac{(p+1)^2}{2}\cdot\bigg\{\int_\Omega\psi\bigg\}\cdot\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2 + 2\|\psi\|_{L^\infty(\Omega)}\cdot\bigg\{\int_\Omega\varphi^p\bigg\}\cdot\int_\Omega\frac{\varphi}{\psi}|\nabla\psi|^2. \tag{3.11}$$

PROOF. We expand

$$\nabla\big(\varphi^{\frac{p+1}{2}}\psi\big) = \frac{p+1}{2}\varphi^{\frac{p-1}{2}}\psi\nabla\varphi + \varphi^{\frac{p+1}{2}}\nabla\psi$$

and use Young's inequality to see that thus

$$\bigg\{\int_\Omega \Big|\nabla\big(\varphi^{\frac{p+1}{2}}\psi\big)\Big|\bigg\}^2 \le 2\cdot\frac{(p+1)^2}{4}\cdot\bigg\{\int_\Omega \varphi^{\frac{p-1}{2}}\psi|\nabla\varphi|\bigg\}^2 + 2\cdot\bigg\{\int_\Omega \varphi^{\frac{p+1}{2}}|\nabla\psi|\bigg\}^2.$$

Since

$$\bigg\{\int_\Omega \varphi^{\frac{p-1}{2}}\psi|\nabla\varphi|\bigg\}^2 \le \bigg\{\int_\Omega \psi\bigg\}\cdot\int_\Omega \varphi^{p-1}\psi|\nabla\varphi|^2$$

and

$$\bigg\{\int_\Omega \varphi^{\frac{p+1}{2}}|\nabla\psi|\bigg\}^2 \le \|\psi\|_{L^\infty(\Omega)}\cdot\bigg\{\int_\Omega \varphi^p\bigg\}\cdot\int_\Omega \frac{\varphi}{\psi}|\nabla\psi|^2$$

by the Cauchy-Schwarz inequality, this already establishes (3.11). $\square$

We can now conveniently accomplish the main step of the analysis in this section:

Lemma 3.7 *Let $p>2$ and $L>0$. Then there exist $\delta_1(p,L)>0$ and $C(p,L)>0$ such that if (1.9) holds with*

$$\|v_0\|_{L^\infty(\Omega)} \le \delta_1(p,L) \qquad \textit{and} \qquad \|\nabla\sqrt{v_0}\|_{L^\infty(\Omega)} \le \delta_1(p,L), \tag{3.12}$$

and if (2.15) is fulfilled for some $\varepsilon\in(0,1)$ and $T\in(0,T_{max,\varepsilon})$, then

$$\int_\Omega u_\varepsilon^p(\cdot,t) + \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} \le \int_\Omega (u_0+1)^p + \frac{1}{2} \qquad \textit{for all } t\in(0,T) \tag{3.13}$$

and

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le 1 \qquad \textit{for all } t\in(0,T) \tag{3.14}$$

as well as

$$\int_0^t\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \le C(p,L) \qquad \textit{for all } t\in(0,T) \tag{3.15}$$

and

$$\int_0^t\int_\Omega u_\varepsilon^{p+1}v_\varepsilon \le C(p,L) \qquad \textit{for all } t\in(0,T) \tag{3.16}$$

and

$$\int_0^t \Big\|\nabla\big(u_\varepsilon^{\frac{p+1}{2}}(\cdot,s)v_\varepsilon(\cdot,s)\big)\Big\|_{L^1(\Omega)}^2 ds \le C(p,L) \qquad \textit{for all } t\in(0,T). \tag{3.17}$$

Proof. For fixed $p>2$, we employ Lemma 3.3 to find $\eta_0(p)>0$ and $c_1(p)>0$ such that whenever u_0, v_0, $\varepsilon\in(0,1)$ and $T\in(0,T_{max,\varepsilon})$ are such that (1.9), (2.14) and (2.15) hold with some $L>0$ and $\eta\in(0,\eta_0(p)]$, and with $\delta_0(p,L,\eta)$ as in Lemma 2.2, we have

$$\begin{aligned} &\frac{d}{dt}\int_\Omega u_\varepsilon^p + \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + 2(7+4\sqrt{2})\int_\Omega u_\varepsilon^2 v_\varepsilon + \eta^2\int_\Omega u_\varepsilon^{p+1}v_\varepsilon \\ &\quad\le\ c_1(p)L\eta^2\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + c_1(p)\cdot\frac{1+L}{\eta^{2(p-1)}}\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t\in(0,T). \end{aligned} \tag{3.18}$$

Given $L>0$, we here fix

$$\eta := \min\Big\{\eta_0(p)\,,\,\frac{1}{\sqrt{2c_1(p)L}}\,,\,1\Big\}, \tag{3.19}$$

and abbreviating $c_2(p,L):=c_1(p)\cdot\frac{1+L}{\eta^{2(p-1)}}$ we let

$$\delta_1(p,L) := \min\Big\{\delta_0(p,L,\eta)\,,\,\Big(\frac{\delta_0(p,L,\eta)}{2}\Big)^{\frac{2}{3}}\,,\,\frac{1}{4\sqrt{|\Omega|}}\,,\,\frac{1}{4c_2(p,L)|\Omega|}\Big\}. \tag{3.20}$$

