

On perturbations in the leading coefficient matrix of time-varying index-1 DAEs

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Time-varying index-1 DAEs and the effect of perturbations in the leading coefficient matrix is investigated. An appropriate class of allowable perturbations is introduced. Robustness of exponential stability with respect to a certain class of perturbations is shown in terms of the Bohl exponent and perturbation operator. Finally, a stability radius involving these perturbations is introduced and investigated. In particular, a lower bound for the stability radius is derived.

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1 Introduction

We study exponential stability and its robustness for time-varying linear differential-algebraic equations (DAEs) of the form

$$E(t)\dot{x} = A(t)x, \tag{1.1}$$

where $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$, $n \in \mathbb{N}$. For brevity, we identify the tuple (E, A) with the DAE (1.1).

DAEs have been discovered to be the appropriate tool for modeling a vast variety of problems in mechanical systems, multibody dynamics, electrical networks, fluid mechanics and chemical engineering [1–5], which often cannot be modeled by standard ordinary differential equations (ODEs).

Index-1 DAEs¹ are relevant in a lot of applications, as the occurring DAEs are very often of index-1. For instance, it is shown in [6] that any passive electrical circuit containing nonlinear and possibly time-varying elements has index less than or equal to two - and the index-2 case is exceptional. Furthermore, so called hybrid models of electrical circuits are always index-1 [7, 8]. Therefore, our approach to index-1 DAEs has a wide area of applications e.g. in electrical engineering, as linear DAEs (E, A) arise as linearizations of nonlinear DAEs $F(t, x, \dot{x}) = 0$ along trajectories [9].

It is the first aim of this paper to introduce a class of allowable perturbations in the leading coefficient and then prove robustness of exponential stability with respect to these perturbations using the Bohl exponent and perturbation operator. The second aim is to introduce a stability radius for time-varying DAEs. The stability radius defined in [10, 11] is defined only with respect to perturbations in the coefficients of A . On the other hand, [12] give a definition for the stability radius involving perturbations in E for time-invariant DAEs. Our definition of the stability radius can be viewed as both: a generalization of the definition given in [12] to time-varying systems and a generalization of the definition given in [10, 11] to a larger set of allowable perturbations with respect to the leading coefficient. We then investigate this stability radius and in particular prove a lower bound. As far as the author is aware, these results are even new for time-invariant DAE systems.

For the proofs and more details on the results in this paper see [13].

2 Index-1 DAEs

In order to define the index-1 property of a DAE (E, A) we introduce the set $\Omega_{E,A}$ of special projector functions as follows.

Definition 2.1 Let $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ be given. Define

$$\Omega_{E,A} := \left\{ Q \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n}) \mid \begin{array}{l} \forall t \geq 0 : Q(t)^2 = Q(t) \wedge \ker E(t) = \text{im } Q(t), \\ E + (E\dot{Q} - A)Q \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbf{G}\mathbf{1}_n(\mathbb{R})) \end{array} \right\}.$$

The following definition of index-1 DAEs coincides with the definition of *index-1 tractability* in [14].

Definition 2.2 The DAE $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ is called *index-1* if, and only if, $\Omega_{E,A} \neq \emptyset$.

As proved in [13] we have, for any $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$,

$$\Omega_{E,A} \neq \emptyset \implies \Omega_{E,A} = \left\{ Q \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n}) \mid \forall t \geq 0 : Q(t)^2 = Q(t) \wedge \ker E(t) = \text{im } Q(t) \right\},$$

which motivates the following algorithm for checking the index-1 property of given real-analytic (E, A) and calculating a corresponding projector $Q \in \Omega_{E,A}$. Algorithm 1 terminates after finitely many steps with either “DAE is not index-1!” or it returns a real-analytic matrix $Q \in \Omega_{E,A}$. The proof for correctness of the algorithm can be found in [13].

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¹ Note that in this article the term “index-1” includes all (implicit) ODE systems, that is all DAE systems which are also referred to as index-0.

Algorithm 1 Calculation of $Q \in \mathcal{Q}_{E,A}$

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1: function  $Q = \mathbf{getQ}(E, A)$ 
2: determine minimal  $r \leq n := \text{size}(E)$  s.t.  $\text{rk } E(t) \leq r$  for all  $t \geq 0$  and real analytic, pointwise nonsingular  $\tilde{Q} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n \times (n-r)}$  s.t.  $E\tilde{Q} = 0$ ;
3: if not( $\forall t \geq 0 : \text{rk } E(t) = r$ ) then
4:   print "DAE is not index-1!" STOP
5: end if
6:  $Q := \tilde{Q}(\tilde{Q}^\top \tilde{Q})^{-1} \tilde{Q}^\top$ ;
7: if  $E + (E\dot{Q} - A)Q \notin \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbf{GL}_n(\mathbb{R}))$  then
8:   print "DAE is not index-1!" STOP
9: end if

