Funnel control for nonlinear systems with higher relative degree

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We consider tracking control for nonlinear multi-input, multi-output systems which have arbitrary strict relative degree and input-to-state stable internal dynamics. For a given reference signal, our aim is to design a controller which achieves that the tracking error evolves within a prespecified performance funnel around the reference signal. To this end, we introduce a new controller which involves the first $r-1$ derivatives of the tracking error, where r is the strict relative degree of the system.

1 Introduction

We consider the following class of nonlinear systems described by functional differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t)\big)\, u(t) \\ y|_{[-h,0]} &= y^0 \in \mathscr{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m), \end{aligned} \tag{1.1}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the strict relative degree, and

(P1): the "disturbance" satisfies $d \in \mathscr{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^p)$, $p \in \mathbb{N}$;

(P2): $f \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m)$, $q \in \mathbb{N}$;

(P3): the "high-frequency gain matrix function" $\Gamma \in \mathscr{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^{m\times m})$ satisfies that $\Gamma + \Gamma^\top$ is pointwise positive definite[1];

(P4): $T : \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm}) \to \mathscr{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ is an operator with the following properties:

a) T maps bounded trajectories to bounded trajectories, i.e, for all $c_1 > 0$, there exists $c_2 > 0$ such that for all $\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm})$,

$$\sup_{t\in[-h,\infty)} \|\zeta(t)\| \leq c_1 \Rightarrow \sup_{t\in[0,\infty)} \|T(\zeta)(t)\| \leq c_2,$$

b) T is causal, i.e, for all $t \geq 0$ and all $\zeta, \xi \in \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm})$,

$$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \Rightarrow T(\zeta){=}T(\xi) \text{ for almost all } \zeta \in [0,t).$$

c) T is locally Lipschitz continuous in the following sense: for all $t \geq 0$ there exist $\tau, \delta, c > 0$ such that for all $\zeta, \Delta\zeta \in \mathscr{C}([-h,\infty) \to \mathbb{R}^{rm})$ with $\Delta\zeta|_{[-h,t)} = 0$ and $\|\Delta\zeta|_{[t,t+\tau]}\|_\infty < \delta$ we have

$$\big\|(T(\zeta+\Delta\zeta) - T(\zeta))|_{[t,t+\tau]}\big\|_\infty \leq c\|\Delta\zeta|_{[t,t+\tau]}\|_\infty.$$

The functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and $y : [-h,\infty) \to \mathbb{R}^m$ are called *input* and *output* of the system (1.1), resp. Systems similar to (1.1) have been studied e.g. in [11, 14, 15, 17]. In the aforementioned references it is shown that the class of systems (1.1) encompasses linear and nonlinear systems with strict relative degree and input-to-state stable internal dynamics (zero dynamics in the linear case) and the operator T allows for infinite-dimensional linear systems, systems with hysteretic effects or nonlinear delay elements, and combinations thereof. Note that the operator T is usually the solution operator of the differential equation describing the internal dynamics of the system and its property (P4a) thus amounts to the input-to-state stability of the internal dynamics.

One important subclass of systems (1.1) are minimum-phase linear time-invariant systems

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \qquad x(0) = x^0 \in \mathbb{R}^n, \tag{1.2}$$

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[1]Note that, analogously, pointwise negative definite $\Gamma + \Gamma^\top$ may be considered by just changing the sign of the input u.

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, which have strict relative degree $r \in \mathbb{N}$ and positive definite high-frequency gain matrix, i.e, $CB = CAB = \ldots = CA^{r-2}B = 0$ and $\Gamma := CA^{r-1}B \in \mathbb{R}^{m\times m}$ is positive definite. The minimum-phase assumption (equivalently, asymptotic stability of the zero dynamics, cf. [4,9]) is characterized by the condition

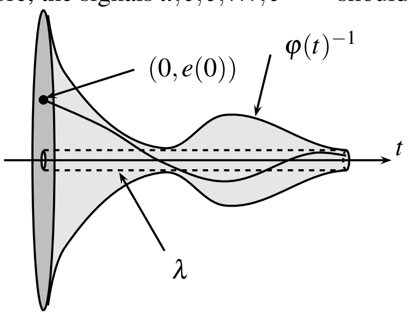
$$\forall \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda \geq 0: \quad \det \begin{bmatrix} \lambda I_n - A & B \\ C & 0 \end{bmatrix} \neq 0.$$

It is known that systems of this type can be transformed into *Byrnes-Isidori normal form*, see [15], and are hence included in the class (1.1). For the corresponding normal form for nonlinear systems see [19] and also the recent paper [5], where a generalization is discussed.