Henceforth assuming that (1.9) be satisfied, and that (3.12) and (2.15) hold with some $\varepsilon\in(0,1)$ and $T\in(0,T_{max,\varepsilon})$, we then first observe that Lemma 2.2 is applicable, because the restriction $\delta_1(p,L)\le\delta_0(p,L,\eta)$ clearly ensures that $\|v_0\|_{L^\infty(\Omega)}\le\delta_0(p,L,\eta)$, and because the second requirement contained in (3.20) guarantees that moreover

$$|\nabla v_0| = 2\sqrt{v_0}|\nabla\sqrt{v_0}| \le 2\delta_1^{\frac{3}{2}}(p,L)\le\delta_0(p,L,\eta) \qquad \text{in } \Omega.$$

In particular, we may thus apply (3.18) to $(u_\varepsilon,v_\varepsilon)$ and T, and combine the outcome with Lemma 3.5, to see that since $c_1(p)L\eta^2\le\frac{1}{2}$ by (3.19), according to our definition of $c_2(p,L)$ we have

$$\begin{aligned} &\frac{d}{dt}\Big\{\int_\Omega u_\varepsilon^p + \int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}\Big\} + \frac{p(p-1)}{4}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_\Omega\frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \eta^2\int_\Omega u_\varepsilon^{p+1}v_\varepsilon \\ &\quad\le\ c_2(p,L)\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t\in(0,T). \end{aligned}$$

After an integration in time, this reveals that

$$\begin{aligned} &\int_\Omega u_\varepsilon^p(\cdot,t) + \int_\Omega\frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} + \frac{p(p-1)}{4}\int_0^t\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_0^t\int_\Omega\frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \eta^2\int_0^t\int_\Omega u_\varepsilon^{p+1}v_\varepsilon \\ &\quad\le\ \int_\Omega(u_0+\varepsilon)^p + \int_\Omega\frac{|\nabla v_0|^2}{v_0} + c_2(p,L)\int_0^t\int_\Omega u_\varepsilon v_\varepsilon \qquad \text{for all } t\in(0,T), \end{aligned} \tag{3.21}$$

where due to (3.12) and the third condition in (3.20),

$$\int_\Omega\frac{|\nabla v_0|^2}{v_0} \le 4|\Omega|\cdot\big\|\nabla\sqrt{v_0}\big\|_{L^\infty(\Omega)}^2 \le 4|\Omega|\delta_1^2(p,L)\le\frac{1}{4},$$

and where thanks to (2.5) and (3.12), the rightmost restriction in (3.20) warrants that

$$c_2(p,L)\int_0^t\int_\Omega u_\varepsilon v_\varepsilon \le c_2(p,L)\int_\Omega v_0 \le c_2(p,L)\delta_1(p,L)|\Omega|\le\frac{1}{4} \qquad \text{for all } t\in(0,T).$$

From (3.21) we therefore obtain that since $\varepsilon < 1$,

$$\int_\Omega u_\varepsilon^p(\cdot,t) + \int_\Omega \frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} + \frac{p(p-1)}{4}\int_0^t\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + \frac{1}{2}\int_0^t\int_\Omega \frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2 + \eta^2\int_0^t\int_\Omega u_\varepsilon^{p+1}v_\varepsilon$$
$$\le \int_\Omega (u_0+1)^p + \frac{1}{2} \qquad \text{for all } t\in(0,T),$$

and that thus, with some suitably large $C(p,L)>0$, both (3.13) and (3.15)-(3.17) hold due to the fact that according to Lemma 3.6, (2.4), (2.15) and (3.12) we have

$$\int_0^t \Big\|\nabla\big(u_\varepsilon^{\frac{p+1}{2}}(\cdot,s)v_\varepsilon(\cdot,s)\big)\Big\|_{L^1(\Omega)}^2 ds$$
$$\le \frac{(p+1)^2|\Omega|\cdot\|v_0\|_{L^\infty(\Omega)}}{2}\cdot\int_0^t\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 + 2\|v_0\|_{L^\infty(\Omega)}\cdot\int_0^t\int_\Omega\frac{u_\varepsilon}{v_\varepsilon}|\nabla v_\varepsilon|^2$$
$$\le \frac{(p+1)^2|\Omega|\delta_1(p,L)}{2}\cdot\frac{4(L+\frac{1}{2})}{p(p-1)} + 2L\delta_1(p,L)\cdot 2\Big(L+\frac{1}{2}\Big) \qquad \text{for all } t\in(0,T).$$

The inequality in (3.14), finally, is an obvious consequence of Lemma 2.2 because of the requirement $\eta\le 1$ contained in (3.19). $\Box$

Through a standard self-map type reasoning, the latter immediately entails the following consequence which now does no longer involve any hypotheses on the solutions to (2.1) themselves.

