ON AN EXTENSION OF A GLOBAL IMPLICIT FUNCTION THEOREM

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Abstract. We study the existence of global implicit functions for equations defined on open subsets of Banach spaces. The partial derivative with respect to the second variable is only required to have a left inverse instead of being invertible. Generalizing known results, we provide sufficient criteria which are easy to check. These conditions essentially rely on the existence of diffeomorphisms between the respective projections of the set of zeros and appropriate Banach spaces, as well as a corresponding growth bound. The projections further allow to consider cases where the global implicit function is not defined on all of the open subset corresponding to the first variable.

1. Introduction

When dealing with nonlinear dynamical systems with constraints, i.e., implicit differential equations of the form

$$F\big(x(t), \dot{x}(t)\big) = 0,$$

where the number of equations does not match the number of variables, it is often necessary to solve this equation for $\dot{x}(t)$, preferably globally in the form $\dot{x}(t) = g\big(x(t)\big)$. This means to find a global implicit function of the equation $F(x, y) = 0$. Numerous results on global implicit function theorems exist, and we mention the relevant literature. However, most results involve conditions which are not easy to check in practice. In the present paper, we provide a novel extension of the global implicit function theorem under conditions which can easily be verified.

In the following we summarize some results on global implicit functions, tailored to be applicable in our framework. We consider equations of the form $F(x, y) = 0$ for which we want to find a unique maximal solution $y(x)$. There are several approaches available in the literature which provide a solution to this problem, see e.g. [10] for an early result. Most works concentrate on the case that the partial derivative $\frac{\partial F}{\partial y}(x, y)$ is invertible for all (x, y), i.e., $F(x, y) = 0$ is locally solvable for $y(x)$ in a neighborhood of every point (a, b) such that $F(a, b) = 0$. We discuss some important work:

- For $F : X \times Y \to \mathbb{R}^l$, where $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$ are open and X is convex, Sandberg [11] provides necessary and sufficient

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conditions for the existence of a unique $g \in \mathcal{C}(X, Y)$ such that $F^{-1}(0) = \{ (x, g(x)) \mid x \in X \}$. However, the conditions are not easy to check; in particular, it needs to be guaranteed that

$$\text{for some } x_0 \in X \text{ there exists exactly one } y_0 \in Y \text{ such that } F(x_0, y_0) = 0. \tag{1.1}$$

Furthermore, the result of Sandberg is not applicable in the case that the maximal solution g is not defined on all of X.

- Using the theory of covering maps, Ichiraku [6] improves the characterization of Sandberg. Nevertheless, the condition (1.1) is still present and the results are only applicable in the case of globally defined g. However, in [6, Thm. 5] it is shown that in the case $X = \mathbb{R}^m$, $Y = \mathbb{R}^n$ and $l = m$ for the existence of a unique solution $g \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$ it is sufficient that $\frac{\partial F}{\partial y}(x, y)$ is invertible for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, condition (1.1) holds and
$$\forall (x, y) \in F^{-1}(0): \quad \left\| \left(\frac{\partial F}{\partial y}(x, y) \right)^{-1} \right\| \cdot \left\| \frac{\partial F}{\partial x}(x, y) \right\| \le M \tag{1.2}$$
for some $M \ge 0$.
- The above result of Ichiraku has in turn be improved by Gutú and Jaramillo [5, Cor. 5.3], who showed that the condition (1.1) can be replaced by the intuitive condition "$F^{-1}(0)$ is connected" and in the condition (1.2) the constant M can be replaced by the term $\omega(\|y\|)$, where $\omega : [0, \infty) \to (0, \infty)$ is a continuous *weight*, which means that ω is nondecreasing and
$$\int_0^\infty \frac{\mathrm{d}t}{\omega(t)} = \infty.$$
These conditions are indeed easy to check. The only drawback is that F needs to be defined on all of $\mathbb{R}^m \times \mathbb{R}^n$ and the solution g is defined on all of $\mathbb{R}^m$.
- A result which is similar to that of Gutú and Jaramillo, but holds for some $X \subseteq \mathbb{R}^m$ which is open, connected and starlike with respect to some $a \in X$ such that $F(a, b) = 0$ for some $b \in Y = \mathbb{R}^n$, has been derived by Cristea [2]. The assumption of connectedness of $F^{-1}(0)$ is not needed, however a version of assumption (1.2) (with $M = \omega(\|y\|)$) is required to hold on all of $X \times \mathbb{R}^n$.
- Yet another approach has been pursued by Zhang and Ge [12] who show that for existence of a unique solution $g \in \mathcal{C}(\mathbb{R}^m, \mathbb{R}^n)$ it is sufficient that the element-wise absolute value of $\frac{\partial F}{\partial y}$ is uniformly strictly diagonally dominant in the sense that there exists $d > 0$ such that
$$\left| \left(\frac{\partial F}{\partial y}(x, y) \right)_{ii} \right| - \sum_{j \ne i} \left| \left(\frac{\partial F}{\partial y}(x, y) \right)_{ij} \right| \ge d$$

for all $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ and all $i = 1, \dots, n$. While this condition is easy to check, it is very restrictive as it already excludes a lot of linear equations $Ax + By = 0$ where B is not strictly diagonally dominant, but invertible.

As discussed above, typical limitations of the approaches are that F needs to be defined on all of $\mathbb{R}^m \times \mathbb{R}^n$ or the solution g is required to be globally defined. In [1] these limitations are resolved as X and Y are assumed to be open and X is connected, and maximal solutions of $F(x, y) = 0$ are considered in every connected component of $F^{-1}(0)$. Assuming that $Z := F^{-1}(0)$ is connected we may then find a solution $g \in \mathcal{C}(\pi_1(Z), Y)$, where $\pi_1 : X \times Y \to X,\ (x, y) \mapsto x$ is the projection onto the first component, provided that $\pi_1(Z)$ is open and simply connected and $\pi_1 : Z \to \pi_1(Z)$ "lifts lines" (for a precise definition see [9, Def. 1.1]). This result can be extended in a straightforward way to the case where $l \geq m$ and $\operatorname{rk} \frac{\partial F}{\partial y}(x, y) = n$ for all $(x, y) \in X \times Y$ since it is only necessary to show that π_1 is locally a homeomorphism, which replaces the condition that $F(x, y) = 0$ is locally solvable for $y(x)$ as in [1, Thm. 4]; then [1, Lem. 1] can still be applied to $\pi_1 : Z \to \pi_1(Z)$. The drawback of this result is that the condition "$\pi_1 : Z \to \pi_1(Z)$ lifts lines" is not easy to check.

