Controlled invariance for DAEs

Thomas Berger[1,*]

[1] Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

We study the concept of locally controlled invariant submanifolds for nonlinear descriptor systems. In contrast to classical approaches, we define controlled invariance as the property of solution trajectories to evolve in a given submanifold whenever they start in it. It is then shown that this concept is equivalent to the existence of a feedback which renders the closed-loop vector field invariant in the descriptor sense. This result is motivated by a preliminary consideration of the linear case.

Local controlled invariance leads to the concept of output zeroing submanifolds. We show that the outcome of the differential-algebraic version of the zero dynamics algorithm yields a locally maximal output zeroing submanifold.

1 Motivation - linear systems

We study controlled invariance for linear descriptor systems governed by differential-algebraic equations (DAEs),

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{1.1}$$

where $E, A \in \mathbb{R}^{l\times n}$ and $B \in \mathbb{R}^{l\times m}$. The set of these systems is denoted by $\Sigma_{l,n,m}$ and we write $[E, A, B] \in \Sigma_{l,n,m}$. Note that we do not assume regularity of the pencil $sE - A$. The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $x : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *state* of the system, resp. The *behavior* of (1.1) is the set

$$\mathfrak{B}_{(1.1)} := \{\, (x,u) \in \mathcal{C}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m) \mid Ex \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^l) \text{ and } (x,u) \text{ satisfies (1.1) for all } t\in\mathbb{R} \,\}.$$

Definition 1.1 Let $[E, A, B] \in \Sigma_{l,n,m}$ and $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace. Then $\mathcal{V}$ is called *controlled invariant*, if

$$\forall\, x^0 \in \mathcal{V}\ \exists\,(x,u)\in\mathfrak{B}_{(1.1)}\ \forall\, t\ge 0 : \quad x\in\mathcal{C}^1(\mathbb{R};\mathbb{R}^n)\ \wedge\ x(0)=x^0\ \wedge\ x(t)\in\mathcal{V}.$$

For ODEs, characterizations of controlled invariance can be found e.g. in [1]; the following is the DAE version.

Theorem 1.2 *For $[E, A, B] \in \Sigma_{l,n,m}$ and a subspace $\mathcal{V} \subseteq \mathbb{R}^n$ the following statements are equivalent:*

(i) *$\mathcal{V}$ is controlled invariant.*
(ii) *$A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B$.*
(iii) *There exists $F \in \mathbb{R}^{m\times n}$ such that $(A + BF)\mathcal{V} \subseteq E\mathcal{V}$.*

For the proofs and more details on the results in the present paper see [2]. Note that a subspace $\mathcal{V}$ satisfying property (ii) in Theorem 1.2 is usually called a (A, E, B)-invariant subspace, see the survey [3] and the references therein.

2 Nonlinear systems

In this section we consider nonlinear descriptor systems governed by DAEs of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}E(x(t)) = f(x(t))+g(x(t))\,u(t),\ \ y(t) = h(x(t)), \tag{2.1}$$

where $X \subseteq \mathbb{R}^n$ is open, $0 \in X$, $f \in \mathcal{C}(X;\mathbb{R}^l)$, $h \in \mathcal{C}(X;\mathbb{R}^p)$, $E \in \mathcal{C}^1(X;\mathbb{R}^l)$ such that $f(0) = 0$, $h(0) = 0$, and $g \in \mathcal{C}(X;\mathbb{R}^{l\times m})$. The set of these systems is denoted by $\Sigma^X_{l,n,m,p}$; and we write $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$.

A trajectory $(x, u, y) \in \mathcal{C}(I;X\times\mathbb{R}^m\times\mathbb{R}^p)$ is called a *solution* of (2.1), if $I = \operatorname{dom} x \subseteq \mathbb{R}$ is an open interval, $E\circ x \in \mathcal{C}^1(I;\mathbb{R}^l)$ and (x, u, y) solves (2.1) for all $t \in I$. A solution (x, u, y) of (2.1) is called *maximal*, if any other solution $(\tilde{x}, \tilde{u}, \tilde{y})$ of (2.1) satisfies

$$J{:=}\operatorname{dom}\tilde{x}\cap\operatorname{dom}x \ne\emptyset \wedge \tilde{x}|_J = x|_J \ \Rightarrow\ \operatorname{dom}\tilde{x}\subseteq\operatorname{dom}x.$$

The *behavior* of (2.1) is the set of maximal solutions

$$\mathfrak{B}_{(2.1)} := \{\, (x,u,y)\in\mathcal{C}(I;X\times\mathbb{R}^m\times\mathbb{R}^p) \mid I\subseteq\mathbb{R} \text{ open interval, } (x,u,y) \text{ is maximal solution of (2.1)} \,\}.$$

The concept of (locally) controlled invariant submanifolds has been introduced by Isidori and Moog [4], see also the textbooks [5, 6]. Loosely speaking, a connected submanifold M is locally controlled invariant, if it is invariant under the flow of the closed-loop vector field $f(x)+g(x)u(x)$ for some feedback $u(x)$. We show that this "classical" definition in terms of feedback is equivalent to the "natural" definition, that (locally) for any initial value in M there exists an input such that the corresponding state trajectory remains in the submanifold M for all times or reaches its boundary in finite time.