Lemma 3.8 *Let $p>2$ and $K>0$. Then there exist $\delta(p,K)>0$ and $C(p,K)>0$ such that whenever u_0 and v_0 satisfy (1.9) and are such that (1.10) and (1.11) are valid, for any choice of $\varepsilon\in(0,1)$ we have*

$$\int_\Omega u_\varepsilon^p(\cdot,t) + \int_\Omega\frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon(\cdot,t)} \le \int_\Omega (u_0+1)^p + 1 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \tag{3.22}$$

and

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le 1 \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \tag{3.23}$$

as well as

$$\int_0^t\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2 \le C(p,K) \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \tag{3.24}$$

and

$$\int_0^t\int_\Omega u_\varepsilon^{p+1}v_\varepsilon \le C(p,K) \qquad \text{for all } t\in(0,T_{max,\varepsilon}) \tag{3.25}$$

and

$$\int_0^t \Big\|\nabla\big(u_\varepsilon^{\frac{p+1}{2}}(\cdot,s)v_\varepsilon(\cdot,s)\big)\Big\|_{L^1(\Omega)}^2 ds \le C(p,K) \qquad \text{for all } t\in(0,T_{max,\varepsilon}). \tag{3.26}$$

Proof. Given $K>0$, we apply Lemma 3.7 to $L := 2^{p-1}K^p + 2^{p-1}|\Omega| + 1$ and let $\delta(p,K) := \delta_1(p,L)$ with $\delta_1(\cdot,\cdot)$ as provided there. Then assuming that u_0 and v_0 comply with (1.9) and are such that (1.10) and (1.11) hold, and letting

$$S_\varepsilon := \Big\{T\in(0,T_{max,\varepsilon})\ \Big|\ \int_\Omega u_\varepsilon^p(\cdot,t) < L \text{ for all } t\in(0,T)\Big\}, \qquad \varepsilon\in(0,1),$$

for each $\varepsilon \in (0,1)$ we then infer from (1.10) and the continuity of u_ε that S_ε is not empty and hence $T_\varepsilon := \sup S_\varepsilon$ is a well-defined element of $(0, T_{max,\varepsilon}] \subset (0,\infty]$. Now since Lemma 3.7 along with (1.10) ensures that

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le \int_\Omega (u_0+1)^p + \frac{1}{2} \le 2^{p-1}\int_\Omega (u_0^p+1) + \frac{1}{2} \le 2^{p-1}K^p + 2^{p-1}|\Omega| + \frac{1}{2} \le L - \frac{1}{2}$$

for all $t \in (0, T_\varepsilon)$, again by continuity of u_ε the hypothesis that T_ε be smaller than $T_{max,\varepsilon}$ is absurd for any such ε. In consequence, we must have $T_\varepsilon = T_{max,\varepsilon}$ for all $\varepsilon \in (0,1)$, so that (3.22) and (3.23)-(3.26) immediately follow upon recalling (3.14)-(3.17). $\square$

4 Global existence of $(u_\varepsilon, v_\varepsilon)$. Further time-dependent regularity properties

As a by-product of (3.22) and (3.23), with only few additional efforts we can now make sure that under the smallness assumption made in Lemma 3.8, each of our solutions to (2.1) is actually global in time.

Lemma 4.1 *Let $p > 2$ and $K > 0$, and with $\delta(p,K)$ taken from Lemma 3.8, suppose that $u_0 \in W^{1,\infty}(\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ are nonnegative with $v_0 \not\equiv 0$ and such that (1.10) and (1.11) hold. Then $T_{max,\varepsilon} = +\infty$ for all $\varepsilon \in (0,1)$.*

Proof. Supposing on the contrary that $T_{max,\varepsilon}$ be finite for some $\varepsilon \in (0,1)$, we first claim that writing $t_0 := \frac{1}{2}T_{max,\varepsilon}$, we could then find $c_1(\varepsilon) > 0$ fulfilling

$$v_\varepsilon(x,t) \ge c_1(\varepsilon) \qquad \text{for all } x \in \Omega \text{ and } t \in (t_0, T_{max,\varepsilon}). \tag{4.1}$$

Indeed, from the inclusion $v_\varepsilon \in C^{2,1}(\overline{\Omega} \times (0, T_{max,\varepsilon}))$ and the strict positivity of v_ε in $\overline{\Omega} \times (0, T_{max,\varepsilon})$, as asserted by Lemma 2.1, from (2.1) it follows that the function $z_\varepsilon := \ln \frac{1}{v_\varepsilon}$ belongs to $C^{2,1}(\overline{\Omega} \times [t_0, T_{max,\varepsilon}))$ and satisfies

$$z_{\varepsilon t} = \Delta z_\varepsilon - |\nabla z_\varepsilon|^2 + u_\varepsilon \le \Delta z_\varepsilon + u_\varepsilon \qquad \text{in } \Omega \times (t_0, T_{max,\varepsilon}).$$

