A JORDAN-LIKE DECOMPOSITION FOR LINEAR RELATIONS IN FINITE-DIMENSIONAL SPACES

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Abstract. A square matrix A has the usual Jordan canonical form that describes the structure of A via eigenvalues and the corresponding Jordan blocks. If A is a linear relation in a finite-dimensional linear space $\mathfrak{H}$ (i.e., A is a linear subspace of $\mathfrak{H} \times \mathfrak{H}$ and can be considered as a multivalued linear operator), then there is a richer structure. In addition to the classical Jordan chains (interpreted in the Cartesian product $\mathfrak{H} \times \mathfrak{H}$), there occur three more classes of chains: chains starting at zero (the chains for the eigenvalue infinity), chains starting at zero and also ending at zero (the singular chains), and chains with linearly independent entries (the shift chains). These four types of chains give rise to a direct sum decomposition (a Jordan-like decomposition) of the linear relation A. In this decomposition there is a completely singular part that has the extended complex plane as eigenvalues; a usual Jordan part that corresponds to the finite proper eigenvalues; a Jordan part that corresponds to the eigenvalue ∞; and a multishift, i.e., a part that has no eigenvalues at all. Furthermore, the Jordan-like decomposition exhibits a certain uniqueness, closing a gap in earlier results. The presentation is purely algebraic, only the structure of linear spaces is used. Moreover, the presentation has a uniform character: each of the above types is constructed via an appropriately chosen sequence of quotient spaces. The dimensions of the spaces are the Weyr characteristics, which uniquely determine the Jordan-like decomposition of the linear relation.

1. Introduction

Let $\mathfrak{H}$ be a finite-dimensional linear space over $\mathbb{C}$ and let A be a linear operator in $\mathfrak{H}$ with $\operatorname{dom} A = \mathfrak{H}$, i.e., A is defined everywhere and admits a representation as a matrix. Then there is at least one eigenvalue $\lambda \in \mathbb{C}$ and to each eigenvalue belong chains of linearly independent vectors $x_1, \dots, x_n$, the so-called Jordan chains

$$(A-\lambda)x_n = x_{n-1},\ (A-\lambda)x_{n-1} = x_{n-2},\ \dots\ ,\ (A-\lambda)x_1 = 0. \tag{1.1}$$

The Jordan canonical form of a matrix offers a decomposition in terms of these Jordan chains. For each $\lambda \in \mathbb{C}$ define the sequence of quotient spaces

$$\ker(A-\lambda),\ \frac{\ker(A-\lambda)^2}{\ker(A-\lambda)},\ \frac{\ker(A-\lambda)^3}{\ker(A-\lambda)^2},\ \cdots \tag{1.2}$$

and the corresponding Weyr characteristic by the sequence

$$\dim\ker(A-\lambda),\ \dim\frac{\ker(A-\lambda)^2}{\ker(A-\lambda)},\ \dim\frac{\ker(A-\lambda)^3}{\ker(A-\lambda)^2},\ \cdots . \tag{1.3}$$

Then the Jordan canonical form is the unique representative of the equivalence class of A with respect to similarity, and it is uniquely determined by the Weyr

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characteristic. Furthermore, two matrices are similar if and only if their Weyr characteristics coincide.

For a recent treatment of the Weyr characteristic of matrices and a historical discussion see [24] (and also [25]). The above condition that $\operatorname{dom} A = \mathfrak{H}$ ensures the existence of at least one eigenvalue. If this condition is not satisfied or if A is a linear relation (multivalued operator), then new phenomena may occur. For instance, it may happen that A has no eigenvalues at all and the Jordan canonical form breaks down.

The purpose of the present note is to derive a general decomposition for a linear relation A in a finite-dimensional space $\mathfrak{H}$ over $\mathbb{C}$, i.e., A is a subspace of $\mathfrak{H} \times \mathfrak{H}$. Linear relations in linear spaces date back to [1], see also [2, 5, 10, 23]. Compared to linear operators, linear relations may have an eigenvalue ∞ with its own Jordan chains. However, in the context of a linear relation A there is also a new feature: it may happen that the usual point spectrum $\sigma_p(A)$ is equal to the extended complex plane. For instance, this is the case when there exists a nontrivial element in $\ker A \cap \operatorname{mul} A$. By splitting off the so-called completely singular part A_S of A, there remains the proper point spectrum $\sigma_\pi(A)$ of A, consisting of finitely many points in $\mathbb{C} \cup \{\infty\}$; see [5]. The main result is the following Jordan-like direct sum decomposition of the linear relation A:

$$A = A_S \oplus J_{\lambda_1}(A) \oplus \cdots \oplus J_{\lambda_l}(A) \oplus J_\infty(A) \oplus A_M, \tag{1.4}$$

where $J_\lambda(A)$ stands for the Jordan part corresponding to $\lambda \in \sigma_\pi(A) = \{\lambda_1, \ldots, \lambda_l\} \cup \{\infty\}$, and A_M is a multishift, i.e., a linear operator without eigenvalues; cf. Theorem 6.1 for the precise meaning of (1.4). For each component in (1.4), there is, parallel to the case of matrices in (1.3), a suitably chosen sequence of quotient spaces with its own Weyr characteristic. The collection of the Weyr characteristics of each part defines the *Weyr characteristic* of the linear relation A; cf. Definition 6.3. It is a complete set of invariants which justifies to view (1.4) as a Jordan-like decomposition: *The decomposition* (1.4) *is uniquely determined by the Weyr characteristic and it is the unique representative of the equivalence class with respect to strict similarity*; cf. Section 6.

In a sense, the present paper can be seen as a completion of the results in [22] with a purely linear algebra approach; see also [5]. In fact, the decomposition derived in [22] exhibits a certain non-uniqueness; cf. Section 6. This is resolved by utilizing the concept of a reducing sum decomposition, which is intrinsically unique; cf. Section 2. To achieve such a decomposition, it is required to use a completely different construction of the subrelations.

