

Funnel control for electrical circuits

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We study the zero dynamics and funnel control for linear passive electrical circuits. We show that asymptotic stability of the zero dynamics can be characterized by criteria on the circuit topology. Thereafter we consider the output regulation problem for electrical circuits by funnel control. We show that for circuits with asymptotically stable zero dynamics, the funnel controller achieves tracking of a class of reference signals within a pre-specified funnel. This result can be relaxed to the case of non-autonomous zero dynamics by requiring the reference trajectory to evolve within a certain subspace.

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1 Introduction

We consider linear differential-algebraic systems of the form

$$\frac{d}{dt}Ex(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad (1.1)$$

where $E, A \in \mathbb{R}^{n,n}$, $B, C^T \in \mathbb{R}^{n,m}$; the set of these square systems (i.e., same number of inputs and outputs) is denoted by $\Sigma_{n,m}$ and we write $[E, A, B, C] \in \Sigma_{n,m}$.

The functions $u, y : \mathbb{R} \rightarrow \mathbb{R}^m$ are called *input* and *output* of the system, respectively. A trajectory $(x, u, y) : \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ is said to be a *solution* of (1.1) if it belongs to the *behavior* of (1.1):

$$\mathfrak{B}_{[E,A,B,C]} := \left\{ (x, u, y) \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m) \mid \begin{array}{l} Ex \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^n) \text{ and } (x, u, y) \\ \text{solves (1.1) for all } t \geq 0 \end{array} \right\}.$$

We investigate systems (1.1) which arise from modified nodal analysis (MNA) models of electrical circuits [1], i.e.,

$$sE - A = \begin{bmatrix} sA_C C A_C^T + A_{\mathcal{R}} \mathcal{G} A_{\mathcal{R}}^T & A_L & A_{\mathcal{V}} \\ -A_L^T & s\mathcal{L} & 0 \\ -A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}, \quad B = C^T = \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I_{n_{\mathcal{V}}} \end{bmatrix}, \quad (1.2)$$

$$x = (\eta^T, i_L^T, i_{\mathcal{V}}^T)^T, \quad u = (i_{\mathcal{I}}^T, v_{\mathcal{V}}^T)^T, \quad y = (-v_{\mathcal{I}}^T, -i_{\mathcal{V}}^T)^T, \quad (1.3)$$

where

$$\left. \begin{array}{l} C \in \mathbb{R}^{n_c, n_c}, \mathcal{G} \in \mathbb{R}^{n_g, n_g}, \mathcal{L} \in \mathbb{R}^{n_L, n_L}, \\ A_C \in \mathbb{R}^{n_e, n_c}, A_{\mathcal{R}} \in \mathbb{R}^{n_e, n_g}, A_L \in \mathbb{R}^{n_e, n_L}, A_{\mathcal{V}} \in \mathbb{R}^{n_e, n_{\mathcal{V}}}, A_{\mathcal{I}} \in \mathbb{R}^{n_e, n_{\mathcal{I}}}, \\ n = n_e + n_L + n_{\mathcal{V}}, \quad m = n_{\mathcal{I}} + n_{\mathcal{V}}. \end{array} \right\} \quad (1.4)$$

The matrices $A_C, A_{\mathcal{R}}, A_L, A_{\mathcal{V}}$ and $A_{\mathcal{I}}$ denote the element-related incidence matrices, C, \mathcal{G} and \mathcal{L} are the matrices expressing the consecutive relations of capacitances, resistances and inductances, η is the vector of node potentials, $i_L, i_{\mathcal{V}}, i_{\mathcal{I}}$ are the vectors of currents through inductances, voltage and current sources, and $v_{\mathcal{V}}, v_{\mathcal{I}}$ are the voltages of voltage and current sources.

