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Funnel Control for Linear DAEs

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We study funnel control for linear differential-algebraic multi-input multi-output systems which are not necessarily regular. We show that the funnel controller (that is a static nonlinear output error feedback) achieves - for a special class of right-invertible systems with asymptotically stable zero dynamics - tracking of a reference signal by the output signal within a pre-specified performance funnel.

1 Introduction

We consider linear constant coefficient DAEs of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t)\,, \tag{1.1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{m\times n}$. The set of these systems is denoted by $\Sigma_{l,n,m}$ and we write $[E, A, B, C] \in \Sigma_{l,n,m}$. In the present paper, we put special emphasis on the non-regular case, i.e., we do not assume that $sE - A$ is *regular*, that is $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.

The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. A trajectory $(x, u, y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is said to be a *solution* of (1.1) if, and only if, it belongs to the *behavior* of (1.1):

$$\mathfrak{B}_{(1.1)} := \left\{ (x, u, y) \in \mathcal{C}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \;\middle|\; Ex \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^l) \text{ and } (x, u, y) \text{ solves (1.1) for all } t \in \mathbb{R} \right\}.$$

Particular emphasis is placed on the *zero dynamics* of (1.1). These are, for $[E, A, B, C] \in \Sigma_{l,n,m}$, defined by $\mathcal{ZD}_{(1.1)} := \{ (x, u, y) \in \mathfrak{B}_{(1.1)} \mid y = 0 \}$. By linearity of (1.1), $\mathcal{ZD}_{(1.1)}$ is a real vector space. The zero dynamics of (1.1) are called

$$\begin{aligned}
\textit{autonomous} \quad &:\Longleftrightarrow \quad \forall\, w_1, w_2 \in \mathcal{ZD}_{(1.1)}\; \forall\, I \subseteq \mathbb{R} \text{ open interval}: \quad w_1|_I = w_2|_I \implies w_1 = w_2\,;\\
\textit{asymptotically stable} \quad &:\Longleftrightarrow \quad \forall\, (x, u, y) \in \mathcal{ZD}_{(1.1)}: \; \lim_{t\to\infty} \big(x(t), u(t)\big) = 0.
\end{aligned}$$

Note that the above definitions are within the spirit of the *behavioral approach* [1] and take into account that the zero dynamics $\mathcal{ZD}_{(1.1)}$ are a linear behavior. In this framework the definition for autonomy of a general behavior was given in [1, Sec. 3.2] and the definition of asymptotic stability in [1, Def. 7.2.1]. (Asymptotically stable) zero dynamics are the vector space of those trajectories of the system which are, loosely speaking, not visible at the output (and tend to zero).

In the present paper we concentrate on systems $[E, A, B, C] \in \Sigma_{l,n,m}$ with autonomous $\mathcal{ZD}_{(1.1)}$ for which the matrix

$$\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)[0, I_m]^\top \in \mathbb{R}^{m\times m} \tag{1.2}$$

exists and satisfies $\Gamma = \Gamma^\top \geq 0$, where $L(s)$ is a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$; it is shown in [2] that autonomous zero dynamics imply left-invertibility of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ and that Γ does not depend on the choice of $L(s)$.

It is the aim of the present paper to show that funnel control is feasible for the class of right-invertible systems $[E, A, B, C] \in \Sigma_{l,n,m}$ with asymptotically stable zero dynamics for which the matrix Γ in (1.2) exists and satisfies $\Gamma = \Gamma^\top \geq 0$. This class encompasses all regular systems with a vector relative degree which is component-wise smaller or equal to one, see [2, App. B]. Furthermore, linear passive electrical circuits may be treated as well within this framework [3].

For the proofs and more details on the results in the present paper see [2].

2 System decomposition and funnel control

We show that right-invertibility, autonomy of the zero dynamics and the existence of Γ in (1.2) allow for a decomposition of the system. The main result of the present paper, Theorem 2.2, is based on this decomposition. A system $[E, A, B, C] \in \Sigma_{l,n,m}$ is called

$$\textit{right-invertible} \quad :\Longleftrightarrow \quad \forall\, y \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^p)\; \exists\, (x, u) \in \mathcal{C}(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m): \; (x, u, y) \in \mathfrak{B}_{(1.1)}.$$