The present paper is organized as follows: The necessary notions of root spaces, chains and reducing sum decompositions for linear relations are recalled in Section 2. The construction of each part in the decomposition (1.4) of the linear relation A follows a uniform pattern: The discussion of the four kinds of sequences of quotient spaces, and the resulting chain structure is the content of Sections 3, 4, and 5; see (3.2), (4.5), (4.21), (4.39), and (5.1). In Section 6 the main decomposition results of the paper are collected and explained in terms of the Weyr characteristic. Section 6 also contains a brief discussion of related literature.

The above characterization of linear relations via their Weyr characteristics is new and allows for a variety of applications, in particular to linear matrix pencils. With any linear pencil one may associate a kernel and a range representation, which are two different linear relations. The relationship between the reducing sum decomposition (1.4) of these linear relations and the Kronecker canonical form of the original matrix pencil is of great interest.

2. Preliminaries

2.1. Linear relations. A linear relation A in a finite dimensional linear space $\mathfrak{H}$ is a subspace of $\mathfrak{H} \times \mathfrak{H}$. In the following a brief review of the usual notions in the context of linear relations is given:

$$\begin{aligned} \operatorname{dom} A &= \{x \in \mathfrak{H} : \exists\, y \in \mathfrak{H} \text{ with } (x,y) \in A\}, \text{ domain}, \\ \ker A &= \{x \in \mathfrak{H} : (x,0) \in A\}, \text{ kernel}, \\ \operatorname{ran} A &= \{y \in \mathfrak{H} : \exists\, x \in \mathfrak{H} \text{ with } (x,y) \in A\}, \text{ range}, \\ \operatorname{mul} A &= \{y \in \mathfrak{H} : (0,y) \in A\}, \text{ multivalued part}. \end{aligned}$$

Note that the inverse of A is a linear relation given by $A^{-1} = \{(y,x) : (x,y) \in A\}$. Hence there are the formal identities $\operatorname{dom} A^{-1} = \operatorname{ran} A$ and $\ker A = \operatorname{mul} A^{-1}$. In addition, recall the following definitions of the product and sum of linear relations A and B; and, in particular, of $A - \lambda$ and λA when $\lambda \in \mathbb{C}$:

$$\begin{aligned} AB &= \{(x,z) \in \mathfrak{H} \times \mathfrak{H} : \exists\, z \in \mathfrak{H} \text{ with } (x,z) \in B, (z,y) \in A\}, \text{ product}, \\ \lambda A &= \{(x, \lambda y) \in \mathfrak{H} \times \mathfrak{H} : (x,y) \in A\}, \\ A + B &= \{(x, y+z) \in \mathfrak{H} \times \mathfrak{H} : \exists\, (x,y) \in A \text{ with } (x,z) \in B\}, \text{ sum}, \\ A - \lambda &= A - \lambda I = \{(x, y - \lambda x) \in \mathfrak{H} \times \mathfrak{H} : (x,y) \in A\}, \end{aligned}$$

where $I = \{(x,x) \in \mathfrak{H} \times \mathfrak{H} : x \in \mathfrak{H}\}$ stands for the identity operator.

2.2. Root spaces and Jordan chains. The usual *point spectrum* $\sigma_p(A)$ is the set of all eigenvalues $\lambda \in \mathbb{C} \cup \{\infty\}$ of the relation A:

$$\sigma_p(A) = \Big\{\ \lambda \in \mathbb{C} \cup \{\infty\}\ :\ \ker(A - \lambda) \neq \{0\},\ \text{if } \lambda \in \mathbb{C}, \quad \text{or } \operatorname{mul} A \neq \{0\},\ \text{if } \lambda = \infty \Big\} \tag{2.1}$$

The *root spaces* $\mathfrak{R}_\lambda(A)$ of A for $\lambda \in \mathbb{C} \cup \{\infty\}$ are linear subspaces of $\mathfrak{H}$ defined by

$$\begin{aligned} \mathfrak{R}_\lambda(A) &= \operatorname{span}\{\ker(A - \lambda)^i : \lambda \in \mathbb{C},\ i \in \mathbb{N}\}, \\ \mathfrak{R}_\infty(A) &= \operatorname{span}\{\operatorname{mul} A^i : i \in \mathbb{N}\}. \end{aligned} \tag{2.2}$$

Note that $x \in \mathfrak{R}_\lambda(A)$, $\lambda \in \mathbb{C}$, if and only if for some $n \in \mathbb{N}$ there exists a chain of elements in $\mathfrak{H} \times \mathfrak{H}$ of the form

$$(x_n, x_{n-1} + \lambda x_n), (x_{n-1}, x_{n-2} + \lambda x_{n-1}), \ldots, (x_2, x_1 + \lambda x_2), (x_1, \lambda x_1) \in A \tag{2.3}$$

such that $x = x_n$, the "endpoint" of (2.3); for all $1 \leq i \leq n$ one has $(x_i, 0) \in (A - \lambda)^i$. If $x_1 \neq 0$, then the chain in (2.3) is said to be a *Jordan chain* for A corresponding to the eigenvalue $\lambda \in \mathbb{C}$. Likewise, $y \in \mathfrak{R}_\infty(A)$ if and only if for some $m \in \mathbb{N}$ there exists a chain of elements in $\mathfrak{H} \times \mathfrak{H}$ of the form

$$(0, y_1),\ (y_1, y_2),\ \ldots\ldots\ldots, (y_{m-2}, y_{m-1}),\ (y_{m-1}, y_m) \in A \tag{2.4}$$

such that $y = y_m$, the "endpoint" of (2.4). If $y_1 \neq 0$, then the chain in (2.4) is said to be a *Jordan chain* for A corresponding to the eigenvalue ∞; for all $1 \leq i \leq m$ one has $(0, y_i) \in A^i$. The *total root space* $\mathfrak{R}_r(A)$ of A is a linear subspace of $\mathfrak{H}$ defined by

$$\mathfrak{R}_r(A) = \operatorname{span}\{\mathfrak{R}_\lambda(A) : \lambda \in \mathbb{C} \cup \{\infty\}\}; \tag{2.5}$$

see (2.2). Clearly, an element belongs to $\mathfrak{R}_r(A)$ if and only if it is the "endpoint" in the above sense of a chain in (2.3) or of a chain in (2.4).