The control objective is to design an output error feedback $u(t) = F(t, e(t), \dot{e}(t), \ldots, e^{(r-1)}(t))$, where $e(t) = y(t) - y_{\text{ref}}(t)$ for some reference trajectory $y_{\text{ref}} \in \mathscr{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, such that in the closed-loop system the tracking error $e(t)$ evolves within a prescribed performance funnel, i.e., $\varphi(t)\|e(t)\| < 1$ for all $t \geq 0$, where φ belongs to

$$\Phi_r := \left\{ \varphi \in \mathscr{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \ldots, \varphi^{(r)} \text{ are bounded, } \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}.$$

Furthermore, the signals $u, e, \dot{e}, \ldots, e^{(r-1)}$ should remain bounded.



The funnel boundary is given by $\varphi(\cdot)^{-1}$, see Fig. 1. It is explicitly allowed that $\varphi(0) = 0$, which means that no restriction on the initial value is imposed as $\varphi(0)\|e(0)\| < 1$. An important property of the class Φ_r is that the boundary of each performance funnel is bounded away from zero, i.e., because of boundedness of φ there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$.

Fig. 1: Error evolution in a funnel with boundary $\varphi(t)^{-1}$

A longstanding open problem in high-gain adaptive control is the treatment of systems with relative degree larger than one, see [13, 16, 20]. We follow the framework of *Funnel Control* which was developed in [14] for systems with relative degree one, see also the survey [16] and the references therein. The funnel controller is an adaptive controller of high-gain type and thus inherently robust. The funnel controller has been successfully applied e.g. in control of industrial servo-systems [12] and voltage and current control of electrical circuits [3]. Funnel control is not restricted to systems of ordinary differential equations, but can also be used for infinite-dimensional systems (see e.g. [18, 21]) and systems of differential-algebraic equations (see e.g. [1, 2]).

We present a simple funnel controller for systems with arbitrary known relative degree r and (in a suitable sense) input-to-state stable internal dynamics. The controller is based on a simple recursion law and involves the first $r-1$ derivatives of the tracking error. The present paper mainly serves as a summary of the journal paper [7], where all proofs can be found.

2 Funnel control

We introduce the following funnel controller for systems of the class (1.1):

$$\boxed{\begin{aligned} e_0(t) &= e(t) = y(t) - y_{\text{ref}}(t), & u(t) &= -k_{r-1}(t)\, e_{r-1}(t), \\ e_1(t) &= \dot{e}_0(t) + k_0(t)\, e_0(t), & k_i(t) &= (1 - \varphi_i(t)^2 \|e_i(t)\|^2)^{-1}, \quad i = 0, \ldots, r-1, \\ e_2(t) &= \dot{e}_1(t) + k_1(t)\, e_1(t), & & \\ &\ \vdots & & \\ e_{r-1}(t) &= \dot{e}_{r-2}(t) + k_{r-2}(t)\, e_{r-2}(t), & & \end{aligned}} \tag{2.1}$$

where the reference signal and funnel functions satisfy:

$$\boxed{y_{\text{ref}} \in \mathscr{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \qquad \varphi_0 \in \Phi_r,\ \ \varphi_1 \in \Phi_{r-1},\ \ldots,\ \ \varphi_{r-1} \in \Phi_1.} \tag{2.2}$$

By a *solution* of (1.1), (2.1) on $[-h, \omega)$ we will mean a function $y \in \mathscr{C}^{r-1}([-h, \omega) \to \mathbb{R}^m)$, $\omega \in (0, \infty]$, with $y|_{[-h,0]} = y^0$ such that $y^{(r-1)}|_{[0,\omega)}$ is absolutely continuous and satisfies the differential equation in (1.1) with u defined in (2.1) for almost all $t \in [0, \omega)$; y is called *maximal*, if it has no right extension that is also a solution. Existence of solutions of functional differential equations has been investigated in [14] for instance.