In the present paper, we provide a generalization of [5, Cor. 5.3] to the case of functions defined only on open subsets and where the partial derivative $\frac{\partial F}{\partial y}$ is only required to have a left inverse instead of being invertible. The crucial assumption is that the projections $\pi_i(Z)$ on the ith component, $i = 1, 2$, are diffeomorphic to some Banach spaces and the transformation of the equation $F(x, y) = 0$ satisfies a generalized version of (1.2). We stress that this assumption in particular implies that $\pi_i(Z)$ must be open and simply connected. The main result is presented in Section 2 and a discussion together with some illustrative examples is given in Section 3.

2. Main result

In this section we state and prove the following main result of the paper.

Theorem 2.1. *Let $X \subseteq \mathcal{U},\ Y \subseteq \mathcal{V}$ be open sets, $\mathcal{U}, \mathcal{V}, \mathcal{Z}$ be Banach spaces, $F \in \mathcal{C}^1(X \times Y, \mathcal{Z})$ and*

$$Z \subseteq \{\ (x, y) \in X \times Y \ \mid F(x, y) = 0\ \}$$

be such that

(i) *Z is path-connected and closed in $X \times Y$;*
(ii) $\forall\, (x, y) \in Z\ \exists\, S(x, y) \in \mathcal{L}(\mathcal{Z}, \mathcal{V}) :\ S(x, y) D_y F(x, y) = \mathrm{id}_{\mathcal{V}}$;[1]
(iii) *for the projections $\pi_i(p_1, p_2) = p_i$ with $i \in \{1, 2\}$, $(p_1, p_2) \in \mathcal{U} \times \mathcal{V}$, there exist diffeomorphisms $\phi : \pi_1(Z) \to \mathcal{X}$, $\psi : \pi_2(Z) \to \mathcal{Y}$ for some*

[1]Here $\mathcal{L}(\mathcal{Z}, \mathcal{V})$ denotes the Banach space of all bounded linear operators $A : \mathcal{Z} \to \mathcal{V}$ and $\mathrm{id}_{\mathcal{V}} : \mathcal{V} \to \mathcal{V},\ v \mapsto v$ is the identity operator on $\mathcal{V}$.

Banach spaces $\mathcal{X},\mathcal{Y}$, and a continuous weight $\omega : [0,\infty) \to (0,\infty)$ such that for all $(x,y) \in Z$ we have

$$\|D\psi(y) \cdot S(x,y)\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \cdot \left\| D_x F(x,y) \cdot \left(D\phi(x)\right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Z})} \leq \omega(\|\psi(y)\|_{\mathcal{Y}}).$$

Then there exists a unique $g \in \mathcal{C}(\pi_1(Z), Y)$ such that

$$\{\ (x, g(x)) \ \mid x \in \pi_1(Z)\ \} = Z,$$

and g is Fréchet-differentiable at every $x \in \pi_1(Z)$.

The proof of Theorem 2.1 requires us to recall the following concepts, which can be found in [5, pp. 77–80].

Definition 2.2. Let Z be a metric space, and let $\mathcal{P}$ be a family of continuous paths in Z. We say that Z is *$\mathcal{P}$-connected*, if the following conditions hold:

(1) If the path $p : [a,b] \to Z$ belongs to $\mathcal{P}$, then the reverse path $\overline{p}$, defined by $\overline{p}(t) = p(a-t+b)$, also belongs to $\mathcal{P}$;
(2) Every two points in Z can be joined by a path in $\mathcal{P}$.

We say that Z is *locally $\mathcal{P}$-contractible* if every point $z_0 \in Z$ has an open neighborhood U which is *$\mathcal{P}$-contractible*, in the sense that there exists a homotopy $H : U \times [0,1] \to U$ between the constant function $U \ni z \mapsto z_0$ and the identity id_U, which satisfies

(a) $H(z_0,t) = z_0$, for all $t \in [0,1]$,
(b) for every $z \in U$, the path $p_z := H(z,t)$ belongs to $\mathcal{P}$.

Further, let Z' also be a metric space and $p : [0,1] \to Z'$ be a path in Z'. We say that a continuous map $f : Z \to Z'$ has the *continuation property for* p, if for every $b \in (0,1]$ and every path $q \in \mathcal{C}([0,b), Z)$ such that $f \circ q = p\restriction_{[0,b)}$, there exists a sequence $\{t_n\}$ in $[0,b)$ convergent to b and such that $\{q(t_n)\}$ converges in Z. Furthermore, a continuous map $f : Z \to Z'$ is called a *covering map*, if every $z' \in Z'$ has an open neighborhood U such that $f^{-1}(U)$ is the disjoint union of open subsets of Z each of which is mapped homeomorphically into U by f.

Proof of Theorem 2.1. We proceed in several steps.

Step 1: We first reduce the original problem to a simpler case. By the existence of ϕ,ψ in assumption (iii) it follows that, for $i \in \{1,2\}$, $\pi_i(Z)$ are open sets in $\mathcal{U},\mathcal{V}$, resp., and since $Z \subseteq \pi_1(Z) \times \pi_2(Z)$, it is no loss of generality to assume $X \times Y = \pi_1(Z) \times \pi_2(Z)$. That is, we search for an implicit function for the restriction $F : \pi_1(Z) \times \pi_2(Z) \to \mathcal{Z}$ instead of $F : X \times Y \to \mathcal{Z}$. Next, we argue that it suffices to prove the theorem for cases in which (i)–(iii) are satisfied with $\phi = \mathrm{id}_{\mathcal{X}}$ and $\psi = \mathrm{id}_{\mathcal{Y}}$. Note that these assumptions imply $\mathcal{U} = X = \pi_1(Z) = \mathcal{X}$ and $\mathcal{V} = Y = \pi_2(Z) = \mathcal{Y}$ since X and Y are open subspaces of $\mathcal{U}$ and $\mathcal{V}$, resp. Having proved this case, we can conclude the general case by considering the function $\tilde{F} = F \circ (\phi^{-1}, \psi^{-1})$ with $\tilde{F} : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$. Next we translate the conditions (i)–(iii) on F to conditions on $\tilde{F}$ for $\tilde{Z} := (\phi,\psi)(Z)$.