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For index-1 $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ and $Q \in \mathcal{Q}_{E,A}$ define

$$P := I - Q, \quad \bar{A} := A - E\dot{Q}, \quad G := E + (E\dot{Q} - A)Q = E - \bar{A}Q. \quad (2.1)$$

Then the DAE (E, A) satisfies:

$$E\dot{x} = Ax \Leftrightarrow E \frac{d}{dt}(Px) = (A + E\dot{P})x \Leftrightarrow (E - \bar{A}Q)(P \frac{d}{dt}(Px) + Qx) = \bar{A}Px \Leftrightarrow P \frac{d}{dt}(Px) = (G^{-1}\bar{A}P - Q)x.$$

It can be shown [13] that the last equation is equivalent to

$$\begin{cases} \frac{d}{dt}(P(t)x) &= (\dot{P}(t) + P(t)G(t)^{-1}\bar{A}(t))P(t)x, \\ Q(t)x &= Q(t)G(t)^{-1}\bar{A}(t)P(t)x. \end{cases} \quad (2.2)$$

It can be seen from (2.2) that, roughly speaking, the solutions of the index-1 DAE (E, A) can be calculated by solving an ODE for Px and then Qx (and therefore x) is given in terms of Px . Therefore, all solutions of the DAE (1.1) are fully determined by the solutions of the ODE (first equation) in (2.2).

It follows from (2.2) that any solution x of (1.1) can be written as $x = Px + Qx = Px + QG^{-1}\bar{A}Px = (I + QG^{-1}\bar{A})Px$. If we are now given the initial condition $x(t_0) = x^0 \in \mathbb{R}^n$, we may observe that $x(t_0) = (I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0)x^0$, and since $R(t_0) := (I + Q(t_0)G(t_0)^{-1}\bar{A}(t_0))P(t_0)$ is idempotent we find that $x(t_0) = x^0$ if, and only if, $x^0 \in \text{im } R(t_0)$. We may further deduce [13] that for $x^0 \in \text{im } R(t_0)$ we have $x(t_0) = x^0$ if, and only if, $E(t_0)(x(t_0) - x^0) = 0$. Thus, we consider initial conditions of the type

$$E(t_0)(x(t_0) - x^0) = 0, \quad t_0 \geq 0, \quad x^0 \in \mathbb{R}^n. \quad (2.3)$$

It is now important that initial value problems (1.1), (2.3) may also be considered for arbitrary $x^0 \in \mathbb{R}^n$ and that it can be shown [13]: The initial value problem (1.1), (2.3) does always have a unique solution.

We use the generalized initial condition (2.3) and the unique solvability of (1.1), (2.3) for all $x^0 \in \mathbb{R}^n$ to define a transition matrix for (1.1). As in the case of ODEs, we may assign the solutions to $x^0 = e_i$ as the columns of a matrix $\Phi(t, t_0)$, that is $\Phi(\cdot, t_0)$ is the unique solution of $E(t) \frac{d}{dt} \Phi(t, t_0) = A(t)\Phi(t, t_0)$, $E(t_0)(\Phi(t_0, t_0) - I) = 0$.

3 Class of allowable perturbations

We consider perturbations of the matrix-valued function E , i.e., for given $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ and perturbation $\Delta_E \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})$ the perturbed system $(E + \Delta_E, A)$. As exponential stability is very sensitive with respect to arbitrary perturbations in the leading term [12] (even in the time-invariant index-1 case) we do not allow for general perturbations Δ_E , but restrict ourselves to the class of perturbations defined in the following.

Definition 3.1 Let $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ be index-1. Then the *set of allowable perturbations* is defined by

$$\mathcal{P}_{E,A} := \left\{ \Delta_E \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n}) \mid \begin{array}{l} \forall t \geq 0 : \ker E(t) = \ker(E(t) + \Delta_E(t)), \\ \exists Q \in \mathcal{Q}_{E,A} : G + \Delta_E(I + \dot{Q}Q) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbf{GL}_n(\mathbb{R})) \text{ for } G \text{ as in (2.1)} \end{array} \right\}$$

Remark 3.2 The definition of the set $\mathcal{P}_{E,A}$ may seem restrictive, in particular the claim for the kernel of E to be preserved, however, perturbations of the algebraic part are still possible. Furthermore, it is usually assumed in the perturbation theory of DAEs that the leading coefficient E is not perturbed at all, see e.g. [10, 11, 15, 16]. Moreover, the condition on perturbations of the leading term to preserve some kernel is not uncommon, as in [12], where time-invariant systems are considered, it is assumed that the left kernel of E is preserved under the perturbation (see proof of [12, Lem. 3.2]). Furthermore, as argued in [12], in practical applications the set of allowable perturbations is limited anyway, restricted by the physical structure of the

considered system. Therefore, as it is widely believed, if the algebraic part of the DAE represents path constraints, then the zero blocks in E are structural and are not subject to disturbances or uncertainties. However, this is not entirely true as it can be deduced from considering a DAE in semi-explicit form:

$$E(t)\dot{x} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A(t)x. \quad (3.1)$$