Definition 2.1 Let $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ and M be a connected submanifold of X such that $0 \in M$. Then M is called *locally controlled invariant*, if there exists an open neighborhood $U \subseteq X$ of the origin in $\mathbb{R}^n$ such that

$$\begin{aligned}&\forall\, x^0\in M\cap U\ \exists\,(x,u,y)\in\mathfrak{B}_{(2.1)},\ x\in\mathcal{C}^1(\operatorname{dom}x;\mathbb{R}^n)\\ &\exists\, t_0\in\operatorname{dom}x,\ x(t_0)=x^0 :\\ &\quad\big(\forall\, t\in\operatorname{dom}x,\ t\ge t_0 :\ x(t)\in M\cap U\big)\ \vee\ \big(\exists\,\hat{t}\in\operatorname{dom}x,\\ &\quad \hat{t}>t_0\ \forall\, t\in[t_0,\hat{t}) :\ x(t)\in M{\cap}U\wedge x(\hat{t})\in\partial(M{\cap}U)\big).\end{aligned}$$

Theorem 2.2 *Let $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ be such that $E \in \mathcal{C}^2(X;\mathbb{R}^l)$, $f \in \mathcal{C}^1(X;\mathbb{R}^l)$ and $g \in \mathcal{C}^1(X;\mathbb{R}^{l\times m})$ and let M be a smooth connected submanifold of X such that $0 \in M$. Suppose that there exists an open neighborhood V of $0 \in X$ such that both $\dim E'(x)T_xM$ and $\dim\big(E'(x)T_xM + \operatorname{im} g(x)\big)$ are constant for $x \in M\cap V$. Then the following statements are equivalent:*

* Corresponding author: email thomas.berger@uni-hamburg.de

(i) *M is locally controlled invariant.*

(ii) *There exists an open neighborhood U of $0 \in X$ such that $f(x) \in E'(x)T_xM + \operatorname{im} g(x)$ for all $x \in M \cap U$.*

(iii) *There exists an open neighborhood U of $0 \in X$ and $u \in \mathcal{C}^1(M \cap U; \mathbb{R}^m)$ such that $f(x) + g(x)u(x) \in E'(x)T_xM$ for all $x \in M \cap U$.*

In the remainder of this paper we consider the zero dynamics of (2.1), which is the set of trajectories $\mathcal{ZD}_{(2.1)} := \{ (x,u,y) \in \mathfrak{B}_{(2.1)} \mid y = 0 \}$. The concept of zero dynamics goes back to Byrnes and Isidori [7] and is studied extensively since then, see e.g. [5,6]. For linear DAEs, the zero dynamics have been investigated in detail recently [8–11]. Zero dynamics are related to the concept of output zeroing submanifolds.

Definition 2.3 Let $[E,f,g,h] \in \Sigma^X_{l,n,m,p}$ and M be a connected submanifold of X such that $0 \in M$. Then M is called *output zeroing*, if M is locally controlled invariant and $h(x) = 0$ for all $x \in M$. An output zeroing submanifold M that is called *locally maximal*, if there exists an open neighborhood U of $0 \in X$ such that any output zeroing submanifold $\tilde{M}$ satisfies $\tilde{M} \cap U \subseteq M \cap U$.

We extend the zero dynamics algorithm developed in [4, 12] to nonlinear DAE systems (2.1).

Theorem 2.4 *Let $[E,f,g,h] \in \Sigma^X_{l,n,m,p}$ be such that E,f,g and h are smooth. Define $M_0 := h^{-1}(0)$ and for any $k \in \mathbb{N}$ the set M_k recursively as follows: Suppose that for some open neighborhood U_{k-1} of $0 \in X$, $M_{k-1} \cap U_{k-1}$ is a submanifold, define $\tilde{M}_{k-1} := \bigcup \{ M_{k-1} \cap U \mid U \subseteq X \text{ open},\ M_{k-1} \cap U \text{ is a submanifold} \}$, let M^c_{k-1} be the connected component of $\tilde{M}_{k-1}$ which contains $0 \in X$ and define $M_k := \left\{ x \in M^c_{k-1} \;\middle|\; f(x) \in E'(x)T_xM^c_{k-1} + \operatorname{im} g(x) \right\}$. Then we have the following:*

(i) *The sequence (M_k) is nested, terminates and satisfies*

$$\exists\, k^* \in \mathbb{N}_0\ \forall\, j \in \mathbb{N}:\ M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_{k^*} \supseteq M^c_{k^*} = M_{k^*+j} = M^c_{k^*+j}.$$

(ii) *If $Z^* := M^c_{k^*}$ satisfies, for some open neighborhood U of $0 \in \mathbb{R}$, that $\dim E'(x)T_xZ^*$ and $\dim\big(E'(x)T_xZ^* + \operatorname{im} g(x)\big)$ are both constant for $x \in Z^* \cap U$, then Z^* is a locally maximal output zeroing submanifold.*

(iii) *There exists an open neighborhood U of $0 \in X$ such that for all open $O \subseteq U$ and all $(x,u,y) \in \mathfrak{B}_{(2.1)}$ with $x \in \mathcal{C}^1(\operatorname{dom} x; X)$ and $x(t) \in O$ for all $t \in \operatorname{dom} x$*

$$(x,u,y) \in \mathcal{ZD}_{(2.1)} \iff x(t) \in Z^* \cap O\ \forall\, t \in \operatorname{dom} x.$$

If the system (2.1) is linear, then the sequence (M_k) becomes an augmented Wong sequence, see [3,13] and the references therein, which is based on the Wong sequences [14–16] and which have their origin in [17].

Output zeroing submanifolds can be exploited to study locally autonomous zero dynamics; the latter have been successively used for the analysis of linear time-varying ODEs in [18] and of linear time-invariant DAEs in [9]. Under the assumption of locally autonomous zero dynamics we aim to derive a local zero dynamics form for nonlinear DAE systems (2.1) which would provide the basis for the application of adaptive control techniques. In particular, we aim to use the results of [19] and show feasibility of funnel control for nonlinear descriptor systems which encompass nonlinear electrical circuits, extending the results for the linear case [20].

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