Using the comparison principle along with known regularization features of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω ([28]), we thus infer that with some $c_2(\varepsilon) > 0$ and $c_3(\varepsilon) > 0$,

$$\begin{aligned}
z_\varepsilon(\cdot,t) &\le e^{(t-t_0)\Delta} z_\varepsilon(\cdot,t_0) + \int_{t_0}^t e^{(t-s)\Delta} u_\varepsilon(\cdot,s)ds \\
&\le \sup_{x\in\Omega} z_\varepsilon(x,t_0) + c_2(\varepsilon)\int_{t_0}^t (t-s)^{-1+\frac{1}{p}} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} ds \\
&\le \sup_{x\in\Omega} z_\varepsilon(x,t_0) + c_3(\varepsilon) \sup_{s\in(t_0,T_{max,\varepsilon})} \|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)} \qquad \text{in } \Omega \text{ for all } t \in (t_0, T_{max,\varepsilon}),
\end{aligned}$$

because $p > 1$. Recalling (3.22), we readily infer (4.1) from this and the continuity of $z_\varepsilon(\cdot,t_0)$ throughout $\overline{\Omega}$.

In order to next make sure that for any $q > p$ we can find $c_4(\varepsilon, q) > 0$ such that

$$\int_\Omega u_\varepsilon^q(\cdot, t) \le c_4(\varepsilon, q) \qquad \text{for all } t \in (t_0, T_{max,\varepsilon}), \tag{4.2}$$

given any such q we go back to (2.1) and use Young's inequality along with (2.4) as well as (3.23) to obtain that

$$\begin{aligned} \frac{1}{q}\frac{d}{dt}\int_\Omega u_\varepsilon^q + (q-1)\int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 &= (q-1)\int_\Omega u_\varepsilon^q v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \ell \int_\Omega u_\varepsilon^q v_\varepsilon \\ &\le \frac{q-1}{2}\int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{q-1}{2}\int_\Omega u_\varepsilon^{q+1} v_\varepsilon |\nabla v_\varepsilon|^2 \\ &\quad + \ell \int_\Omega u_\varepsilon^{q+1} + \ell \int_\Omega v_\varepsilon^{q+1} \\ &\le \frac{q-1}{2}\int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\quad + c_5(q)\int_\Omega u_\varepsilon^{q+1} + c_6(q) \qquad \text{for all } t \in (t_0, T_{max,\varepsilon}) \end{aligned} \tag{4.3}$$

with $c_5(q) := \frac{q-1}{2}\|v_0\|_{L^\infty(\Omega)} + \ell$ and $c_6(q) := \ell|\Omega| \cdot \|v_0\|_{L^\infty(\Omega)}^{q+1}$. Here, a combination of a Poincaré type inequality with (3.22) and (4.1) shows that with some $c_7(\varepsilon, q) > 0$ and $c_8(q) > 0$ we have

$$\begin{aligned} c_5(q)\int_\Omega u_\varepsilon^{q+1} &= c_5(q)\|u_\varepsilon^{\frac{q+1}{2}}\|_{L^2(\Omega)}^2 \\ &\le \frac{2(q-1)c_1(\varepsilon)}{(q+1)^2}\|\nabla u_\varepsilon^{\frac{q+1}{2}}\|_{L^2(\Omega)}^2 + c_7(\varepsilon, q)\|u_\varepsilon^{\frac{q+1}{2}}\|_{L^{\frac{2p}{q+1}}(\Omega)}^2 \\ &= \frac{(q-1)c_1}{2}\int_\Omega u_\varepsilon^{q-1}|\nabla u_\varepsilon|^2 + c_7(\varepsilon, q)\cdot\Big\{\int_\Omega u_\varepsilon^p\Big\}^{\frac{q+1}{p}} \\ &\le \frac{q-1}{2}\int_\Omega u_\varepsilon^{q-1} v_\varepsilon|\nabla u_\varepsilon|^2 + c_8(\varepsilon, q) \qquad \text{for all } t \in (t_0, T_{max,\varepsilon}), \end{aligned}$$

so that from (4.3) we infer that

$$\frac{1}{q}\frac{d}{dt}\int_\Omega u_\varepsilon^q \le c_8(\varepsilon, q) + c_6(q) \qquad \text{for all } t \in (t_0, T_{max,\varepsilon}),$$

and that hence indeed (4.2) is valid with some appropriately large $c_4(\varepsilon, q) > 0$.

Recalling (3.23), we may therefore invoke a standard result on L^∞ bounds in scalar parabolic equations involving nonlinear diffusion of porous medium type ([22, Lemma A.1]) to obtain $c_9(\varepsilon) > 0$ such that

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le c_9(\varepsilon) \qquad \text{for all } t \in (t_0, T_{max,\varepsilon}),$$

which in view of (2.3) is incompatible with our hypothesis and hence shows that, in fact, we must have $T_{max,\varepsilon} = \infty$ for any $\varepsilon \in (0,1)$. $\Box$

We next prepare an argument concerned with the time derivative of the product $u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon$, as appearing in (3.26), by documenting the following consequence of (3.22) and (3.23) when applied in the course of an analysis of $\int_\Omega u_\varepsilon^q$ for $q \in (0,1)$.