Equation (3.1) consists of n_1 differential equations and $n_2 = n - n_1$ algebraic constraints. Changing any of the zeros in the second column of E would involve derivatives of x_2 and therefore inevitably change the structure of the system - so these zero blocks are structural. However, the zero block in the lower left corner is not. If we change this block to E_{21} for instance, then the second equation now reads $E_{21}(t)\dot{x}_1 = A_{21}(t)x_1 + A_{22}(t)x_2$ and incorporating the first equation gives

$$0 = (A_{21} - E_{21}A_{11})(t)x_1 + (A_{22} - E_{21}A_{12})(t)x_2,$$

so the system has again the same structure as before. This shows that we have to distinguish between perturbations which change the structure of the system and perturbations which change the structure of the matrices E and A . What is desired is that the structure of the system is preserved under perturbations and indeed, in the above example, changing the lower left block in E does not change the kernel of E . This shows that for semi-explicit DAEs, the perturbations which preserve the kernel of E (and may change anything else) are those which preserve the (physical) structure of the system.

4 Bohl exponent

The *Bohl exponent* [17] of an index-1 DAE $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ with transition matrix $\Phi(\cdot, \cdot)$ is defined as

$$k_B(E, A) = \inf \left\{ \rho \in \mathbb{R} \mid \exists N_\rho > 0 \forall t \geq s \geq 0 : \|\Phi(t, s)\| \leq N_\rho e^{\rho(t-s)} \right\}.$$

(E, A) is called *exponentially stable* if, and only if, there exist $\mu, M > 0$ such that for all initial times $t_0 \geq 0$ and all $t \geq t_0$ it holds $\|\Phi(t, t_0)\| \leq M e^{-\mu(t-t_0)}$. It is easy to observe that $k_B(E, A) < 0$ if, and only if, (E, A) is exponentially stable. We have the following theorem on robustness of the Bohl exponent under perturbations which preserve the kernel of E .

Theorem 4.1 *Let $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ be index-1 and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Let P, \bar{A}, G be as in (2.1). Suppose that \dot{P}, G^{-1} and $(I - \dot{P}Q)G^{-1}\bar{A}$ are bounded and that $k_B(E, A) \neq -\infty$. Then, for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all $\Delta_E \in \mathcal{P}_{E,A}$ which satisfy $\|\Delta_E\|_\infty < \delta$ it holds that*

$$k_B(E + \Delta_E, A) \leq k_B(E, A) + \varepsilon.$$

The proof of Theorem 4.1 can be found in [13].

As a system (E, A) is exponentially stable if, and only if, its Bohl exponent is negative, Theorem 4.1 does also state that exponential stability of index-1 DAEs is robust with respect to perturbations in $\mathcal{P}_{E,A}$. However, Theorem 4.1 does only state that the perturbation has to be sufficiently small in order to preserve exponential stability. In the following section we provide a calculable upper bound on the perturbation such that exponential stability is preserved.

5 Perturbation operator

The *perturbation operator* (see [18] for ODEs) is defined as

$$L_{t_0} : L^2([t_0, \infty); \mathbb{R}^n) \rightarrow L^2([t_0, \infty); \mathbb{R}^n), f(\cdot) \mapsto x(\cdot), \text{ where } x(\cdot) \text{ solves } E(t)\dot{x} = A(t)x + f(t), E(t_0)x(t_0) = 0.$$

Lemma 5.1 *Let $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ be index-1 and exponentially stable. Let $Q \in \mathfrak{Q}_{E,A}$, P, G be as in (2.1), and suppose that PG^{-1} and QG^{-1} are bounded. Then we have:*

(i) *For all $t_0 \geq 0$ the operator L_{t_0} is bounded.*

(ii) *$t_0 \mapsto \|L_{t_0}\|$ is monotonically nonincreasing on $\mathbb{R}_{\geq 0}$, i.e., $\|L_{t_0}\| \geq \|L_{t_1}\|$ for $0 \leq t_0 \leq t_1$.*

The proof of Lemma 5.1 can be found in [10, 11]. We show now that a calculable bound on the perturbation such that exponential stability is preserved can be given in terms of the inverse norm of the perturbation operator.