(i)' We have that $\tilde{Z}$ is path-connected and closed in $\mathcal{X} \times \mathcal{Y}$ if, and only if, Z is path-connected and closed in $X \times Y$.

(ii)' Define

$$\tilde{S} : \tilde{Z} \to \mathcal{Y},\ (\tilde{x}, \tilde{y}) \mapsto \big(D(\psi^{-1})(\tilde{y})\big)^{-1} \cdot S(\phi^{-1}(\tilde{x}), \psi^{-1}(\tilde{y})).$$

With the identification $(x, y) = (\phi^{-1}(\tilde{x}), \psi^{-1}(\tilde{y}))$ we obtain for all $(x, y) \in Z$ that

$$\begin{aligned} D_y F(x, y) &= D_y\big(\tilde{F}(\phi(x), \psi(y))\big) \\ &= (D_{\tilde{y}}\tilde{F})(\phi(x), \psi(y)) \cdot D\psi(y) \\ &= D_{\tilde{y}}\tilde{F}(\tilde{x}, \tilde{y}) \big(D(\psi^{-1})(\tilde{y})\big)^{-1}, \end{aligned}$$

where the latter equality is a consequence of the inverse function theorem. Using this we find that

$$\begin{aligned} & S(x, y) D_y F(x, y) = \mathrm{id}_{\mathcal{Y}} \\ \iff\ & S\big(\phi^{-1}(\tilde{x}), \psi^{-1}(\tilde{y})\big) \cdot \Big(D_{\tilde{y}}\tilde{F}\Big)\big(\tilde{x}, \tilde{y}\big) \cdot \big(D(\psi^{-1})(\tilde{y})\big)^{-1} = \mathrm{id}_{\mathcal{Y}} \\ \iff\ & \underbrace{\big(D(\psi^{-1})(\tilde{y})\big)^{-1} \cdot S\big(\phi^{-1}(\tilde{x}), \psi^{-1}(\tilde{y})\big)}_{=\tilde{S}(\tilde{x}, \tilde{y})} \cdot D_{\tilde{y}}\tilde{F}(\tilde{x}, \tilde{y}) = \mathrm{id}_{\mathcal{Y}}. \end{aligned}$$

(iii)' Similar to the computations above we obtain that, for $(x, y) \in Z$,

$$D_{\tilde{x}}\tilde{F}(\tilde{x}, \tilde{y}) = D_x F(x, y) \cdot \big(D\phi(x)\big)^{-1}.$$

Therefore, we have for all continuous weights $\omega : [0, \infty) \to (0, \infty)$ and all $(x, y) \in Z$ that, omitting the spaces in the subscripts of the norms,

$$\begin{aligned} & \|D\psi(y) \cdot S(x, y)\| \cdot \Big\|D_x F(x, y) \cdot \big(D\phi(x)\big)^{-1}\Big\| \leq \omega(\|\psi(y)\|) \\ \iff\ & \Big\|\tilde{S}(\tilde{x}, \tilde{y})\Big\| \cdot \Big\|D_{\tilde{x}}\tilde{F}(\tilde{x}, \tilde{y})\Big\| \leq \omega(\|\tilde{y}\|). \end{aligned}$$

Now, define the projections $\tilde{\pi}_i(p_1, p_2) = p_i$ with $i \in \{1, 2\}$, $(p_1, p_2) \in \mathcal{X} \times \mathcal{Y}$. We recapitulate the situation with the following commuting diagram:

$$\begin{array}{ccccc} & X = \pi_1(Z) & \xrightarrow{\ \phi\ } & \tilde{\pi}_1(\tilde{Z}) \subseteq \mathcal{X} & \\ {}^{\pi_1}\nearrow & & & & \nwarrow^{\tilde{\pi}_1} \\ X \times Y \supseteq Z & & \xrightarrow{\ (\phi, \psi)\ } & & \tilde{Z} \subseteq \mathcal{X} \times \mathcal{Y} \\ {}_{\pi_2}\searrow & & & & \swarrow_{\tilde{\pi}_2} \\ & Y = \pi_2(Z) & \xrightarrow{\ \psi\ } & \tilde{\pi}_2(\tilde{Z}) \subseteq \mathcal{Y} & \end{array}$$

For the conclusion, consider $\tilde{g} := \psi \circ g \circ \phi^{-1}$ together with the equality $\pi_1(Z) = \phi^{-1}(\tilde{\pi}_1(\tilde{Z}))$.

Step 2: By Step 1, in the following we assume that $X = \pi_1(Z)$ and $Y = \pi_2(Z)$ as well as $\phi = \mathrm{id}_{\mathcal{X}}$ and $\psi = \mathrm{id}_{\mathcal{Y}}$. We show that $\pi_1 : Z \to \pi_1(Z)$

is a local homeomorphism between connected metric spaces. Clearly, Z and $\pi_1(Z)$ are metric spaces and since Z is path-connected, $\pi_1(Z)$ is path-connected as well. To show that $\pi_1 : Z \to \pi_1(Z)$ is a local homeomorphism, let $(a, b) \in Z$, i.e., $F(a, b) = 0$. Applying the implicit function theorem, see e.g. [3, Thm. 10.2.1], yields open neighborhoods $U \subseteq X$ of a, $V \subseteq Y$ of b, and $g \in \mathcal{C}^1(U, V)$ such that

$$\{ \, (x, g(x)) \, \mid x \in U \, \} = \{ \, (x, y) \in U \times V \, \mid S(a, b)F(x, y) = 0 \, \} \, .$$

Consider the restriction $\hat{\pi}_1 : Z \cap (U \times V) \to \pi_1\big(Z \cap (U \times V)\big)$. Then $\hat{\pi}_1$ is injective since $\hat{\pi}_1(x_1, y_1) = \hat{\pi}_1(x_2, y_2)$ for some $(x_1, y_1), (x_2, y_2) \in Z \cap (U \times V)$ gives $x_1 = x_2$ and $S(a, b)F(x_1, y_1) = S(a, b)F(x_2, y_2)$, thus $y_1 = g(x_1) = g(x_2) = y_2$. Therefore, $\hat{\pi}_1$ is bijective and continuous. Furthermore, it is easy to see that $\hat{\pi}_1$ is an open map, and hence it is a homeomorphism.