We assume that the given electrical circuit is connected; otherwise, since the components of connectivity do not physically interact, one might consider them separately. Furthermore, we consider circuits with *passive* devices. These assumptions lead to the following assumptions on an MNA model (1.2)–(1.4) of the circuit.

$$\text{(A1)} \quad \text{rk} [A_C \quad A_{\mathcal{R}} \quad A_L \quad A_{\mathcal{V}} \quad A_{\mathcal{I}}] = n_e,$$

$$\text{(A2)} \quad C = C^T > 0, \mathcal{L} = \mathcal{L}^T > 0, \mathcal{G} + \mathcal{G}^T > 0.$$

Note that the transfer function $G(s) = C(sE - A)^{-1}B$ of an MNA model (1.2), (1.4) is positive real and reciprocal, see [2].

It is the aim of the present paper to show that stability of the zero dynamics or, more generally, stability of the invariant zeros of a circuit can be characterized by sufficient criteria on the circuit topology. These criteria are inspired by the results of Rianza and Tischendorf [3] on asymptotic stability of circuits. Stability of the zero dynamics is exploited for output regulation

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by funnel control introduced by Ilchmann, Ryan and Sangwin [4]: We show that this assumption is sufficient for feasibility of a closed-loop control law (the *funnel controller*) that guarantees that the output evolves inside a prescribed domain (the so-called *funnel*) around some given reference trajectory. This result can be extended to circuits with possibly non-autonomous zero dynamics: Under the condition of stability of the invariant zeros we show that funnel control is feasible; the drawback is that the reference trajectory must be restricted to a certain subspace in order to account for the non-autonomous part of the zero dynamics. The latter has the physical interpretation that the reference trajectory satisfies Kirchhoff's laws pointwise.

For the proofs and more details on the results in the present paper see [5, 6].

2 Zero dynamics

In the present paper we play particular emphasis on the *zero dynamics* of (1.1). These are, for $[E, A, B, C] \in \Sigma_{n,m}$, defined by $\mathcal{ZD}_{[E,A,B,C]} := \{ (x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \mid y = 0 \}$. By linearity of (1.1), $\mathcal{ZD}_{[E,A,B,C]}$ is a real vector space. The zero dynamics of (1.1) are called

$$\begin{aligned} \text{autonomous} & \quad :\iff \forall w_1, w_2 \in \mathcal{ZD}_{[E,A,B,C]} \forall I \subseteq \mathbb{R} \text{ open interval} : w_1|_I = w_2|_I \implies w_1 = w_2 ; \\ \text{asymptotically stable} & \quad :\iff \forall (x, u, y) \in \mathcal{ZD}_{[E,A,B,C]} : \lim_{t \rightarrow \infty} (x(t), u(t)) = 0. \end{aligned}$$

Note that the above definitions are within the spirit of the *behavioral approach* [7] and take into account that the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are a linear behavior. (Asymptotically stable) zero dynamics are the vector space of those trajectories of the system which are, loosely speaking, not visible at the output (and tend to zero). Asymptotically stable zero dynamics are characterized in [8] via stabilizability, detectability and the location of transmission zeros of the system $[E, A, B, C]$.

For the purpose of topological criteria for asymptotically stable zero dynamics we need to introduce some graph theoretical notation: For a graph \mathcal{G} , a spanning subgraph \mathcal{K} of \mathcal{G} (i.e., \mathcal{K} has the same node set as \mathcal{G}), a subgraph \mathcal{L} of \mathcal{G} , and a path ℓ of \mathcal{G} we call \mathcal{L} a \mathcal{K} -*cutset*, if \mathcal{L} is a subgraph of \mathcal{K} , deleting all branches from \mathcal{L} in \mathcal{G} results in a disconnected subgraph and if not every branch is deleted, then the subgraph is still connected; ℓ is called a \mathcal{K} -*loop*, if ℓ is a loop of \mathcal{K} .

In the following we will use expressions like $\mathcal{V}\mathcal{L}$ -loop for a \mathcal{K} -loop in the circuit graph where the branch set of \mathcal{K} consists only of branches corresponding to voltage sources and/or inductors. Likewise, a $\mathcal{I}\mathcal{C}$ -cutset is a \mathcal{K} -cutset in the circuit graph where the branch set of \mathcal{K} consist only of branches corresponding to current sources and/or capacitors.

We have the following characterization of autonomous zero dynamics.