Lemma 4.2 *Let $p > 2$, and assume that (1.9), (1.10) and (1.11) hold with some $K > 0$ and with $\delta(p, K)$ as in Lemma 3.8. Then for all $q \in (0,1)$ and any $T > 0$ there exists $C(q,T) > 0$ such that*

$$\int_0^T \int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 \le C(q,T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{4.4}$$

Proof. We again integrate by parts in the first equation from (2.1) and use Young's inequality together with (3.23) and (2.4) to find that

$$\begin{aligned} -\frac{1}{q}\frac{d}{dt}\int_\Omega u_\varepsilon^q + (1-q)\int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 \\ &= (1-q)\int_\Omega u_\varepsilon^q v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon - \ell \int_\Omega u_\varepsilon^q v_\varepsilon \\ &\le \frac{1-q}{2}\int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1-q}{2}\int_\Omega u_\varepsilon^{q+1} v_\varepsilon |\nabla v_\varepsilon|^2 \\ &\le \frac{1-q}{2}\int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{1-q}{2}\|v_0\|_{L^\infty(\Omega)}\int_\Omega u_\varepsilon^{q+1} \\ &\le \frac{1-q}{2}\int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 + c_1(q) \qquad \text{for all } t > 0, \end{aligned} \tag{4.5}$$

where $c_1(q) := \frac{1-q}{2}\|v_0\|_{L^\infty(\Omega)} \cdot \sup_{\varepsilon\in(0,1)} \sup_{t>0} \int_\Omega u_\varepsilon^{q+1}(\cdot,t)$ is finite due to Lemma 3.8 and the fact that $q+1 < 2 < p$. As the same source also provides $c_2(q) > 0$ such that $\frac{1}{q}\int_\Omega u_\varepsilon^q(\cdot,t) \le c_2(q)$ for all $t > 0$ and each $\varepsilon \in (0,1)$, an integration of (4.5) shows that

$$\frac{1-q}{2}\int_0^T \int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 \le c_1(q)T + c_2(q) \qquad \text{for all } \varepsilon \in (0,1),$$

and hence yields (4.4). $\square$

By suitable interpolation with the information already observed in Lemma 3.8, the latter indeed entails the following time regularity feature of $u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon$.

Lemma 4.3 *Let $p > 2$, and suppose that (1.9), (1.10) and (1.11) are satisfied with some $K > 0$ and with $\delta(p,K)$ as given by Lemma 3.8. Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \Big\| \partial_t \big(u_\varepsilon^{\frac{p+1}{2}}(\cdot,t) v_\varepsilon(\cdot,t) \big) \Big\|_{(W^{3,2}(\Omega))^\star} dt \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{4.6}$$

Proof. Drawing on the continuity of the embedding $W^{3,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$, we fix $c_1 > 0$ such that $\|\psi\|_{L^\infty(\Omega)} + \|\nabla\psi\|_{L^\infty(\Omega)} \le c_1$ for all $\psi \in C^3(\overline{\Omega})$ fulfilling $\|\psi\|_{W^{3,2}(\Omega)} \le 1$. Fixing any such ψ and an arbitrary $t>0$, for $\varepsilon\in(0,1)$ we then use (2.1) to compute

$$\begin{aligned}
\int_\Omega \partial_t\big(u_\varepsilon^{\frac{p+1}{2}}v_\varepsilon\big)\cdot\psi &= \frac{p+1}{2}\int_\Omega u_\varepsilon^{\frac{p-1}{2}}v_\varepsilon\psi\cdot\Big\{\nabla\cdot(u_\varepsilon v_\varepsilon\nabla u_\varepsilon)-\nabla\cdot(u_\varepsilon^2v_\varepsilon\nabla v_\varepsilon)+\ell u_\varepsilon v_\varepsilon\Big\} \\
&\quad +\int_\Omega u_\varepsilon^{\frac{p+1}{2}}\psi\cdot\big\{\Delta v_\varepsilon-u_\varepsilon v_\varepsilon\big\} \\
&= -\frac{p+1}{2}\int_\Omega \nabla\big(u_\varepsilon^{\frac{p-1}{2}}v_\varepsilon\psi\big)\cdot\big\{u_\varepsilon v_\varepsilon\nabla u_\varepsilon-u_\varepsilon^2v_\varepsilon\nabla v_\varepsilon\big\} \\
&\quad +\frac{(p+1)\ell}{2}\int_\Omega u_\varepsilon^{\frac{p+1}{2}}v_\varepsilon^2\psi \\
&\quad -\int_\Omega\nabla\big(u_\varepsilon^{\frac{p+1}{2}}\psi\big)\cdot\nabla v_\varepsilon-\int_\Omega u_\varepsilon^{\frac{p+3}{2}}v_\varepsilon\psi \\
&= -\frac{p+1}{2}\sum_{i=1}^{6}I_i(\varepsilon)+\sum_{i=7}^{10}I_i(\varepsilon), \qquad (4.7)
\end{aligned}$$