Step 3: Let

$$\mathcal{P} := \mathcal{C}^1\big([0, 1], \pi_1(Z)\big)$$

and observe that $\pi_1(Z)$ is $\mathcal{P}$-connected and locally $\mathcal{P}$-contractible since it is open. We show that π_1 has the continuation property for every path in $\mathcal{P}$, that is, for all $q_1 \in \mathcal{P}$, all $b \in (0, 1]$ and all $q_2 \in \mathcal{C}([0, b), Y)$ such that $(q_1(t), q_2(t)) \in Z$ for all $t \in [0, b)$ there exists a sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0, b)$ with $\lim_{n\to\infty} t_n = b$ such that $\big(q_2(t_n)\big)_{n\in\mathbb{N}}$ converges and

$$\lim_{n\to\infty} (q_1(t_n), q_2(t_n)) \in Z.$$

First note that q_2 is differentiable at any $t \in [0, b)$, since there exists a local implicit function as in Step 2, so that $q_2(s) = g(q_1(s))$ for all s in a neighborhood of t. Since g and q_1 are differentiable we obtain $\dot{q}_2(t) = Dg(q_1(t))\dot{q}_1(t)$. Moreover, it can be seen that the derivative is continuous at each point in $[0, b)$. Then, using property (iii), it can be proved by only a slight modification of the proof of [5, Cor. 5.3] that for any sequence $(t_n)_{n\in\mathbb{N}} \subseteq [0, b)$ with $\lim_{n\to\infty} t_n = b$ the sequence $\big(q_2(t_n)\big)_{n\in\mathbb{N}}$ is a Cauchy sequence and hence converges in $Y = \mathcal{V}$. Since Z is closed in $X \times Y = \mathcal{U} \times \mathcal{V}$ by (i) we thus obtain $\lim_{n\to\infty}(q_1(t_n), q_2(t_n)) \in Z$.

Step 4: We show that $\pi_1 : Z \to \pi_1(Z)$ is a homeomorphism. By [5, Thm. 2.6] and Step 3 we may infer that π_1 is a covering map. Since $\pi_1(Z) = \phi^{-1}(\mathcal{X})$ is in particular simply connected by (iii) it follows from [7, Prop. A.79] that $\pi_1 : Z \to \pi_1(Z)$ is a homeomorphism.

Step 5: By Step 4 we have $\big(x \mapsto (x, g(x)) = \pi_1^{-1}(x)\big) \in \mathcal{C}(\pi_1(Z), Z)$ which uniquely defines the desired function $g \in \mathcal{C}(\pi_1(Z), Y)$. Since $\pi_1(Z)$ is in particular open by condition (iii), for all $x \in \pi_1(Z)$ we have that g coincides with any solution provided by the implicit function theorem as in Step 1 in a neighborhood of x. The implicit function theorem provides Fréchet-differentiability of the local solution, thus g is Fréchet-differentiable at x. □

We like to emphasize that Z in Theorem 2.1 may only be a subset of the zero set $F^{-1}(0)$. This allows to exclude points (x, y) in $F^{-1}(0)$ at which $D_yF(x,y)$ has not left inverse, or, actually, to exclude open sets containing such points so that Z is closed (alternatively, one may restrict the sets X and Y). Then a global implicit function may still exist in each connected component of Z, provided the growth bound in (iii) is satisfied.

Remark 2.3. An important question that arises is whether the growth bound in condition (iii) in Theorem 2.1 is independent of the choice of the diffeomorphisms ϕ and ψ. Use the notation from Theorem 2.1, assume that conditions (i)–(iii) are satisfied and let $\hat{\phi} : \pi_1(Z) \to \hat{\mathcal{X}}$ and $\hat{\psi} : \pi_2(Z) \to \hat{\mathcal{Y}}$ be diffeomorphisms for some Banach spaces $\hat{\mathcal{X}}, \hat{\mathcal{Y}}$. Then, omitting the subscripts indicating the spaces corresponding to the norms, we have the estimate

$$\begin{aligned}
&\left\| D\hat{\psi}(y) \cdot S(x,y) \right\| \cdot \left\| D_xF(x,y) \cdot \left(D\hat{\phi}(x)\right)^{-1} \right\| \\
&\leq \| D\psi(y) \cdot S(x,y) \| \cdot \left\| D\hat{\psi}(y) \cdot \left(D\psi(y)\right)^{-1} \right\| \\
&\quad \cdot \left\| D_xF(x,y) \cdot \left(D\phi(x)\right)^{-1} \right\| \cdot \left\| D\phi(x) \cdot \left(D\hat{\phi}(x)\right)^{-1} \right\| \\
&\leq \omega(\|\psi(y)\|_{\mathcal{Y}}) \cdot \left\| D\hat{\psi}(y) \cdot \left(D\psi(y)\right)^{-1} \right\| \cdot \left\| D\phi(x) \cdot \left(D\hat{\phi}(x)\right)^{-1} \right\|
\end{aligned}$$

for all $(x,y) \in Z$. If the last term satisfies

$$\begin{aligned}
\forall\, (x,y) \in Z :\ \omega(\|\psi(y)\|_{\mathcal{Y}}) \cdot \left\| D\hat{\psi}(y) \cdot \left(D\psi(y)\right)^{-1} \right\| \cdot \left\| D\phi(x) \cdot \left(D\hat{\phi}(x)\right)^{-1} \right\| \\
\leq \hat{\omega}(\|\hat{\psi}(y)\|_{\hat{\mathcal{Y}}})
\end{aligned}$$

for some continuous weight $\hat{\omega}$, then the growth bound in condition (iii) would indeed be independent of ϕ and ψ. However, it is still an open problem whether this is true (or a counterexample exists) and remains for future research.

3. Examples and discussion

In this section we discuss the assumptions in Theorem 2.1 and provide some illustrative examples. First, we provide a practical example occurring in the modelling of electrical circuits, where the projection $\pi_1(Z)$ (on which the implicit function is defined) is a proper subset of X.