Proposition 2.1 (Autonomous zero dynamics) *Let $[E, A, B, C] \in \Sigma_{n,m}$ with (1.2), (1.4) be an MNA model of an electrical circuit and suppose that (A1) and (A2) hold. Then the following statements are equivalent.*

- (i) *The zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are autonomous.*
- (ii) $\text{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = n + m.$
- (iii) $\ker \begin{bmatrix} A_C & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{I}} \end{bmatrix}^{\top} = \{0\}$ and $\ker A_{\mathcal{I}} = \{0\}.$
- (iv) *The circuit neither contains \mathcal{I} -loops nor \mathcal{V} -cutsets.*

Note that autonomy of the zero dynamics implies left invertibility of the system $[E, A, B, C]$, but in general not vice versa, see [9]; equivalence holds true for systems with regular $sE - A$. Autonomous zero dynamics are furthermore equivalent to "left invertibility in the strong sense" [10], provided that $[E^{\top}, A^{\top}, C^{\top}]$ has full row rank.

In order to characterize asymptotic stability of the (non-autonomous part of the) zero dynamics we need the concept of an invariant zero, which is, loosely speaking, a generalized eigenvalue of $\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}$, see e.g. [11].

Definition 2.2 (Invariant zeros) *Let $[E, A, B, C] \in \Sigma_{n,m}$. Then $\lambda \in \mathbb{C}$ is called *invariant zero* of $[E, A, B, C]$ if*

$$\text{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} < \text{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}.$$

Extending the results by Riaza and Tischendorf [3], who obtained sufficient topological criteria for asymptotic stability of a circuit, we get the following result on the location of invariant zeros.

Corollary 2.3 (Location of invariant zeros) *Let $[E, A, B, C] \in \Sigma_{n,m}$ with (1.2), (1.4) be an MNA model of an electrical circuit and suppose that (A1) and (A2) hold. Furthermore, suppose that at least one of the following two assertions holds:*

- (i) *The circuit neither contains $\mathcal{I}\mathcal{L}$ -loops except for \mathcal{I} -loops, nor $\mathcal{V}\mathcal{C}\mathcal{L}$ -cutsets except for $\mathcal{V}\mathcal{L}$ -cutsets.*
- (ii) *The circuit neither contains $\mathcal{V}\mathcal{C}$ -cutsets except for \mathcal{V} -cutsets, nor $\mathcal{I}\mathcal{C}\mathcal{L}$ -loops except for $\mathcal{I}\mathcal{C}$ -loops.*

Then all invariant zeros of $[E, A, B, C]$ are contained in \mathbb{C}_- .

Combining Proposition 2.1 and Corollary 2.3 we may characterize asymptotically stable zero dynamics.

Theorem 2.4 (Asymptotically stable zero dynamics) *Let $[E, A, B, C] \in \Sigma_{n,m}$ with (1.2), (1.4) be an MNA model of an electrical circuit such that (A1) and (A2) hold. Then the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are asymptotically stable if, and only if,*

- a) $\mathcal{ZD}_{[E,A,B,C]}$ are autonomous and
- b) all invariant zeros of $[E, A, B, C]$ are contained in \mathbb{C}_- .

Furthermore, suppose that at least one of the following two assertions holds:

- (i) The circuit neither contains \mathcal{IL} -loops, nor \mathcal{VCL} -cutsets except for \mathcal{VL} -cutsets with at least one inductor.
- (ii) The circuit neither contains \mathcal{VC} -cutsets, nor \mathcal{ICL} -loops except for \mathcal{IC} -loops with at least one capacitor.

Then the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are asymptotically stable.

3 Funnel control

In this section we consider funnel control for systems $[E, A, B, C] \in \Sigma_{l,n,m}$. The aim is to achieve tracking of a reference trajectory by the output signal with prescribed transient behavior. We use the notation $\mathcal{B}^\ell(\mathbb{R}_{\geq 0}; \mathbb{R}^n) = \{f \in \mathcal{C}^\ell(\mathbb{R}_{\geq 0}; \mathbb{R}^n) \mid \frac{d^i}{dt^i} f \text{ is bounded for } i = 0, \dots, \ell\}$ for $\ell \in \mathbb{N}_0 \cup \{\infty\}$. For any function φ belonging to

$$\Phi := \left\{ \varphi \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \cap \mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}) \mid \varphi(0) = 0, \varphi(s) > 0 \text{ for all } s > 0 \text{ and } \liminf_{s \rightarrow \infty} \varphi(s) > 0 \right\}$$