where

$$I_1(\varepsilon):=\frac{p-1}{2}\int_\Omega u_\varepsilon^{\frac{p-1}{2}}v_\varepsilon^2|\nabla u_\varepsilon|^2\psi,\qquad I_2(\varepsilon):=\int_\Omega u_\varepsilon^{\frac{p+1}{2}}v_\varepsilon(\nabla u_\varepsilon\cdot\nabla v_\varepsilon)\psi\qquad\text{and}$$

$$I_3(\varepsilon):=\int_\Omega u_\varepsilon^{\frac{p+1}{2}}v_\varepsilon^2\nabla u_\varepsilon\cdot\nabla\psi,$$

where

$$I_4(\varepsilon):=-\frac{p-1}{2}\int_\Omega u_\varepsilon^{\frac{p+1}{2}}v_\varepsilon^2(\nabla u_\varepsilon\cdot\nabla v_\varepsilon)\psi,\qquad I_5(\varepsilon):=-\int_\Omega u_\varepsilon^{\frac{p+3}{2}}v_\varepsilon|\nabla v_\varepsilon|^2\psi\qquad\text{and}$$

$$I_6(\varepsilon):=-\int_\Omega u_\varepsilon^{\frac{p+3}{2}}v_\varepsilon^2\nabla v_\varepsilon\cdot\nabla\psi,$$

and where

$$I_7(\varepsilon):=-\frac{p+1}{2}\int_\Omega u_\varepsilon^{\frac{p-1}{2}}(\nabla u_\varepsilon\cdot\nabla v_\varepsilon)\psi\quad\text{and}\quad I_8(\varepsilon):=-\int_\Omega u_\varepsilon^{\frac{p-1}{2}}\nabla v_\varepsilon\cdot\nabla\psi$$

as well as

$$I_9(\varepsilon):=\frac{(p+1)\ell}{2}\int_\Omega u_\varepsilon^{\frac{p+1}{2}}v_\varepsilon^2\psi\quad\text{and}\quad I_{10}(\varepsilon):=-\int_\Omega u_\varepsilon^{\frac{p+3}{2}}v_\varepsilon\psi.$$

In order to estimate $I_1(\varepsilon)$ appropriately, we pick any $q\in(0,1)$ and observe that then $q-1\le\frac{p-1}{2}\le p-1$, so that an interpolation based on Young's inequality shows that thanks to (2.4) and the fact that $\varepsilon<1$, writing $c_2:=\|v_0\|_{L^\infty(\Omega)}$ we have

$$\begin{aligned}
|I_1(\varepsilon)| &\le \frac{(p-1)c_1}{2}\int_\Omega(u_\varepsilon^{p-1}+u_\varepsilon^{q-1})v_\varepsilon^2|\nabla u_\varepsilon|^2 \\
&\le \frac{(p-1)c_1c_2}{2}\int_\Omega u_\varepsilon^{p-1}v_\varepsilon|\nabla u_\varepsilon|^2+\frac{(p-1)c_1c_2}{2}\int_\Omega u_\varepsilon^{q-1}v_\varepsilon|\nabla u_\varepsilon|^2. \qquad (4.8)
\end{aligned}$$

Next, several further applications of Young's inequality, again combined with the restriction $\varepsilon < 1$, reveal that

$$|I_7(\varepsilon)| \le (p+1)c_1 \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{(p+1)c_1}{8} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon}, \tag{4.9}$$

that since $|\nabla v_\varepsilon| \le 1$ by (3.23), and since $p+1 \ge \max\left\{1\,,\,\frac{p+3}{2}\,,\,\frac{p+1}{2}\right\}$, we have

$$\begin{aligned} |I_2(\varepsilon)| &\le c_1 \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{c_1}{4} \int_\Omega u_\varepsilon^2 v_\varepsilon \\ &\le c_1 \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{c_1}{4} \int_\Omega u_\varepsilon^{p+1} v_\varepsilon + \frac{c_1 c_2 |\Omega|}{4} \end{aligned} \tag{4.10}$$

and that

$$\begin{aligned} |I_3(\varepsilon)| + |I_4(\varepsilon)| &\le \frac{(p+1)c_1}{2} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{(p+1)c_1}{8} \int_\Omega u_\varepsilon^2 v_\varepsilon^3 \\ &\le \frac{(p+1)c_1}{2} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \frac{(p+1)c_1 c_2^2}{8} \int_\Omega u_\varepsilon^{p+1} v_\varepsilon + \frac{(p+1)c_1 c_2^3 |\Omega|}{8} \end{aligned} \tag{4.11}$$

as well as

$$\begin{aligned} &|I_5(\varepsilon)| + |I_6(\varepsilon)| + |I_9(\varepsilon)| + |I_{10}(\varepsilon)| \\ &\quad \le c_1 \int_\Omega u_\varepsilon^{\frac{p+3}{2}} v_\varepsilon + c_1 \int_\Omega u_\varepsilon^{\frac{p+3}{2}} v_\varepsilon^2 + \frac{(p+1)\ell c_1}{2} \int_\Omega u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon^2 + c_1 \int_\Omega u_\varepsilon^{\frac{p+3}{2}} v_\varepsilon \\ &\quad \le \Big(c_1 + c_1 c_2 + \frac{(p+1)\ell c_1 c_2}{2} + c_1\Big) \int_\Omega u_\varepsilon^{p+1} v_\varepsilon + \Big(c_1 c_2 + c_1 c_2^2 + \frac{(p+1)\ell c_1 c_2^2}{2} + c_1 c_2\Big) \cdot |\Omega| \end{aligned} \tag{4.12}$$