Theorem 5.2 *Let $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ be index-1 and exponentially stable and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Let P, \bar{A}, G be as in (2.1) and suppose that \dot{P}, G^{-1} and $G^{-1}\bar{A}$ are bounded. Furthermore, let $\Delta_E \in \mathcal{P}_{E,A}$ be bounded and such that $\|\Delta_E(t)\| < \|(I - \dot{P}(t)Q(t))G(t)^{-1}\|^{-1}$ for all $t \geq 0$. Set $\kappa_1 := \|P\|_\infty(\|\dot{P}\|_\infty + \|(I - \dot{P}Q)G^{-1}\bar{A}\|_\infty) \geq 0$ and $\kappa_2 := \|(I - \dot{P}Q)G^{-1}\|_\infty > 0$. If*

$$\lim_{t_0 \rightarrow \infty} \left\| \Delta_E|_{[t_0, \infty)} \right\|_\infty < \frac{\alpha}{\kappa_1 + \kappa_2 \alpha}, \quad \text{where } \alpha = \min \left\{ \lim_{t_0 \rightarrow \infty} \|L_{t_0}\|^{-1}, \|QG^{-1}\|_\infty^{-1} \right\},$$

then the perturbed system $(E + \Delta_E, A)$ is exponentially stable.

The proof of Theorem 5.2 can be found in [13].

6 Stability radius

The bound on the perturbation obtained in Theorem 5.2 now rises the question for the distance to instability. To this end, we introduce and investigate a *stability radius* (introduced in [19] for ODEs) in this section. In order to define the stability radius we introduce, for given index-1 $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$, the following sets, where $\mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n})$ denotes the set of all continuous and bounded functions $f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n \times 2n}$:

$$\begin{aligned} \mathcal{I}(E, A) &:= \{ [\Delta_E, \Delta_A] \in \mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n}) \mid (E + \Delta_E, A + \Delta_A) \text{ is index-1} \}, \\ \mathcal{K}(E, A) &:= \{ [\Delta_E, \Delta_A] \in \mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n}) \mid \forall t \geq 0 : \ker E(t) = \ker(E(t) + \Delta_E(t)) \}, \\ \mathcal{S} &:= \{ (E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2 \mid (E, A) \text{ is exponentially stable} \}. \end{aligned}$$

Definition 6.1 Let $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ be index-1. Then the *stability radius* of (E, A) is the number

$$r(E, A) := \inf \{ \|\Delta_E, \Delta_A\|_{\infty} \mid [\Delta_E, \Delta_A] \in \mathcal{K}(E, A) \wedge ([\Delta_E, \Delta_A] \notin \mathcal{I}(E, A) \vee (E + \Delta_E, A + \Delta_A) \notin \mathcal{S}) \}.$$

For more comments on and properties of the stability radius see [13]. The following theorem provides a lower bound on the stability radius in terms of the inverse norm of the perturbation operator.

Theorem 6.2 Let $(E, A) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})^2$ be index-1 and exponentially stable and suppose that there exists a bounded $Q \in \mathfrak{Q}_{E,A}$. Let P, \bar{A}, G be as in (2.1) and suppose that \dot{P}, G^{-1} and $G^{-1}\bar{A}$ are bounded. Set $\kappa_1 := \|P\|_{\infty} \left(\left\| \begin{bmatrix} \dot{P} \\ I \end{bmatrix} \right\|_{\infty} + \left\| \begin{bmatrix} (I - \dot{P}Q)G^{-1}\bar{A} \\ -QG^{-1}\bar{A} \end{bmatrix} \right\|_{\infty} \right) \geq 0$ and $\kappa_2 := \left\| \begin{bmatrix} (I - \dot{P}Q)G^{-1} \\ -QG^{-1} \end{bmatrix} \right\|_{\infty} > 0$. Then

$$\frac{\alpha}{\kappa_1 + \kappa_2 \alpha} \leq r(E, A), \quad \text{where } \alpha = \min \left\{ \lim_{t_0 \rightarrow \infty} \|L_{t_0}\|^{-1}, \|QG^{-1}\|_{\infty}^{-1} \right\}.$$

The proof of Theorem 6.2 can be found in [13].

While Theorem 6.2 already provides a robustness result, we may also deduce from it that, roughly speaking, the set of exponentially stable index-1 systems is open.

Corollary 6.3 Let $Q \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times n})$ such that Q and \dot{Q} are bounded and $Q(t)^2 = Q(t)$ for all $t \geq 0$. Define

$$\mathcal{K}_Q := \left\{ [E, A] \in \mathcal{B}(\mathbb{R}_{\geq 0}; \mathbb{R}^{n \times 2n}) \mid Q \in \mathfrak{Q}_{E,A} \text{ and } F := (E - (A - E\dot{Q})Q)^{-1} \text{ and } F(A - E\dot{Q}) \text{ are bounded} \right\}$$

and $\mathcal{S}_Q := \mathcal{K}_Q \cap \mathcal{S}$. Then \mathcal{S}_Q is open in $\overline{\mathcal{K}_Q}$.

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