2.3. **Singular chains.** The *singular chain subspace* $\mathfrak{R}_c(A)$ of A is a linear subspace of the total root space $\mathfrak{R}_r(A)$ defined by

$$\mathfrak{R}_c(A) = \mathfrak{R}_0(A) \cap \mathfrak{R}_\infty(A); \tag{2.6}$$

cf. [22]. Note that $u \in \mathfrak{R}_c(A)$ if and only if for some $k \in \mathbb{N}$ there is a chain of elements of the form

$$(0, u_k),\ (u_k, u_{k-1}),\ \ldots\ldots\ldots, (u_2, u_1),\ (u_1, 0) \in A \tag{2.7}$$

such that $u = u_l$ for some $1 \leq l \leq k$. The chain in (2.7) is said to be a *singular chain* for A. It is clear from (2.7) that $\mathfrak{R}_c(A) \subset \operatorname{dom} A \cap \operatorname{ran} A$, and that $\mathfrak{R}_c(A) \neq \{0\}$ implies that $\ker A \cap \mathfrak{R}_c(A)$ and $\operatorname{mul} A \cap \mathfrak{R}_c(A)$ are non-trivial. The singular chain space $\mathfrak{R}_c(A)$ can also be written as follows (for a proof, see [5]): for any $\lambda, \mu \in \mathbb{C} \cup \{\infty\}$ with $\lambda \neq \mu$ one has

$$\mathfrak{R}_c(A) = \mathfrak{R}_\lambda(A) \cap \mathfrak{R}_\mu(A) \tag{2.8}$$

so that, in particular,

$$\mathfrak{R}_c(A) \subset \mathfrak{R}_\lambda(A), \quad \lambda \in \mathbb{C} \cup \{\infty\}. \tag{2.9}$$

2.4. **Proper point spectrum.** In order to discuss a reducing sum decomposition (see Definition 2.1 below) in terms of $\mathfrak{R}_c(A)$ and $\mathfrak{R}_r(A)$, one needs to consider a certain restriction of the point spectrum of A. It is clear that if $\mathfrak{R}_c(A) \neq \{0\}$, then $\mathfrak{R}_\lambda(A) \neq \{0\}$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$, so that $\sigma_p(A) = \mathbb{C} \cup \{\infty\}$. In fact, it is known that, due to finite-dimensionality,

$$\sigma_p(A) = \mathbb{C} \cup \{\infty\} \quad \text{if and only if} \quad \mathfrak{R}_c(A) \neq \{0\},$$

see [22, Prop. 3.2, Thm. 4.4]. The *proper point spectrum*, see [5], is a subset of the point spectrum $\sigma_p(A)$ and defined by

$$\sigma_\pi(A) = \{\lambda \in \sigma_p(A) :\ \mathfrak{R}_\lambda(A) \setminus \mathfrak{R}_c(A) \neq \emptyset\}, \tag{2.10}$$

cf. (2.9). The elements in $\sigma_\pi(A)$ are called the *proper eigenvalues* of A. As a consequence of (2.10), observe that

$$\mathfrak{R}_r(A) = \operatorname{span}\{\mathfrak{R}_\lambda(A) :\ \lambda \in \sigma_\pi(A)\} \cup \mathfrak{R}_c(A). \tag{2.11}$$

Note that if $\mathfrak{R}_c(A) = \{0\}$, then $\sigma_\pi(A) = \sigma_p(A)$. Entries of chains belonging to different proper eigenvalues in $\sigma_\pi(A)$ are linearly independent and hence

$$|\sigma_\pi(A)| \leq \dim \mathfrak{H},$$

see [5], so that $\sigma_\pi(A)$ is a finite set, since $\mathfrak{H}$ is assumed to be finite-dimensional. The proper point spectrum $\sigma_\pi(A)$ of a linear relation A will be the substitute for the usual point spectrum $\sigma_p(A)$ in the operator case.

2.5. **Shift chains.** To complete the description of the structure of a linear relation A, one needs to go beyond the total root space $\mathfrak{R}_r(A)$. A collection of linearly independent elements $x_1, \dots x_n$ in $\mathfrak{H}$ is called a *shift chain* if

$$(x_1, x_2), \dots, (x_{n-1}, x_n) \in A. \tag{2.12}$$

Shift chains, in a sense, extend the notions of singular and Jordan chains:

- if additionally $(x_n, 0) \in A$, then (2.12) is a Jordan chain at 0,
- if additionally $(0, x_1) \in A$, then (2.12) is a Jordan chain at ∞,
- if additionally $(0, x_1), (x_n, 0) \in A$, then (2.12) is a singular chain.

On the other hand, they exhibit completely different spectral properties: the linear relation spanned by the elements in (2.12) is an operator without point spectrum in $\mathbb{C}$. A linear relation A in a finite-dimensional linear space $\mathfrak{H}$ is said to be a *multishift* if A has no eigenvalues in $\mathbb{C} \cup \{\infty\}$ (i.e., if A is a linear operator without eigenvalues in $\mathbb{C}$). It will be shown that there exists a linear subspace $\mathfrak{R}_m(A) \subset \mathfrak{H}$, spanned by entries of shift chains, such that it complements the subspace $\mathfrak{R}_r(A)$, and the graph restriction of A to $\mathfrak{R}_m(A)$ is given by

$$A_M = A \cap (\mathfrak{R}_m(A) \times \mathfrak{R}_m(A)); \tag{2.13}$$

cf. Theorem 5.2. The relation A_M in (2.13) is a multishift.

2.6. **Reducing sum decompositions.** Here is a brief review of reducing sum decompositions for linear relations in a linear space $\mathfrak{H}$. Recall that subspaces $\mathfrak{H}_j \subset \mathfrak{H}$ for $j = 1, \dots, n$ of $\mathfrak{H}$ are said to form a *direct sum*, denoted by

$$\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots \oplus \mathfrak{H}_n \tag{2.14}$$

if $0 = x_1 + x_2 + \ldots + x_n$ with elements $x_j \in \mathfrak{H}_j$ implies that $x_j = 0$ for $j = 1, \dots, n$. In particular, each $x \in \mathfrak{H}_1 + \mathfrak{H}_2 + \cdots + \mathfrak{H}_n$ admits a sum $x = x_1 + x_2 + \ldots + x_n$ with unique elements $x_j \in \mathfrak{H}_j$ for $j = 1, \dots, n$.