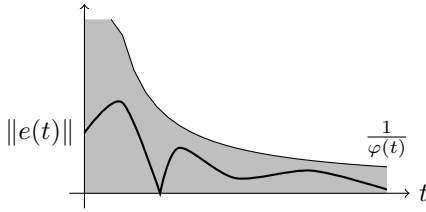


Fig. 1: Error evolution in a funnel \mathcal{F}_φ with boundary $\varphi(t)^{-1}$.

we associate the *performance funnel* $\mathcal{F}_\varphi := \{(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t) \|e\| < 1\}$, see Figure 1. The control objective is feedback control so that the tracking error $e = y - y_{\text{ref}}$, where y_{ref} is the reference signal, evolves within \mathcal{F}_φ and all variables are bounded. To ensure this, we introduce the *funnel controller*:

$$u(t) = -k(t) e(t), \quad k(t) = \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}. \quad (3.1)$$

If we assume asymptotically stable zero dynamics, we see intuitively that, in order to maintain the error evolution within the funnel, high gain values may only be required if the norm $\|e(t)\|$ of the error is close to the funnel boundary $\varphi(t)^{-1}$: $k(\cdot)$ increases if necessary to exploit the high-gain property of the system and decreases if a high gain is not necessary. This intuition underpins the choice of the gain $k(t)$ in (3.1). The control design (3.1) has two advantages: $k(\cdot)$ is non-monotone and (3.1) is a static time-varying proportional output feedback of striking simplicity.

The concept of funnel control has been introduced in [4] and investigated for linear DAEs recently in [6, 9, 12–14] which provide the basis for the investigation in the present paper; nonlinear DAEs have also been treated in [15].

Before we state and prove feasibility of funnel control for electrical circuits, we need to define consistency of the initial value of the closed-loop system and solutions of the latter. We also define what “feasibility of funnel control” will mean.

Definition 3.1 (Consistent initial value) *Let $[E, A, B, C] \in \Sigma_{n,m}$, $\varphi \in \Phi$ and $y_{\text{ref}} \in \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. An initial value $x^0 \in \mathbb{R}^n$ is called *consistent* for (1.1), (3.1) if there exists a solution of the initial value problem (1.1), (3.1), $x(0) = x^0$, i.e., a function $x \in \mathcal{C}^1([0, \omega); \mathbb{R}^n)$ for some $\omega \in (0, \infty]$, such that $x(0) = x^0$ and x satisfies (1.1), (3.1) for all $t \in [0, \omega)$.*

In the following we define feasibility of funnel control for a system on a set of reference trajectories. For reference trajectories we allow signals in $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, whereas in [9] signals in $\mathcal{B}^\nu(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ are allowed and $\nu \in \mathbb{N}$ is a number which can be calculated out of a certain system decomposition. To avoid the details of this calculation we restrict ourselves to the case of $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.

Definition 3.2 (Feasibility of funnel control) *Let $[E, A, B, C] \in \Sigma_{n,m}$ and $\mathcal{S} \subseteq \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ be a set of reference trajectories. We say that *funnel control is feasible* for $[E, A, B, C]$ on \mathcal{S} , if for all $\varphi \in \Phi$, any reference signal $y_{\text{ref}} \in \mathcal{S}$ and any consistent initial value $x^0 \in \mathbb{R}^n$ the application of the funnel controller (3.1) to (1.1) yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution $x(\cdot)$,*

- (i) $x(\cdot)$ is bounded and the corresponding tracking error $e(\cdot) = Cx(\cdot) - y_{\text{ref}}(\cdot)$ evolves uniformly within the performance funnel \mathcal{F}_φ ; more precisely,

$$\exists \varepsilon > 0 \forall t > 0 : \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon. \quad (3.2)$$

- (ii) the corresponding gain function $k(\cdot)$ given by (3.1) is bounded.