Example 3.1. Consider two diodes $\mathcal{D}_1$, $\mathcal{D}_2$ with associated currents i_1, i_2 and voltages u_1, u_2. Following [8, Eq. (39.46)], we can model their constitutive relations as

$$i_1(t) = a_1 \left(e^{\frac{u_1(t)}{b_1}} - 1 \right), \quad i_2(t) = a_2 \left(e^{\frac{u_2(t)}{b_2}} - 1 \right), \quad t \in \mathbb{R}, \tag{3.1}$$

for some constants $a_1, a_2, b_1, b_2 > 0$. We may further impose the restrictions

$$u_1(t) \in (u_1^{\min}, u_1^{\max}) =: Y_1,\ u_2(t) \in (u_2^{\min}, u_2^{\max}) =: Y_2, \quad t \in \mathbb{R}, \tag{3.2}$$

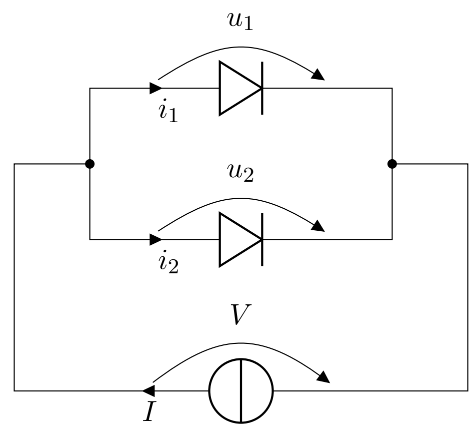

FIGURE 1. Circuit containing two diodes.

reflecting physical properties, e.g. the regions of operation of the corresponding devices that are being modelled. From (3.1) and (3.2) we can also derive restrictions

$$i_1(t) \in (i_1^{\min}, i_1^{\max}) =: X_1,\ \ i_2(t) \in (i_2^{\min}, i_2^{\max}) =: X_2, \quad t \in \mathbb{R}. \tag{3.3}$$

Such restrictions are incorporated in a natural manner in the port-Hamiltonian modelling of nonlinear electrical circuits in the context of resistive relations, see e.g. [4]. Next, we consider a parallel connection of the two diodes and add a current source with current I and voltage V as depicted in Figure 1, i.e., we have

$$\begin{aligned} &i_1(t) + i_2(t) = I(t), \\ &u_1(t) = u_2(t) = V(t), \quad \text{for } t \in \mathbb{R}. \end{aligned} \tag{3.4}$$

From the constitutive relations, it is clear that we can describe i_1, i_2 and hence I as a function of V. However, the converse is not evident, i.e., how V is given in terms of the current I. Invoking (3.1) and (3.4), I and V satisfy the relation

$$I = a_1 \left(e^{\frac{V}{b_1}} - 1 \right) + a_2 \left(e^{\frac{V}{b_2}} - 1 \right) =: f(V).$$

Further by (3.2) and (3.4), V has to satisfy $V \in Y_1 \cap Y_2 =: Y$, whereas I has to satisfy $I \in X_1 + X_2 = (i_1^{\min} + i_2^{\min}, i_1^{\max} + i_2^{\max}) =: X$ by (3.3) and (3.4). Note that both X and Y are open intervals and we exclude the trivial case that Y is empty. Recapitulating, with $F : X \times Y \to \mathbb{R}, (I, V) \mapsto I - f(V)$ and

$$Z := \{\ (I, V) \in X \times Y \ \mid F(I, V) = 0\ \},$$

we seek the existence of a function $g \in \mathcal{C}(\pi_1(Z), Y)$ such that

$$\{\ (I, g(I)) \ \mid I \in \pi_1(Z)\ \} = Z.$$

The existence of such a function is obviously equivalent to the invertibility of f on $\pi_2(Z) \subseteq Y$, which holds true since its derivative is positive. Nevertheless, we check the assumptions (i)-(iii) of Theorem 2.1 in order to illustrate it. Since Z is the zero set of a continuous function, it is relatively closed in $X \times Y$, i.e., (i) holds. For (ii), note that

$$D_V F(I,V) = -f'(V) = -\tfrac{a_1}{b_1}\exp\left(\tfrac{V}{b_1}\right) - \tfrac{a_2}{b_2}\exp\left(\tfrac{V}{b_2}\right) < 0.$$

It remains to find diffeomorphisms $\phi : \pi_1(Z) \to \mathbb{R}$, $\psi : \pi_2(Z) \to \mathbb{R}$ and a continuous weight $\omega : [0,\infty) \to (0,\infty)$ such that the growth bound in (iii) is satisfied. Choose any diffeomorphism $\phi : \pi_1(Z) \to \mathbb{R}$, which exists since $\pi_1(Z) = X$ is an open interval. Define $\psi := \phi \circ f$, which is a diffeomorphism since f is invertible on $\pi_2(Z)$. Then $\psi' = (\phi' \circ f) \cdot f'$. Further, let $\omega(t) = t+1$ for $t \in [0,\infty)$ and note that $S(I,V) = \big(D_V F(I,V)\big)^{-1} = \big(-f'(V)\big)^{-1}$ for $(I,V) \in Z$. Recalling that $I = f(V)$ for all $(I,V) \in Z$ we find

$$\begin{aligned}
&\|D\psi(V) \cdot S(I,V)\| \cdot \left\| D_I F(I,V) \cdot \big(D\phi(I)\big)^{-1} \right\| \\
&= \left|\phi'(I) \cdot f'(V) \cdot \big(-f'(V)\big)^{-1}\right| \cdot \left|\phi'(I)^{-1}\right| = \frac{|\phi'(I)|}{|\phi'(I)|} = 1 \\
&\le |\psi(V)| + 1 = \omega(\|\psi(V)\|),
\end{aligned}$$

for all $(I,V) \in Z$, proving (iii).

We like to highlight that none of the assumptions (i)–(iii) in Theorem 2.1 can be omitted in general. It is clear that connectedness of Z in (i) and local solvability guaranteed by (ii) are indispensable. Counterexamples in finite dimension are constructed for (iii) in the following examples. Condition (iii) basically consists of two parts. The first one is to check whether $\pi_i(Z)$, $i = 1,2$, are diffeomorphic to some Banach spaces. The second part is the growth bound involving the diffeomorphisms, the partial derivative $D_x F$ and the left inverse of $D_y F$.

First, we like to discuss why we chose the projections of Z as the domains of the diffeomorphisms in Theorem 2.1, whereas intuitively one could consider the open sets X and Y as the domains.

Remark 3.2. In a possible different formulation of Theorem 2.1 one could choose diffeomorphisms $\tilde{\phi} : X \to \mathcal{X}$ and $\tilde{\psi} : Y \to \mathcal{Y}$ and then consider, *mutatis mutandis*, the corresponding growth bound in condition (iii). This would relax the assumptions on the projections $\pi_i(Z)$, which would then not necessarily need to be open and simply connected. However, for the proof technique to be feasible we need to additionally require that $\pi_1(Z)$ is simply connected. Indeed, the proof is analogous, but the modified theorem does not cover basic examples.