A linear relation A in a linear space $\mathfrak{H}$ is a linear subspace of $\mathfrak{H} \times \mathfrak{H}$. The componentwise sum $A_1 \widehat{+} A_2$ of linear relations A_1 and A_2 in a linear space $\mathfrak{H}$ is defined as the sum of the subspaces in $\mathfrak{H} \times \mathfrak{H}$:

$$A_1 \widehat{+} A_2 = \{(x+u, y+v) \in \mathfrak{H} \times \mathfrak{H} : (x, y) \in A_1, \, (u, v) \in A_2\}.$$

If a sum $A_1 \widehat{+} A_2 \widehat{+} \cdots \widehat{+} A_n$ of linear relations A_j in $\mathfrak{H}$ is direct, it is denoted by

$$A_1 \oplus A_2 \oplus \cdots \oplus A_n.$$

For any linear subspace $\mathfrak{X} \subset \mathfrak{H}$ one defines the *graph restriction* of A to $\mathfrak{X}$ as $A' = A \cap (\mathfrak{X} \times \mathfrak{X})$, so that A' is a linear relation in $\mathfrak{X}$.

Define the linear space $\mathfrak{H}(A)$ by $\mathfrak{H}(A) = \operatorname{dom} A + \operatorname{ran} A$. Then clearly,

$$A \subset \mathfrak{H}(A) \times \mathfrak{H}(A) \subset \mathfrak{H} \times \mathfrak{H}. \tag{2.15}$$

Hence A coincides with its graph restriction to $\mathfrak{H}(A)$. Moreover, one sees that $\mathfrak{H}(A)$ is the smallest subspace $\mathfrak{X} \subset \mathfrak{H}$ with the property that $A \subset \mathfrak{X} \times \mathfrak{X} \subset \mathfrak{H} \times \mathfrak{H}$.

Definition 2.1. *Let A be a linear relation in a linear space $\mathfrak{H}$. If*

$$A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \tag{2.16}$$

is a direct sum of its graph restrictions A_j to subspaces $\mathfrak{H}_j \subset \mathfrak{H}$ for $j = 1, \dots, n$, which form a direct sum $\mathfrak{H}_1 \oplus \mathfrak{H}_2 \oplus \cdots \oplus \mathfrak{H}_n$, then the decomposition (2.16) *is called a reducing sum decomposition of A with respect to $(\mathfrak{H}_1, \dots, \mathfrak{H}_n)$.*

Instead of calling (2.16) a reducing sum decomposition of A with respect to $(\mathfrak{H}_1, \dots, \mathfrak{H}_n)$, it is frequently called a reducing sum decomposition of A with respect to (2.14).

Lemma 2.2. *Let A be a linear relation in $\mathfrak{H}$ with a reducing sum decomposition* (2.16) *with respect to $(\mathfrak{H}_1, \dots, \mathfrak{H}_n)$. If $\mathfrak{H} = \operatorname{dom} A + \operatorname{ran} A$, then*

$$\mathfrak{H}_j = \operatorname{dom} A_j + \operatorname{ran} A_j, \quad j = 1, \dots, n. \tag{2.17}$$

Proof. It follows from (2.14) and (2.16) that

$$\operatorname{dom} A = \operatorname{dom} A_1 \oplus \cdots \oplus \operatorname{dom} A_n, \quad \operatorname{ran} A = \operatorname{ran} A_1 \oplus \cdots \oplus \operatorname{ran} A_n,$$

so that since $\operatorname{dom} A_j + \operatorname{ran} A_j \subset \mathfrak{H}_j$ for $j = 1, \dots, n$, one has

$$\operatorname{dom} A + \operatorname{ran} A = (\operatorname{dom} A_1 + \operatorname{ran} A_1) \oplus \cdots \oplus (\operatorname{dom} A_n + \operatorname{ran} A_n) \subset \mathfrak{H}_1 \oplus \cdots \oplus \mathfrak{H}_n \subset \mathfrak{H}.$$

Therefore, the identity $\mathfrak{H} = \operatorname{dom} A + \operatorname{ran} A$ implies that (2.17) holds. □

Note that Lemma 2.2 implies that for any reducing sum decomposition (2.16) with respect to $(\mathfrak{H}_1, \dots, \mathfrak{H}_n)$ one has that, since $A \subset \mathfrak{H}(A) \times \mathfrak{H}(A)$, $\mathfrak{H}_1 \oplus \cdots \oplus \mathfrak{H}_n = \mathfrak{H}(A)$.

Finally, it is emphasized that any reducing sum decomposition is intrinsically unique when the decomposition $\mathfrak{H}(A) = \mathfrak{H}_1 \oplus \cdots \oplus \mathfrak{H}_n$ is fixed, as the linear relations A_j, $1 \leq j \leq n$, are defined as the graph restrictions of A to $\mathfrak{H}_j$.

3. The completely singular part of a linear relation

Let A be a linear relation in a finite-dimensional linear space $\mathfrak{H}$. The *completely singular part* A_S of A is a linear relation defined as the graph restriction of A to the singular chain subspace $\mathfrak{R}_c(A)$:

$$A_S = A \cap (\mathfrak{R}_c(A) \times \mathfrak{R}_c(A)). \tag{3.1}$$

A linear relation A in $\mathfrak{H}$ is called *completely singular* if $A = A_S$. In this section it will be shown that A_S is spanned by singular chains as in (2.7). The existence of such a basis was established in [22, Thm. 7.2]. The new proof in Theorem 3.2 below is more in line with similar constructions in later sections and it reveals some additional properties of the basis elements and the connection to the Weyr characteristic.