As, similarly,

$$\begin{aligned} |I_8(\varepsilon)| &\le c_1 \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \frac{c_1}{4} \int_\Omega u_\varepsilon^{p-1} v_\varepsilon \\ &\le c_1 \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \frac{c_1}{4} \int_\Omega u_\varepsilon^{p+1} v_\varepsilon + \frac{c_1 c_2 |\Omega|}{4}, \end{aligned}$$

we may collect (4.8)-(4.12) to infer from (4.7) that there exists $c_3 > 0$ fulfilling

$$\begin{aligned} \Big\| \partial_t \big(u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon\big) \Big\|_{(W^{3,2}(\Omega))^\star} &\le c_3 \cdot \Big\{ \int_\Omega u_\varepsilon^{p-1} v_\varepsilon |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^{q-1} v_\varepsilon |\nabla u_\varepsilon|^2 \\ &\qquad + \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} + \int_\Omega u_\varepsilon^{p+1} v_\varepsilon + 1 \Big\} \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1). \end{aligned}$$

In view of (3.24), (4.4), (3.22) and (3.25), a time integration therefore readily leads to (4.6). $\square$

Now the intention to turn compactness features of $\big(u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon\big)_{\varepsilon \in (0,1)}$ into knowledge on u_ε seems promising only when accomanied by appropriate information on positivity of the weight functions v_ε appearing therein. Unlike all previous arguments in this manuscript, the following observation in this regard relies on our overall positivity assumption on v_0 in an indispensable manner.

Lemma 4.4 *Let $p > 2$ and $u_0 \in W^{1,\infty}(\Omega)$ be a nonnegative function fulfilling (1.10) for some $K > 0$, and assume that $v_0 \in W^{1,\infty}(\Omega)$ is positive in $\overline{\Omega}$ and satisfies (1.11) with $\delta(p, K)$ as in Lemma 3.8. Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_\Omega \ln \frac{\|v_0\|_{L^\infty(\Omega)}}{v_\varepsilon(\cdot, t)} \le C(T) \qquad \textit{for all } t \in (0, T) \textit{ and } \varepsilon \in (0, 1). \tag{4.13}$$

PROOF. According to the second equation in (2.1) and Young's inequality,

$$\frac{d}{dt} \int_\Omega \ln \frac{\|v_0\|_{L^\infty(\Omega)}}{v_\varepsilon} = -\frac{d}{dt} \int_\Omega \ln v_\varepsilon = -\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + \int_\Omega u_\varepsilon \le \int_\Omega u_\varepsilon^p + |\Omega| \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0, 1),$$

which upon an integration in time already yields (4.13), because $\ln \frac{\|v_0\|_{L^\infty(\Omega)}}{v_0}$ belongs to $L^1(\Omega)$ by positivity of v_0 in $\overline{\Omega}$. □

5 The limit $\varepsilon \searrow 0$. Proof of Theorem 1.2

After the above preparations, the extraction of a subsequence converging to a global weak solution of (1.4) now becomes rather straightforward:

Lemma 5.1 *Assume that $p > 2$, that $K > 0$ and that u_0 and v_0 are such that (1.9), (1.10) and (1.11) hold with $\delta(p, K)$ as in Lemma 3.8. Then there exist $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, 1)$ and functions*

$$\begin{cases} u \in L^\infty((0, \infty); L^p(\Omega)) & \textit{and} \\ v \in L^\infty((0, \infty); W^{1,\infty}(\Omega)) \end{cases} \tag{5.1}$$

such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that $u \ge 0$ and $v > 0$ a.e. in $\Omega \times (0, \infty)$, that

$$u_\varepsilon \to u \qquad \textit{a.e. in } \Omega \times (0, \infty) \textit{ and in } L^q_{loc}(\overline{\Omega} \times [0, \infty)) \textit{ for all } q \in [1, p), \tag{5.2}$$

$$v_\varepsilon \to v \qquad \textit{a.e. in } \Omega \times (0, \infty) \textit{ and in } L^q_{loc}(\overline{\Omega} \times [0, \infty)) \textit{ for all } q \in [1, \infty) \qquad \textit{and} \tag{5.3}$$

$$\nabla v_\varepsilon \overset{\star}{\rightharpoonup} \nabla v \qquad \textit{in } L^\infty(\Omega \times (0, \infty)) \tag{5.4}$$

as $\varepsilon = \varepsilon_j \searrow 0$, and that (u, v) forms a global weak solution of (1.4) in the sense of Definition 1.1.