For $F : \mathbb{R} \times (-1,1) \to \mathbb{R}$, $(x,y) \mapsto x - y$ and $Z := F^{-1}(0)$ it is easy to check that conditions (i) and (ii) are satisfied. The growth bound in condition (iii) reads

$$\begin{aligned} &\forall\,(x,y)\in Z:\ |\tilde\psi'(y)|\cdot|\tilde\phi'(x)^{-1}|\le\omega(|\tilde\psi(y)|)\\ \iff\ &\forall\,y\in(-1,1):\ |\tilde\phi'(y)|\ge\frac{|\tilde\psi'(y)|}{\omega(|\tilde\psi(y)|)}\end{aligned}$$

for some continuous weight ω. Note that $\tilde\phi'((-1,1))$ is bounded and $\tilde\phi(y)\neq 0$ for all $y\in\mathbb{R}$, hence

$$\int_{-1}^{1}|\tilde\phi'(y)|\mathrm{d}y=\left|\int_{-1}^{1}\tilde\phi'(y)\mathrm{d}y\right|<\infty.$$

Then the change of variables theorem with the substitution $t=\tilde\psi(y)$ together with the inverse function theorem yields that

$$\begin{aligned}\infty>\int_{-1}^{1}|\tilde\phi'(y)|\mathrm{d}y\ge\int_{-1}^{1}\frac{|\tilde\psi'(y)|}{\omega(|\tilde\psi(y)|)}\mathrm{d}y=\int_{-\infty}^{\infty}\frac{\left|\tilde\psi'\big(\tilde\psi^{-1}(t)\big)\right|}{\omega(|t|)}\left|(\tilde\psi^{-1})'(t)\right|\mathrm{d}t\\ =2\int_{0}^{\infty}\frac{1}{\omega(t)}\mathrm{d}t=\infty,\end{aligned}$$

a contradiction.

Nevertheless, a global implicit function obviously exists. The assumptions of Theorem 2.1 are satisfied since $\pi_1(Z)=\pi_2(Z)=(-1,1)$ and we may choose $\phi=\psi$, with which the growth bound holds true.

We continue by presenting an example where in assumption (iii) it is not possible to find suitable diffeomorphisms and, at the same time, a global implicit function does not exist.

Example 3.3. Consider

$$F:\mathbb{R}^2\times\mathbb{R}\to\mathbb{R}^2,\ (x_1,x_2,y)\mapsto\begin{pmatrix}x_1-\cos y\\ x_2-\sin y\end{pmatrix},\qquad Z:=F^{-1}(0).$$

Then assumptions (i) and (ii) in Theorem 2.1 are satisfied. Since $\pi_1(Z)=S^1$, the unit circle in $\mathbb{R}^2$, there is no Banach space $\mathcal{X}$ such that $\pi_1(Z)$ is diffeomorphic to $\mathcal{X}$. Indeed, no global implicit function can exist, since $y\mapsto(\cos y,\sin y)$ is not injective on $\mathbb{R}$.

In the next example the growth bound in condition (iii) is not satisfied for any suitable choice of diffeomorphisms and, at the same time, a global implicit function does not exist.

Example 3.4. We choose $F:X\times Y\to\mathbb{R}^2=\mathcal{Z}$ as a function of the type $F(x,y)=x-\tilde F(y)$ and $Z:=F^{-1}(0)$. This means that the existence of a global implicit function is equivalent to $\tilde F$ being injective. We further set $X\times Y=\mathbb{R}^2\times(0,1)^2$ and construct $\tilde F$ by successively defining its restrictions $\tilde F\restriction_{(0,\delta)\times(0,1)}=\tilde F_1$ and $\tilde F\restriction_{(\delta,1)\times(0,1)}=\tilde F_2$ for some $0<\delta\le\frac{1}{2}$. Choose $\varepsilon,\alpha>0$,

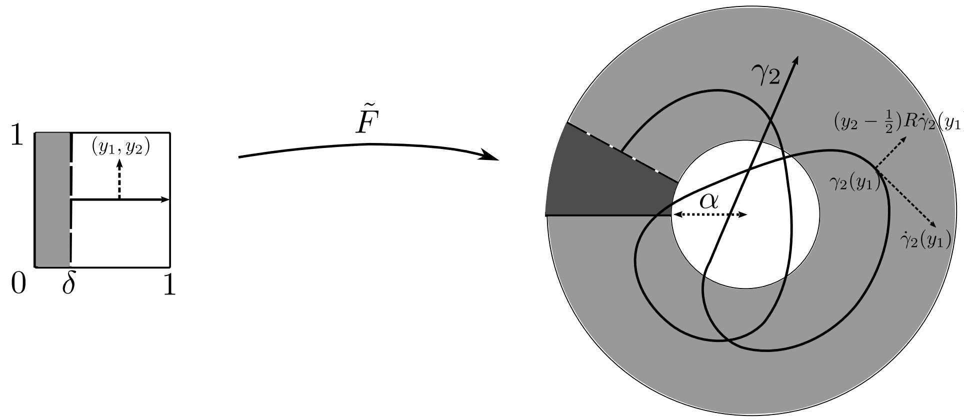

FIGURE 2. Illustration of the construction of $\tilde{F}$.

and consider the non-injective function

$$\tilde{F}_1 : (0,\delta) \times (0,1) \to \mathbb{R}^2,\ (y_1, y_2) \mapsto \begin{pmatrix} (\alpha + y_2)\sin\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) \\ (\alpha + y_2)\cos\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) \end{pmatrix}.$$

Observe that $\operatorname{im} \tilde{F}_1 = B_{\alpha+1}(0) \setminus \overline{B_\alpha(0)}$, where $B_\alpha(z)$ denotes the open ball with radius α around $z \in \mathbb{R}^2$, i.e., the image of $\tilde{F}_1$ is an annulus. Next, define $\tilde{F}_2$ similarly to $\tilde{F}_1$ using elementary functions such that the hole of the annulus is filled as displayed in Fig. 2, i.e., $\overline{B_\alpha(0)} \subset \operatorname{im} \tilde{F}_2 \subset B_{\alpha+1}(0)$. Note that $\tilde{F}_2$ can be chosen such that the resulting composition $\tilde{F}$ is differentiable everywhere. Overall, we have constructed a *non-injective* function $\tilde{F}$.