The construction of the basis of singular chains is based on an appropriate choice of quotient spaces involving $\mathfrak{R}_c(A)$. First, recall that $\ker A^k \subset \ker A^{k+1}$ for all $k \geq 1$. The sequence of quotient spaces $\mathfrak{K}_k(A)$ is defined by

$$\mathfrak{K}_1(A) = \ker A \cap \mathfrak{R}_c(A), \quad \mathfrak{K}_k(A) := \frac{\ker A^k \cap \mathfrak{R}_c(A)}{\ker A^{k-1} \cap \mathfrak{R}_c(A)}, \quad k \geq 2. \tag{3.2}$$

Indeed, since the denominator is included in the numerator, each quotient space $\mathfrak{K}_k(A)$, $k \geq 2$, is well defined. The *Weyr characteristic* of A with respect to the sequence of quotient spaces in (3.2) is defined as the sequence $(B_k)_{k \geq 1}$ with

$$B_k := \dim \mathfrak{K}_k(A), \quad k \geq 1. \tag{3.3}$$

Observe that if $\mathfrak{R}_c(A) = \{0\}$, then $B_k = 0$ for all $k \geq 1$. In this case A_S is trivial.

Now the case $\mathfrak{R}_c(A) \neq \{0\}$ will be considered. Then the sequence in (3.3) is not trivial, although ultimately the entries are zero. To see this, observe that since the linear space $\mathfrak{H}$ is finite-dimensional the number

$$d = \min\big\{k \in \mathbb{N} : \ker A^{k+1} \cap \mathfrak{R}_c(A) = \ker A^k \cap \mathfrak{R}_c(A)\big\} \tag{3.4}$$

is well defined.

Lemma 3.1. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ and assume $\mathfrak{R}_c(A) \neq \{0\}$. Let $d \geq 1$ be given by* (3.4)*, then for $k > d$ one has*

$$\ker A^k \cap \mathfrak{R}_c(A) = \ker A^d \cap \mathfrak{R}_c(A) \quad \textit{and, hence,} \quad B_k = 0.$$

Moreover, $B_1 \geq 1$.

Proof. Due to (3.4) it suffices to show that $\ker A^k \cap \mathfrak{R}_c(A) = \ker A^{k+1} \cap \mathfrak{R}_c(A)$ for some $k \in \mathbb{N}$ implies that $\ker A^{k+1} \cap \mathfrak{R}_c(A) = \ker A^{k+2} \cap \mathfrak{R}_c(A)$. Therefore, let $x \in \ker A^{k+2} \cap \mathfrak{R}_c(A)$. Then there exist $x_1, \dots, x_{k+1}$ such that

$$(x, x_{k+1}), (x_{k+1}, x_k)(x_k, x_{k-1}), \dots, (x_1, 0) \in A. \tag{3.5}$$

As $x \in \mathfrak{R}_c(A)$, also $x_j \in \mathfrak{R}_c(A)$ for $1 \leq k \leq k+1$. Moreover, $x_{k+1} \in \ker A^{k+1}$ and, by assumption, $x_{k+1} \in \ker A^k \cap \mathfrak{R}_c(A)$. Thus there exist $x'_1, \dots, x'_{k-1}$ such that

$$(x_{k+1}, x'_{k-1}), (x'_{k-1}, x'_{k-2}), \dots, (x'_1, 0) \in A.$$

In combination with (3.5) one obtains

$$(x, x_{k+1}), (x_{k+1}, x'_{k-1}), (x'_{k-1}, x'_{k-2}), \dots, (x'_1, 0) \in A$$

and $x \in \ker A^{k+1} \cap \mathfrak{R}_c(A)$ follows. This shows

$$\ker A^{k+2} \cap \mathfrak{R}_c(A) \subset \ker A^{k+1} \cap \mathfrak{R}_c(A).$$

The opposite inclusion follows from the fact that $\ker A^{k+1} \subset \ker A^{k+2}$.

To see that $B_1 \geq 1$, observe that $\mathfrak{R}_c(A) \neq \{0\}$ implies $\ker A \cap \mathfrak{R}_c(A) \neq \{0\}$. □

Theorem 3.2. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ and assume $\mathfrak{R}_c(A) \neq \{0\}$. Let $d \geq 1$ be given by* (3.4)*, then the Weyr characteristic $(B_k)_{k\geq 1}$ in* (3.3) *satisfies*

$$B_1 \geq B_2 \geq \cdots \geq B_d \geq 1 \quad \textit{and} \quad B_k = 0, \quad k > d. \tag{3.6}$$

Moreover, there exist singular chains for A of the following form

$$\begin{array}{rr} (0, x_d^i),\ (x_d^i, x_{d-1}^i),\ (x_{d-1}^i, x_{d-2}^i),\ \dots,\ (x_2^i, x_1^i),\ (x_1^i, 0), & 1 \leq i \leq B_d, \\ (0, x_{d-1}^i),\ (x_{d-1}^i, x_{d-2}^i),\ \dots,\ (x_2^i, x_1^i),\ (x_1^i, 0), & B_d + 1 \leq i \leq B_{d-1}, \\ \ddots \qquad \vdots \qquad \vdots & \vdots \\ (0, x_2^i),\ (x_2^i, x_1^i),\ (x_1^i, 0), & B_3 + 1 \leq i \leq B_2, \\ (0, x_1^i),\ (x_1^i, 0), & B_2 + 1 \leq i \leq B_1, \end{array} \tag{3.7}$$

where $\{[x_k^1], \dots, [x_k^{B_k}]\}$ is a basis of $\mathfrak{K}_k(A)$, $1 \leq k \leq d$, and, consequently, the elements in the set $\big\{x_k^i : 1 \leq i \leq B_k,\ 1 \leq k \leq d\big\}$ are linearly independent in $\mathfrak{H}$. Furthermore, the completely singular part A_S defined in (3.1) *admits the representation*

$$\begin{aligned} A_S = \operatorname{span}\big\{&(0, x_k^i),\ (x_k^i, x_{k-1}^i),\ \dots,\ (x_2^i, x_1^i),\ (x_1^i, 0) : \\ &B_{k+1} + 1 \leq i \leq B_k,\ 1 \leq k \leq d\big\}. \end{aligned} \tag{3.8}$$

In particular, $\operatorname{dom} A_S = \operatorname{ran} A_S = \mathfrak{R}_c(A)$ *and the total dimension of* A_S *is*

$$\dim A_S = 2B_1 + B_2 + B_3 + \ldots + B_{d-1} + B_d. \tag{3.9}$$

Proof. The main tool in the proof is the existence of linear relations $A_k \subset \mathfrak{K}_k(A) \times \mathfrak{K}_{k-1}(A)$, $2 \leq k \leq d$, which are injective, i.e., $\ker A_k = \{0\}$. From this, it follows that the sequence $(B_k)_{k\geq 1}$ is nonincreasing. The singular chains for A will be constructed via suitably chosen bases in each of the quotient spaces $\mathfrak{K}_k(A)$, $1 \leq k \leq d$, beginning with $\mathfrak{K}_d(A)$ and working backwards to $\mathfrak{K}_1(A)$. This procedure is carried out in a number of steps.