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Theorem 2.1 *Let $[E,A,B,C] \in \Sigma_{l,n,m}$ be right-invertible and have autonomous $\mathcal{ZD}_{(1.1)}$. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in* (1.2) *exists. Then there exists $T \in \mathbf{Gl}_n(\mathbb{R})$ such that, for any $(x,u,y) \in \mathfrak{B}_{(1.1)} \cap \left(\mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}^0(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^m)\right)$ and $Tx = (x_1^\top, y^\top, x_3^\top)^\top \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^{k+m+n_3})$, (Tx,u,y) solves*

$$\dot{x}_1(t) = Q\,x_1(t) + A_{12}\,y(t), \qquad \Gamma\,\dot{y}(t) = A_{22}y(t) + \Psi(x_1(0),y)(t) + u(t), \qquad x_3(t) = \sum_{k=0}^{\nu-1} N^k E_{32}\,y^{(k+1)}(t),$$

where $N \in \mathbb{R}^{n_3 \times n_3}$, $n_3 = n-k-m$, is nilpotent with $N^\nu = 0$ and $N^{\nu-1} \neq 0$, $\nu \in \mathbb{N}$, $Q \in \mathbb{R}^{k\times k}$, $E_{22}, A_{22} \in \mathbb{R}^{m\times m}$ and all other matrices are of appropriate sizes. Furthermore, $\Psi : \mathbb{R}^k \times \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^m) \to \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^m)$ is defined by $\Psi(x_1^0, y(\cdot))(t) = A_{21}e^{Qt}x_1^0 + \int_0^t A_{21}e^{Q(t-\tau)}A_{12}y(\tau)\,\mathrm{d}\tau$ and, if $\mathcal{ZD}_{(1.1)}$ are asymptotically stable, then $\sigma(Q) \subseteq \mathbb{C}_-$.

Now, we consider funnel control for systems $[E,A,B,C] \in \Sigma_{l,n,m}$. The aim is to achieve tracking of a reference trajectory by the output signal with prescribed transient behavior. We use the notation $\mathcal{B}^\ell(\mathbb{R}_{\geq 0};\mathbb{R}^n) = \{ f \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0};\mathbb{R}^n) \,\big|\, \frac{\mathrm{d}^i}{\mathrm{d}t^i} f \text{ is bounded for } i = 0,\ldots,\ell \}$. For any function φ belonging to

$$\Phi^\mu := \left\{ \varphi \in \mathcal{C}^\mu(\mathbb{R}_{\geq 0};\mathbb{R}) \cap \mathcal{B}^1(\mathbb{R}_{\geq 0};\mathbb{R}) \;\middle|\; \varphi(0) = 0,\ \ \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \liminf_{s\to\infty} \varphi(s) > 0 \right\}$$

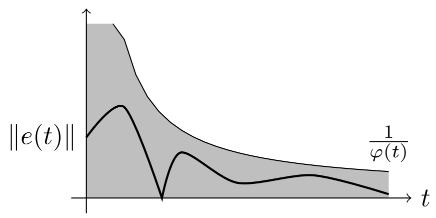

Fig. 1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$.

for $\mu \in \mathbb{N}$, we associate the *performance funnel* $\mathcal{F}_\varphi := \left\{ (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \,\middle|\, \varphi(t)\|e\| < 1 \right\}$, see Figure 1. The control objective is feedback control so that the tracking error $e = y - y_{\mathrm{ref}}$, where y_{ref} is the reference signal, evolves within $\mathcal{F}_\varphi$ and all variables are bounded. To ensure this, we introduce, for $\hat{k} > 0$, the *funnel controller*:

$$u(t) = -k(t)\,e(t), \quad k(t) = \frac{\hat{k}}{1 - \varphi(t)^2\|e(t)\|^2}\,. \tag{2.1}$$

If we assume asymptotically stable zero dynamics, we see intuitively that, in order to maintain the error evolution within the funnel, high gain values may only be required if the norm $\|e(t)\|$ of the error is close to the funnel boundary $\varphi(t)^{-1}$: $k(\cdot)$ increases if necessary to exploit the high-gain property of the system and decreases if a high gain is not necessary. This intuition underpins the choice of the gain $k(t)$ in (2.1). The control design (2.1) has two advantages: $k(\cdot)$ is non-monotone and (2.1) is a static time-varying proportional output feedback of striking simplicity.

In [4, 5] funnel control has been proved to work for two important classes of DAE systems. We generalize these results in the following.

Theorem 2.2 *Let $[E,A,B,C] \in \Sigma_{l,n,m}$ be right-invertible and have asymptotically stable zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in* (1.2) *exists and satisfies $\Gamma = \Gamma^\top \geq 0$. Using the notation from Theorem 2.1, let $\varphi \in \Phi^{\nu+1}$ define a performance funnel $\mathcal{F}_\varphi$. Then, for any reference signal $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$, any consistent initial value $x^0 \in \mathbb{R}^n$, and initial gain $\hat{k} > \left\| \lim_{s\to\infty} \left([0, I_m] L(s) [0, I_m]^\top + s\Gamma \right) \right\|$, the application of the funnel controller* (2.1) *to* (1.1) *yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution $x(\cdot)$, $x(\cdot)$ is bounded and the corresponding tracking error $e(\cdot) = Cx(\cdot) - y_{\mathrm{ref}}(\cdot)$ evolves uniformly within the performance funnel $\mathcal{F}_\varphi$; more precisely,*

$$\exists\, \varepsilon > 0\ \forall\, t > 0 : \ \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon\,.$$

The assumption of asymptotically stable zero dynamics is characterized in [6] via stabilizability, detectability and the location of transmission zeros of the system.

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