PROOF. From (2.4) and (3.23) we know that

$$(v_\varepsilon)_{\varepsilon \in (0,1)} \text{ is bounded in } L^\infty((0, \infty); W^{1,\infty}(\Omega)),$$

while in view of the boundedness of $(u_\varepsilon v_\varepsilon)_{\varepsilon \in (0,1)}$ in $L^2_{loc}(\Omega \times (0, T))$ for all $T > 0$, as implied by (3.22) and (2.4) due to the inequality $p \ge 2$, it can readily be verified that

$$(v_{\varepsilon t})_{\varepsilon \in (0,1))} \text{ is bounded in } L^2\big((0, T); (W^{1,2}(\Omega))^\star\big) \qquad \text{for all } T > 0.$$

Apart from that, a combination of (3.26) with (3.25) shows that

$$\big(u_\varepsilon^{\frac{p+1}{2}} v_\varepsilon\big)_{\varepsilon \in (0,1)} \text{ is bounded in } L^2((0, T); W^{1.1}(\Omega)) \qquad \text{for all } T > 0,$$

while Lemma 4.3 asserts that

$$\Big(\partial_t\big(u_\varepsilon^{\frac{p+1}{2}}v_\varepsilon\big)\Big)_{\varepsilon\in(0,1)} \text{ is bounded in } L^1\big((0,T);(W^{3,2}(\Omega))^\star\big) \qquad \text{for all } T>0.$$

Two applications of an Aubin-Lions type lemma ([24]) therefore enable us to pick $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$, and such that with two nonnegative functions $z\in L^1_{loc}(\overline{\Omega}\times[0,\infty))$ and $v\in L^\infty((0,\infty);W^{1,\infty}(\Omega))$ we have (5.3) and (5.4) as well as

$$u_\varepsilon^{\frac{p+1}{2}}v_\varepsilon\to z \qquad \text{a.e. in } \Omega\times(0,\infty) \text{ and in } L^1_{loc}(\overline{\Omega}\times[0,\infty)) \tag{5.5}$$

as $\varepsilon=\varepsilon_j\searrow 0$. Since $v\le\|v_0\|_{L^\infty(\Omega)}$ by (2.4) and (5.3), and since thus Lemma 4.4 together with Fatou's lemma guarantees that $\ln v$ belongs to $L^1_{loc}(\overline{\Omega}\times[0,\infty))$, and that hence, in particular, v is actually positive a.e. in $\Omega\times(0,\infty)$, letting $u:=\big(\frac{z}{v}\big)^{\frac{2}{p+1}}$ we obtain an a.e. in $\Omega\times(0,\infty)$ well-defined nonnegative function u for which we have $u_\varepsilon=\big(\frac{z_\varepsilon}{v_\varepsilon}\big)^{\frac{2}{p+1}}\to u$ a.e. in $\Omega\times(0,\infty)$ as $\varepsilon=\varepsilon_j\searrow 0$ according to (5.5) and (5.3). Since (3.22) asserts that $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((0,\infty);L^p(\Omega))$, this limit must as well belong to this space, and must moreover satisfy (5.2) as a consequence of the Vitali convergence theorem.

Now given $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ such that $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega\times(0,\infty)$, from (2.1) we obtain that

$$\begin{aligned}
-\int_0^\infty\int_\Omega u_\varepsilon\varphi_t-\int_\Omega u_0\varphi(\cdot,0) \quad = \quad & \frac{1}{2}\int_0^\infty\int_\Omega u_\varepsilon^2\nabla v_\varepsilon\cdot\nabla\varphi \\
&+\frac{1}{2}\int_0^\infty\int_\Omega u_\varepsilon^2 v_\varepsilon\Delta\varphi+\int_0^\infty\int_\Omega u_\varepsilon^2 v_\varepsilon\nabla v_\varepsilon\cdot\nabla\varphi \\
&+\ell\int_0^\infty\int_\Omega u_\varepsilon v_\varepsilon\varphi \qquad \text{for all } \varepsilon\in(0,1),
\end{aligned}$$

where we may use that by (5.2) and (5.3), as $\varepsilon=\varepsilon_j\searrow 0$ we have

$$u_\varepsilon\to u, \quad u_\varepsilon^2\to u^2 \quad u_\varepsilon^2 v_\varepsilon\to u^2 v \quad \text{and} \quad u_\varepsilon v_\varepsilon\to uv \qquad \text{in } L^1_{loc}(\overline{\Omega}\times[0,\infty)),$$

to infer (1.7) upon taking $\varepsilon=\varepsilon_j\searrow 0$ and utilizing (5.4). Since (1.8) can similarly be derived for any $\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$, and since the regularity requirements in Definition 1.1 are clearly implied by (5.1), it thus follows that indeed (u,v) solves (1.4) in the claimed sense. □

Our main result has thereby been achieved already:

Proof of Theorem 1.2. We only need to employ Lemma 5.1 and combine (2.4) and (3.23) with (1.11). □

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