Step 1: Let $1 \leq k \leq d$ and let $x \in [x] \in \mathfrak{K}_k(A)$. Then there exists $y \in \mathfrak{H}$ such that

$$(x, y) \in A, \quad x \in \ker A^k \cap \mathfrak{R}_c(A), \quad y \in \ker A^{k-1} \cap \mathfrak{R}_c(A).$$

For $k = 1$ this means that $(x, 0) \in A$. To see the implication, note that by definition $x \in \ker A^k \cap \mathfrak{R}_c(A)$. Since $x \in \ker A^k$, there is some $y \in \ker A^{k-1}$ such that $(x, y) \in A$ and $(y, 0) \in A^{k-1}$. Since $x \in \mathfrak{R}_c(A)$, it follows that $y \in \mathfrak{R}_c(A)$. Therefore one concludes that $y \in \ker A^{k-1} \cap \mathfrak{R}_c(A)$.

Step 2: Define the linear relation $A_k \subset \mathfrak{K}_k(A) \times \mathfrak{K}_{k-1}(A)$, $2 \leq k \leq d$, as follows:

$$A_k := \{([x], [y]) \in \mathfrak{K}_k(A) \times \mathfrak{K}_{k-1}(A) : \exists\, (x', y') \in A \text{ with } [x'] = [x] \text{ and } [y'] = [y]\}. \tag{3.10}$$

By Step 1 it is clear that A_k is defined on all of $\mathfrak{K}_k(A)$, $1 \leq k \leq d$.

Moreover, A_k is injective for $k \geq 2$, that is, $\ker A_k = \{0\}$. To see this, let $([x], [0]) \in A_k$. Then there exists $(x', y') \in A$ with $[x] = [x'] \in \mathfrak{K}_k(A)$ and $[y'] = [0] \in \mathfrak{K}_{k-1}(A)$. Hence $x' \in \ker A^k \cap \mathfrak{R}_c(A)$ and $y' \in \ker A^{k-2} \cap \mathfrak{R}_c(A)$. As $y' \in \mathfrak{R}_c(A)$ there exists $z' \in \mathfrak{R}_c(A)$ with $(z', y') \in A$. Thus, $z' \in \ker A^{k-1}$ and $(x' - z', 0) \in A$. Since $\ker A \subset \ker A^{k-1}$, it follows that

$$x' = x' - z' + z' \in \ker A + \ker A^{k-1} \subset \ker A^{k-1}.$$

This gives $[x] = [x'] = 0$ and shows that A_k is injective. As a consequence, the sequence $(B_k)_{k\geq 1}$ is nonincreasing. Recall that $B_k = 0$ for $k > d$ by Lemma 3.1.

Step 3: The construction of the singular chains for A is associated with the quotient spaces $\mathfrak{K}_d(A), \ldots, \mathfrak{K}_1(A)$, where $d \geq 1$. Since $\dim \mathfrak{K}_d = B_d$, let

$$\{[v_d^1], \ldots, [v_d^{B_d}]\} \tag{3.11}$$

be some basis for $\mathfrak{K}_d(A)$.

First assume that $d = 1$. In this case $(v_1^i, 0) \in A$, $1 \leq i \leq B_1$. As $v_1^i \in \mathfrak{R}_c(A)$, there exist $z_2^i \in \mathfrak{R}_c(A)$ with $(z_2^i, v_1^i) \in A$, so that $z_2^i \in \ker A^2 \cap \mathfrak{R}_c(A) = \ker A \cap \mathfrak{R}_c(A)$ by (3.4). Hence $(z_2^i, 0) \in A$, and it follows that $(0, v_1^i) = (z_2^i, v_1^i) - (z_2^i, 0) \in A$. Thus with the choice $x_1^i = v_1^i$, $1 \leq i \leq B_1$, the theorem has been proved when $d = 1$.

Next assume that $d \geq 2$. Then there are $d - 1$ linear relations

$$A_d \subset \mathfrak{K}_d(A) \times \mathfrak{K}_{d-1}(A), \quad A_{d-1} \subset \mathfrak{K}_{d-1}(A) \times \mathfrak{K}_{d-2}(A), \quad \ldots, \quad A_2 \subset \mathfrak{K}_2(A) \times \mathfrak{K}_1(A),$$

of the form (3.10). With the choice (3.11) it follows from Step 1 that there are elements v_{d-1}^i such that for $1 \leq i \leq B_d$:

$$(v_d^i, v_{d-1}^i) \in A, \quad v_d^i \in \ker A^d \cap \mathfrak{R}_c(A), \quad v_{d-1}^i \in \ker A^{d-1} \cap \mathfrak{R}_c(A).$$