Observe that $\operatorname{im} \tilde{F} = \pi_1(Z) = B_{\alpha+1}(0)$ and $\pi_2(Z) = (0,1)^2$. Then the three conditions on F translate to $\tilde{F}$ as follows:

(i') the graph of $\tilde{F}$ is connected;
(ii') $\forall\, y \in (0,1)^2 :\ \operatorname{rk} D\tilde{F}(y) = 2\,$;
(iii') there exist diffeomorphisms $\phi : B_{\alpha+1}(0) \to \mathcal{X}, \psi : (0,1)^2 \to \mathcal{Y}$ for some Banach spaces $(\mathcal{X}, \|\cdot\|_{\mathcal{X}}), (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$, and a continuous weight $\omega : [0,\infty) \to (0,\infty)$ such that for all $y \in (0,1)^2$ we have

$$\left\| D\psi(y) \cdot D\tilde{F}_1(y)^{-1} \right\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \cdot \left\| \left(D\phi(\tilde{F}_1(y))\right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \leq \omega(\|\psi(y)\|_{\mathcal{Y}}).$$

Note that (i') is guaranteed by our choice of $Y = (0,1)^2$ and the continuity of $\tilde{F}$, which holds by construction. For (ii') note that

$$D\tilde{F}_1(y_1, y_2) = \begin{bmatrix} (\alpha + y_2)\frac{2\pi}{\delta}(1+\varepsilon)\cos\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) & \sin\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) \\ -(\alpha + y_2)\frac{2\pi}{\delta}(1+\varepsilon)\sin\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) & \cos\left(\frac{2\pi}{\delta}(1+\varepsilon)y_1\right) \end{bmatrix},$$

and $\det\left(\tilde{F}_1(y_1, y_2)\right) = (\alpha + y_2)\frac{2\pi}{\delta}(1+\varepsilon) \neq 0$. Further, the use of the constant α (large enough) guarantees that $\operatorname{rk} D\tilde{F}(y) = 2$ when filling $\overline{B_\alpha(0)}$ as displayed in Fig 2. Hence, (ii') is satisfied. Next, we show that (iii') is not satisfied, although, obviously, both $\pi_1(Z) = B_{\alpha+1}(0)$ and $\pi_2(Z) = (0,1)^2$ are diffeomorphic to some Banach spaces $\mathcal{X}$ and $\mathcal{Y}$. Without loss of generality, we may assume that $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$.

We show that condition (iii') is not satisfied for any diffeomorphisms $\phi : B_{\alpha+1}(0) \to \mathbb{R}^2$ and $\psi : (0,1)^2 \to \mathbb{R}^2$ by considering two cases. Let $(\hat{y}_1, \hat{y}_2) := \psi^{-1}(0,0) \in (0,1)^2$.

Case 1: Assume that $\hat{y}_1 \le \delta$. We show that the growth bound fails for $\tilde{F}_1$. Seeking a contradiction, assume that we have

$$\left\| D\psi(y) \cdot D\tilde{F}_1(y)^{-1} \right\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \cdot \left\| \left(D\phi(\tilde{F}_1(y)) \right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \le \omega(\|\psi(y)\|_{\mathcal{Y}})$$

for all $y \in (0,\delta) \times (0,1)$ and some weight ω. Although we did not specify the norms on $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, using the weight property and the equivalence of all norms on $\mathbb{R}^{n^2}, \mathbb{R}^n$, respectively, guarantees the existence of positive constants c_1, c_2 such that

$$\begin{aligned} &c_1 \left\| D\psi(y) \cdot D\tilde{F}_1(y)^{-1} \right\|_F \cdot \left\| \left(D\phi(\tilde{F}_1(y)) \right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \\ &\le \left\| D\psi(y) \cdot D\tilde{F}_1(y)^{-1} \right\|_{\mathcal{L}(\mathcal{Z},\mathcal{Y})} \cdot \left\| \left(D\phi(\tilde{F}_1(y)) \right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \\ &\le \omega(\|\psi(y)\|_{\mathcal{Y}}) \le \omega(c_2 \|\psi(y)\|_2), \end{aligned}$$

where $\|\cdot\|_F$ is the Frobenius norm. Observing that $\tilde{\omega}(\cdot) := c_1^{-1}\omega(c_2\,\cdot)$ again defines a weight, we obtain

$$\left\| D\psi(y) \cdot D\tilde{F}_1(y)^{-1} \right\|_F \cdot \left\| \left(D\phi(\tilde{F}_1(y)) \right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \le \tilde{\omega}(\|\psi(y)\|_2)$$

for all $y \in (0,\delta) \times (0,1)$. In order to simplify the computations, choose $\delta = \frac{1}{2}$ and $\alpha = \varepsilon = 1$. Since $\tilde{F}_1\left((0,\frac{1}{2}) \times \{\hat{y}_2\}\right) = (1+\hat{y}_2)S^1$ is a compact subset of $B_2(0)$ and $B_2(0) \ni z \mapsto \|D\phi(z)\|$ is a continuous mapping we have

$$\exists\, \gamma > 0\ \forall\, y \in (0,\tfrac{1}{2}) \times \{\hat{y}_2\} : \ \left\| D\phi(\tilde{F}_1(y)) \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \le \gamma.$$

This gives $\left\| \left(D\phi(\tilde{F}_1(y)) \right)^{-1} \right\|_{\mathcal{L}(\mathcal{X},\mathcal{Y})} \ge \gamma^{-1}$ for all $y \in (0,\frac{1}{2}) \times \{\hat{y}_2\}$. Accordingly, we may calculate that for all $y_1 \in (0,\frac{1}{2})$ we have

$$D\tilde{F}_1(y_1,\hat{y}_2)^{-1} = \begin{bmatrix} \frac{1}{8\pi(1+\hat{y}_2)} \cos(8\pi y_1) & -\frac{1}{8\pi(1+\hat{y}_2)} \sin(8\pi y_1) \\ \sin(8\pi y_1) & \cos(8\pi y_1) \end{bmatrix}$$

and hence

$$\begin{aligned} \left\| D\psi(y_1,\hat{y}_2) \cdot D\tilde{F}_1(y_1,\hat{y}_2)^{-1} \right\|_F &= \sqrt{\tfrac{1}{64\pi^2(1+\hat{y}_2)^2} \left(\tfrac{\partial \psi_1}{\partial y_1}^2 + \tfrac{\partial \psi_2}{\partial y_1}^2 \right) + \tfrac{\partial \psi_1}{\partial y_2}^2 + \tfrac{\partial \psi_2}{\partial y_2}^2} \\ &\ge \tfrac{1}{8\pi(1+\hat{y}_2)} \left\| \tfrac{\partial \psi}{\partial y_1}(y_1,\hat{y}_2) \right\|_2 . \end{aligned}$$