Remark 2.1 We stress that while the derivatives $\dot{e}_0,\dots,\dot{e}_{r-2}$ appear in (2.1), they only serve as short-hand notations and may be resolved in terms of the tracking error, the funnel functions and the derivatives of these. For the case $r=2$ this reads

$$u(t) = -k_1(t)(\dot{e}(t)+k_0(t)e(t)), \qquad k_0(t) = \left(1-\varphi_0^2(t)\|e(t)\|^2\right)^{-1},$$
$$k_1(t) = \left(1-\varphi_1^2(t)\|\dot{e}(t)+k_0(t)e(t)\|^2\right)^{-1}.$$

We show that the funnel controller (2.1) is feasible.

Theorem 2.2 *Consider a system* (1.1) *with strict relative degree $r\in\mathbb{N}$ and properties (P1)-(P4). Let y_{ref} and $\varphi_0,\dots,\varphi_{r-1}$ be as in* (2.2) *and $y|_{[-h,0]} = y^0 \in \mathscr{W}^{r-1,\infty}([-h,0]\to\mathbb{R}^m)$ an initial value such that $e_0,\dots,e_{r-1}$ as defined in* (2.1) *satisfy*

$$\varphi_i(0)\|e_i(0)\| < 1 \quad \textit{for} \quad i=0,\dots,r-1. \tag{2.3}$$

Then the application of the funnel controller (2.1) *to* (1.1) *yields an initial-value problem, which has a solution, and every maximal solution $y:[-h,\omega)\to\mathbb{R}^m$, $\omega\in(0,\infty]$, has the following properties:*

(i) *The solution is global (i.e., $\omega=\infty$).*

(ii) *The input $u:\mathbb{R}_{\ge 0}\to\mathbb{R}^m$, the gain functions $k_0,\dots,k_{r-1}:\mathbb{R}_{\ge 0}\to\mathbb{R}$ and $y,\dots,y^{(r-1)}:\mathbb{R}_{\ge 0}\to\mathbb{R}^m$ are bounded.*

(iii) *The functions $e_0,\dots,e_{r-1}:\mathbb{R}_{\ge 0}\to\mathbb{R}^m$ evolve in their respective performance funnels and are uniformly bounded away from the funnel boundaries in the following sense:*

$$\forall\, i=0,\dots,r-1\ \exists\,\varepsilon_i>0\ \forall\, t>0:\ \|e_i(t)\| \le \varphi_i(t)^{-1}-\varepsilon_i.$$

We stress that a drawback of our approach, which still needs to be resolved, is that the derivatives of the output must be available for the controller. However, this is not satisfied in several applications, and it may even be hard to obtain suitable estimates of the output derivatives. A first approach to treat this problem using a "funnel pre-compensator" has been developed in [6, 8] for systems with relative degree $r=2$ or $r=3$.

3 Simulations

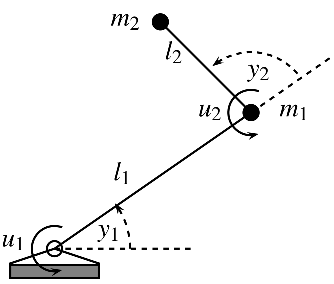


Fig. 2: Planar rigid revolute joint robotic manipulator.

To illustrate the funnel controller (2.1) for a nonlinear multi-input, multi-output system we consider an example of a robotic manipulator from [10], see also [12, Ch. 13], as depicted in Fig. 2. The robotic manipulator is planar, rigid, with revolute joints and has two degrees of freedom.

The two joints are actuated by u_1 and u_2 (in N m). We assume that the links are massless, have lengths l_1 and l_2 (in m), resp., and point masses m_1 and m_2 (in kg) are attached to their ends. The two outputs are the joint angles y_1 and y_2 (in rad) and the equations of motion are given by

$$M(y(t))\ddot{y}(t)+C(y(t),\dot{y}(t))\dot{y}(t)+G(y(t)) = u(t) \tag{3.1}$$

with initial value $(y(0),\dot{y}(0)) = \left(0\,\mathrm{rad}^2, 0\,(\mathrm{rad/s})^2\right)$, inertia matrix $M:\mathbb{R}^2\to\mathbb{R}^{2\times 2}$, $(y_1,y_2)\mapsto M(y_1,y_2)$, centrifugal and Coriolis force matrix $C:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^{2\times 2}$, $(y_1,y_2,v_1,v_2)\mapsto C(y_1,y_2,v_1,v_2)$ and gravity vector $G:\mathbb{R}^2\to\mathbb{R}^2$, $(y_1,y_2)\mapsto G(y_1,y_2)$ with