As $v_d^i \in \mathfrak{R}_c(A)$, there exists $z_{d+1}^i \in \mathfrak{R}_c(A)$ such that $(z_{d+1}^i, v_d^i) \in A$, and since $v_d^i \in \ker A^d$ this implies $z_{d+1}^i \in \ker A^{d+1}$. Therefore, by (3.4) one concludes that $z_{d+1}^i \in \ker A^d \cap \mathfrak{R}_c(A)$. Thus there exist numbers $\alpha_d^{i,j}$, $j = 1, \dots, B_d$, with

$$z_{d+1}^i = y_{d-1}^i + \sum_{j=1}^{B_d} \alpha_d^{i,j} v_d^j,$$

where $y_{d-1}^i \in \ker A^{d-1} \cap \mathfrak{R}_c(A)$. Hence, one finds $u_{d-1}^i \in \ker A^{d-2}$ with

$$(y_{d-1}^i, u_{d-1}^i) \in A,$$

and

$$\left(z_{d+1}^i, u_{d-1}^i + \sum_{j=1}^{B_d} \alpha_d^{i,j} v_{d-1}^j \right) = \left(y_{d-1}^i + \sum_{j=1}^{B_d} \alpha_d^{i,j} v_d^j, u_{d-1}^i + \sum_{j=1}^{B_d} \alpha_d^{i,j} v_{d-1}^j \right) \in A,$$

which, via $(z_{d+1}^i, v_d^i) \in A$, implies

$$\left(0, v_d^i - u_{d-1}^i - \sum_{j=1}^{B_d} \alpha_d^{i,j} v_{d-1}^j \right) \in A.$$

This result suggests to define the elements x_d^i, $1 \le i \le B_d$, by

$$x_d^i := v_d^i - u_{d-1}^i - \sum_{j=1}^{B_d} \alpha_d^{i,j} v_{d-1}^j, \quad 1 \le i \le B_d.$$

Clearly, $x_d^i \in \ker A^d \cap \mathfrak{R}_c(A)$ for $1 \le i \le B_d$, and they provide a basis for the quotient space $\mathfrak{K}_d(A)$:

$$\text{span}\{[x_d^1], \dots, [x_d^{B_d}]\} = \mathfrak{K}_d(A) \quad \text{with} \quad (0, x_d^i) \in A, \quad 1 \le i \le B_d. \tag{3.12}$$

Again, by Step 1, with the elements x_d^i from (3.12) there exist elements x_{d-1}^i such that for $1 \le i \le B_d$:

$$(x_d^i, x_{d-1}^i) \in A, \quad x_d^i \in \ker A^d \cap \mathfrak{R}_c(A), \quad x_{d-1}^i \in \ker A^{d-1} \cap \mathfrak{R}_c(A).$$

Observe that by definition

$$([x_d^i], [x_{d-1}^i]) \in A_d,$$

so that by Step 2 the elements $[x_{d-1}^i]$, $1 \le i \le B_d$, are linearly independent in $\mathfrak{K}_{d-1}(A)$. Since $\dim \mathfrak{K}_{d-1}(A) = B_{d-1} \ge B_d$, one can enlarge the family $\{[x_{d-1}^1], \dots, [x_{d-1}^{B_d}]\}$ by choosing elements $[v_{d-1}^{B_d+1}], \dots, [v_{d-1}^{B_{d-1}}]$ such that

$$\text{span}\{[x_{d-1}^1], \dots, [x_{d-1}^{B_d}], [v_{d-1}^{B_d+1}], \dots, [v_{d-1}^{B_{d-1}}]\} = \mathfrak{K}_{d-1}(A).$$

By Step 1 there exist elements v_{d-2}^i such that for $B_d + 1 \le i \le B_{d-1}$:

$$(v_{d-1}^i, v_{d-2}^i) \in A, \quad v_{d-1}^i \in \ker A^{d-1} \cap \mathfrak{R}_c(A), \quad v_{d-2}^i \in \ker A^{d-2} \cap \mathfrak{R}_c(A).$$

Similar to the procedure leading to x_d^i it is then possible to find elements $x_{d-1}^i \in \ker A^{d-1} \cap \mathfrak{R}_c(A)$, $B_d + 1 \le i \le B_{d-1}$, and to obtain a basis for the quotient space $\mathfrak{K}_{d-1}(A)$:

$$\{[x_{d-1}^1], \dots, [x_{d-1}^{B_{d-1}}]\} \quad \text{with} \quad \begin{array}{ll} (x_d^i, x_{d-1}^i) \in A & \text{for} \quad 1 \le i \le B_d, \\ (0, x_{d-1}^i) \in A & \text{for} \quad B_d + 1 \le i \le B_{d-1}, \end{array} \tag{3.13}$$

so that the basis $\{[x_{d-1}^1], \dots, [x_{d-1}^{B_{d-1}}]\}$ of $\mathfrak{K}_{d-1}(A)$ is in the desired form.

Continuing in this way by induction, one finds as successors to (3.13), that for $1 \leq j \leq d-1$ there exist elements

$$x_{d-j}^i \in \ker A^{d-j} \cap \mathfrak{R}_c(A), \quad 1 \leq i \leq B_{d-j},$$

which give a basis for the quotient space $\mathfrak{K}_{d-j}(A)$:

(3.14)

$$\{[x_{d-j}^1], \dots, [x_{d-j}^{B_{d-j}}]\} \quad \text{with} \quad \begin{array}{l} (x_{d-j+1}^i, x_{d-j}^i) \in A \quad \text{for} \quad 1 \leq i \leq B_{d-j+1}, \\ (0, x_{d-j}^i) \in A \quad \text{for} \quad B_{d-j+1}+1 \leq i \leq B_{d-j}, \end{array}$$

so that the basis $\{[x_{d-j}^1], \dots, [x_{d-j}^{B_{d-j}}]\}$ of $\mathfrak{K}_{d-j}(A)$ is in the desired form. Note that for $j = d-1$ the construction implies that $(x_1^i, 0) \in A$, $1 \leq i \leq B_1$. Hence the assertion concerning the existence of singular chains for the completely singular part A_S has been proved.