Remark 3.3 (Bound for the gain) If funnel control is feasible as stated in Definition 3.2, then the gain function k is bounded in a way that, for all $t_0 > 0$, $\sup_{t \geq t_0} |k(t)| \leq 1/(1 - (1 - \varepsilon \lambda_{t_0})^2)$, where ε is given in (3.2) and $\lambda_{t_0} := \inf_{t \geq t_0} \varphi(t) > 0$ for all $t_0 > 0$. A proof for this can be found in [9].

As a preliminary result, we show feasibility of funnel control for positive real systems $[E, A, B, C] \in \Sigma_{n,m}$ (which in particular encompass the class of MNA models (1.2), (1.4)) with asymptotically stable zero dynamics.

Proposition 3.4 (Funnel control for systems with stable zero dynamics) *Let $[E, A, B, C] \in \Sigma_{n,m}$ be such that $E = E^\top \geq 0$, $A + A^\top \leq 0$, and $B = C^\top$. Further, assume that the zero dynamics of $[E, A, B, C]$ are asymptotically stable. Then funnel control is feasible for $[E, A, B, C]$ on $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.*

We may improve the above by relaxing the assumption of asymptotically stable zero dynamics (which in particular requires autonomous zero dynamics) to the assumption of asymptotic stability of the autonomous part of the zero dynamics; the latter is characterized by the stability of the invariant zeros of the system. This leads to the following result, where we show that funnel control is feasible for circuits where all invariant zeros are located in \mathbb{C}_- . However, the drawback is that the reference trajectory must be restricted to a certain subspace in order to account for the non-autonomous part of the zero dynamics.

Theorem 3.5 (Funnel control for circuits) *Let $[E, A, B, C] \in \Sigma_{n,m}$ with (1.2), (1.4) be an MNA model of an electrical circuit and suppose that (A1) and (A2) hold. Assume that the system $[E, A, B, C]$ does not have any invariant zeros on the imaginary axis. Let $Z_{\mathcal{CRLI}}$ be a matrix with full column rank such that $\text{im } Z_{\mathcal{CRLI}} = \ker [A_C \ A_{\mathcal{R}} \ A_L \ A_{\mathcal{I}}]^\top$. Then funnel control is feasible for $[E, A, B, C]$ on $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \text{im } A_{\mathcal{I}}^\top \times \ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}})$.*

The subspace restriction

$$y_{\text{ref}}(t) \in \text{im } A_{\mathcal{I}}^\top \times \ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} \quad \forall t \geq 0 \quad (3.3)$$

on the reference signal can be interpreted as follows: Since we do no longer assume autonomous zero dynamics it is, by Proposition 2.1, possible that the circuit contains \mathcal{I} -loops and \mathcal{V} -cutsets and we have to check the consequences of these two configurations. If the circuit contains a \mathcal{V} -cutset, then, by Kirchhoff's current law, the currents of the voltage sources in the \mathcal{V} -cutset sum up to zero. Since both are components of the output y this leads to an output constraint. Clearly, the reference trajectory has to satisfy this restriction as well if we want funnel control to be feasible. Likewise, if the circuit contains an \mathcal{I} -loop, then Kirchhoff's voltage law implies that the voltages of the current sources in the \mathcal{I} -loop sum up to zero which again leads to an output constraint. Condition (3.3) therefore means that, in a sense, the reference signal has to satisfy Kirchhoff's laws pointwise, see also Figure 2.

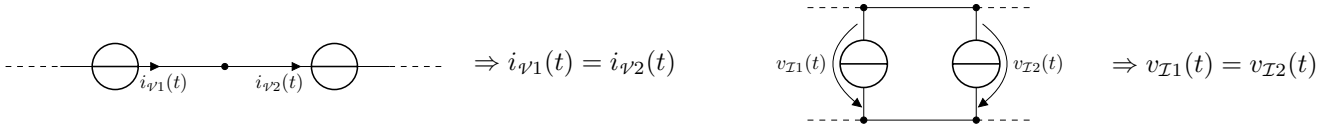


Fig. 2: Interpretation of condition (3.3) in terms of Kirchhoff's laws

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