$$\begin{aligned}
M(y_1,y_2) &:= \begin{bmatrix} m_1 l_1^2 + m_2(l_1^2+l_2^2+2l_1l_2\cos(y_2)) & m_2(l_2^2+l_1l_2\cos(y_2)) \\ m_2(l_2^2+l_1l_2\cos(y_2)) & m_2 l_2^2 \end{bmatrix},\\
C(y_1,y_2,v_1,v_2) &:= \begin{bmatrix} -2m_2l_1l_2\sin(y_2)v_1 & -m_2l_1l_2\sin(y_2)v_2 \\ -m_2l_1l_2\sin(y_2)v_1 & 0 \end{bmatrix},\\
G(y_1,y_2) &:= g\begin{pmatrix} m_1l_1\cos(y_1)+m_2(l_1\cos(y_1)+l_2\cos(y_1+y_2)) \\ m_2l_2\cos(y_1+y_2) \end{pmatrix},
\end{aligned}$$

where $g=9.81\,\mathrm{m/s^2}$ is the acceleration of gravity. If we multiply system (3.1) with the pointwise positive definite matrix $M(y(t))^{-1}$, we see that the resulting system belongs to the class (1.1) with $r=m=2$.

For the simulation, we choose the parameters $m_1 = m_2 = 1\,\mathrm{kg}$, $l_1 = l_2 = 1\,\mathrm{m}$ and the reference trajectories $y_{\mathrm{ref},1}(t) = \sin t\,\mathrm{rad}$ and $y_{\mathrm{ref},2}(t) = \sin 2t\,\mathrm{rad}$. For the controller (2.1) we choose the funnel functions

$$\varphi_0(t) = (e^{-2t} + 0.1)^{-1}, \quad \varphi_1(t) = (3e^{-2t} + 0.1)^{-1}.$$

The initial errors lie within the respective funnel boundaries, i.e., conditions (2.3) are satisfied, thus Theorem 2.2 yields that funnel control is feasible.

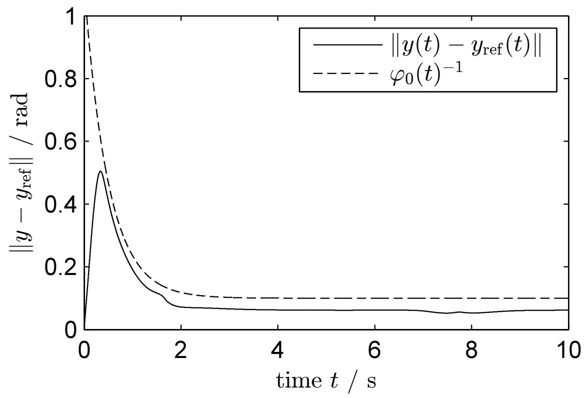


Fig. 3: Funnel and tracking error

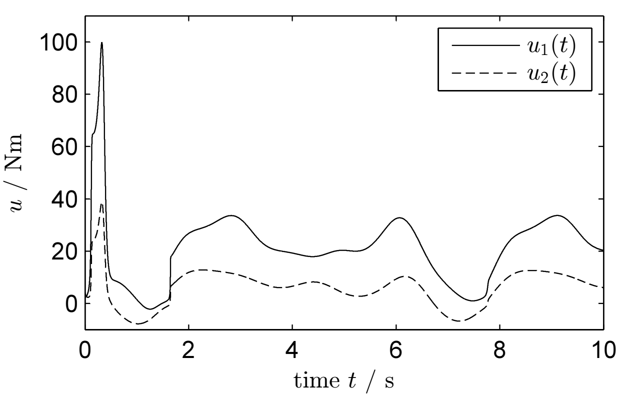


Fig. 4: Input components

The simulation of the funnel controller (2.1) applied to (3.1) over the time interval $0 - 10\,\mathrm{s}$ has been performed in MATLAB (solver: *ode45*, rel. tol: 10^{-14}, abs. tol: 10^{-10}) and is depicted in Fig. 3 (tracking error) and Fig. 4 (input components).

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