Note that for all $y_1 \in (0,\hat{y}_1)$ we have that $\|\psi(y_1,\hat{y}_2)\|_2 > 0$ and, because $\lim_{y_1 \to 0} \|\psi(y_1,\hat{y}_2)\|_2 = \infty$, the set

$$\mathcal{S} := \left\{ \, y_1 \in (0,\hat{y}_1) \;\middle|\; \tfrac{\partial}{\partial y_1} \|\psi(y_1,\hat{y}_2)\|_2^2 < 0 \, \right\}$$

has compact complement $(0,\hat{y}_1)\setminus\mathcal{S}$. Furthermore, for all $y_1\in(0,\hat{y}_1)$ we have that

$$\begin{aligned}\tfrac{1}{2}\left|\tfrac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2\right| &= \left|\psi(y_1,\hat{y}_2)^\top \tfrac{\partial\psi}{\partial y_1}(y_1,\hat{y}_2)\right| \\ &\le \|\psi(y_1,\hat{y}_2)\|_2\|\tfrac{\partial\psi}{\partial y_1}(y_1,\hat{y}_2)\|_2 \\ &\le 8\pi(1+\hat{y}_2)\|\psi(y_1,\hat{y}_2)\|_2\left\|D\psi(y_1,\hat{y}_2)\cdot D\tilde{F}_1(y_1,\hat{y}_2)^{-1}\right\|_F \\ &\le 8\pi(1+\hat{y}_2)\gamma\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}(\|\psi(y_1,\hat{y}_2)\|_2).\end{aligned}$$

With

$$\xi := \int_{(0,\hat{y}_1)\setminus\mathcal{S}} \frac{\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 < \infty$$

and the substitutions $t=\|\psi(y_1,\hat{y}_2)\|_2^2$ and $u=\sqrt{t}$ we may then derive

$$\begin{aligned}16\pi\gamma\hat{y}_1 &\ge \int_0^{\hat{y}_1} \frac{\left|\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2\right|}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 \ge \int_{\mathcal{S}} \frac{\left|\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2\right|}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 \\ &= \int_{\mathcal{S}} \frac{-\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 - \xi + \xi \\ &= \int_0^{\hat{y}_1} \frac{-\frac{\partial}{\partial y_1}\|\psi(y_1,\hat{y}_2)\|_2^2}{\|\psi(y_1,\hat{y}_2)\|_2\,\tilde{\omega}\big(\|\psi(y_1,\hat{y}_2)\|_2\big)}\,\mathrm{d}y_1 + \xi \\ &= \int_{\|\psi(\hat{y}_1,\hat{y}_2)\|_2^2}^{\infty} \frac{1}{\sqrt{t}\tilde{\omega}(\sqrt{t})}\,\mathrm{d}t + \xi = 2\int_0^\infty \frac{1}{\tilde{\omega}(u)}\,\mathrm{d}u + \xi = \infty,\end{aligned}$$

a contradiction.

Case 2: Assume that $\hat{y}_1>\delta$. We show that the growth bound fails for $\tilde{F}_2$, which is similar to Case 1. To this end, we render the definition of $\tilde{F}_2$ more precisely. First define the curve

$$\gamma_1 : (0,\delta)\to\mathbb{R}^2,\ t\mapsto \begin{pmatrix}(\alpha+\frac{1}{2})\sin\left(\frac{2\pi}{\delta}(1+\varepsilon)t\right)\\ (\alpha+\frac{1}{2})\cos\left(\frac{2\pi}{\delta}(1+\varepsilon)t\right)\end{pmatrix}$$

and the (rotation) matrix $R:=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$. Then $\tilde{F}_1$ can alternatively be written as

$$\tilde{F}_1(y_1,y_2)=\gamma_1(y_1)+(y_2-\tfrac{1}{2})R\dot{\gamma}_1(y_1)$$

for all $(y_1,y_2)\in(0,\delta)\times(0,1)$. In view of this, we may choose a curve $\gamma_2 : (\delta,1)\to\mathbb{R}^2$ as depicted in Fig. 2 such that

$$\tilde{F}_2(y_1,y_2)=\gamma_2(y_1)+(y_2-\tfrac{1}{2})R\dot{\gamma}_2(y_1)$$

for all $(y_1,y_2)\in(\delta,1)\times(0,1)$ and, as mentioned before, $\overline{B_\alpha(0)}\subset\operatorname{im}\tilde{F}_2\subset B_{\alpha+1}(0)$ and $D\tilde{F}_2(y_1,y_2)$ is invertible for all $(y_1,y_2)\in(\delta,1)\times(0,1)$. Omitting the details, the same arguments as in Case 1 may now be applied to arrive at a contradiction. In particular, fixing $y_2=\hat{y}_2$ leads to a curve $\tilde{\gamma}_2(y_1)=\gamma_2(y_1)+(\hat{y}_2-\frac{1}{2})R\dot{\gamma}_2(y_1)$ along which the growth bound is violated.

Remarks 3.5. Finally, we like to point out that, while Theorem 2.1 is already quite general, still it does not cover all relevant cases. Consider

$$F : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3,\ (x_1, x_2, y_1, y_2) \mapsto \begin{pmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_1^2 + x_2^2 - 1 \end{pmatrix},$$

then

$$Z := F^{-1}(0) = \left\{\ (x_1, x_2, y_1, y_2) \in \mathbb{R}^4 \ \middle|\ x_1^2 + x_2^2 = 1,\ y_1 = x_1,\ y_2 = x_2\ \right\}$$

and $\pi_1(Z), \pi_2(Z)$ are both the unit circle in $\mathbb{R}^2$, i.e., closed subsets which are not simply connected, for which it is not possible to satisfy condition (iii). However, a global implicit function obviously exists. Further research is necessary to cover examples of this type.

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