Step 4: It follows from the construction in Step 3 that $\{[x_k^1], \dots, [x_k^{B_k}]\}$ is a basis of $\mathfrak{K}_k(A)$, $1 \leq k \leq d$. Then the representatives

$$\left\{x_k^i : 1 \leq i \leq B_k, \ 1 \leq k \leq d\right\}$$

are linearly independent in $\mathfrak{H}$. To see this, assume that for some $c_k^i \in \mathbb{C}$

$$\sum_{k=1}^{d} \sum_{i=1}^{B_k} c_k^i x_k^i = 0.$$

Forming equivalence classes in $\mathfrak{K}_d(A)$, it follows by definition that $[x_k^i] = [0] \in \mathfrak{K}_d(A)$ for $i = 1, \dots, B_k$ and $k = 1, \dots, d-1$, and since $\{[x_d^1], \dots, [x_d^{B_d}]\}$ is a basis of $\mathfrak{K}_d(A)$ the reduced equality

$$\sum_{i=1}^{B_d} c_d^i [x_d^i] = 0$$

implies that $c_d^1 = \dots c_d^{B_d} = 0$. Forming equivalence classes in $\mathfrak{K}_{d-1}$ and proceeding in a similar way, it ultimately follows that $c_k^i = 0$ for all the coefficients, which proves the claim.

Step 5: It will be shown that (3.8) holds. In fact, by the construction in Step 3 it suffices to show that A_S is contained in the right-hand side of (3.8). To this end, let $(x, y) \in A_S$, so that $(x, y) \in A$ and $x, y \in \mathfrak{R}_c(A)$. If $y = 0$, then

$$x \in \ker A \cap \mathfrak{R}_c(A) = \mathfrak{K}_1(A) = \operatorname{span}\{x_1^1, \dots, x_1^{B_1}\}$$

and the assertion is shown. If $y \neq 0$, then $y \in \ker A^k$ for some $1 \leq k \leq d$ and hence $x \in \ker A^{k+1}$. Choose maximal $k \geq 1$ with the property $[y] \in \mathfrak{K}_k(A) \setminus \{[0]\}$. If $k = d$, then

$$y = y_{d-1} + \sum_{i=1}^{B_d} \gamma^i x_d^i,$$

where $\gamma^i \in \mathbb{C}$, $i = 1, \dots, B_d$, and $y_{d-1} \in \ker A^{d-1} \cap \mathfrak{R}_c(A)$. It follows that

$$(x, y) = (x, y_{d-1}) + \sum_{i=1}^{B_d} \gamma^i \left(0, x_d^i\right),$$

and it remains to show that (x, y_{d-1}) is contained in the right-hand side of (3.8). Note that $x \in \ker A^{d+1} \cap \mathfrak{R}_c(A) = \ker A^d \cap \mathfrak{R}_c(A)$ by (3.4). To continue by an inductive argument, assume now that $k < d$. Then one can write y as

$$y = y_{k-1} + \sum_{i=1}^{B_k} \gamma^i x_k^i,$$

where $\gamma^i \in \mathbb{C}$, $i = 1, \dots, B_k$, and $y_{k-1} \in \ker A^{k-1} \cap \mathfrak{R}_c(A)$. Hence, there exists $y_k \in \ker A^k \cap \mathfrak{R}_c(A)$ with $(y_k, y_{k-1}) \in A$ and it follows that

$$\left(y_k + \sum_{i=1}^{B_{k+1}} \gamma^i x_{k+1}^i,\, y \right) = (y_k, y_{k-1}) + \sum_{i=1}^{B_{k+1}} \gamma^i (x_{k+1}^i, x_k^i) + \sum_{i=B_{k+1}+1}^{B_k} \gamma^i (0, x_k^i) \in A,$$

where $(0, x_k^i) \in A$ for $B_{k+1} + 1 \le i \le B_k$ was used, see Step 3. Since $[y_k] = [0] \in \mathfrak{K}_{k+1}(A)$, this gives

$$\left(\left[\sum_{i=1}^{B_{k+1}} \gamma^i x_{k+1}^i \right], [y] \right) \in A_{k+1},$$

whereas it is clear that $([x], [y]) \in A_{k+1}$. By Step 2, the linear relation A_{k+1} is injective, so that

$$[x] = \left[\sum_{i=1}^{B_{k+1}} \gamma^i x_{k+1}^i \right]$$

in $\mathfrak{K}_{k+1}(A)$ and therefore $x = x_k + \sum_{i=1}^{B_{k+1}} \gamma^i x_{k+1}^i$ with some $x_k \in \ker A^k \cap \mathfrak{R}_c(A)$. Hence, one obtains

$$(x, y) = (x_k, y_{k-1}) + \sum_{i=1}^{B_{k+1}} \gamma^i \left(x_{k+1}^i, x_k^i \right) + \sum_{i=B_{k+1}+1}^{B_k} \gamma^i \left(0, x_k^i \right).$$

From this it follows that to show $(x, y) \in (\ker A^{k+1} \cap \mathfrak{R}_c(A)) \times (\ker A^k \cap \mathfrak{R}_c(A))$ is a linear combination of elements from the right-hand side of (3.8), it is sufficient to show the same for $(x_k, y_{k-1}) \in (\ker A^k \cap \mathfrak{R}_c(A)) \times (\ker A^{k-1} \cap \mathfrak{R}_c(A))$. By repeating the above procedure one arrives at elements $(x_1, y_0) = (x_1, 0)$, where $x_1 \in \ker A \cap \mathfrak{R}_c(A) = \mathfrak{K}_1(A)$. Obviously, x_1 can be written as a linear combination of the elements x_i^1, $i = 1, \dots, B_1$, and therefore, $(x_1, 0)$ is a linear combination of elements of the form $(x_i^1, 0)$. Thus (x, y) belongs to the right-hand side of (3.8).

Step 6: It remains to show (3.9), which directly follows from (3.7). □

4. The root part of a linear relation

Let A be a linear relation in a finite-dimensional linear space $\mathfrak{H}$, and let A_S be its completely singular part (3.1). The linear relation A_λ, $\lambda \in \mathbb{C} \cup \{\infty\}$, is defined as the graph restriction of A to the root space $\mathfrak{R}_\lambda(A)$:

(4.1) $$A_\lambda = A \cap (\mathfrak{R}_\lambda(A) \times \mathfrak{R}_\lambda(A)).$$

By (2.9) one has that $A_S \subset A_\lambda$, $\lambda \in \mathbb{C} \cup \{\infty\}$. The *root part* A_R of A is a linear relation defined as the graph restriction of A to the total root subspace $\mathfrak{R}_r(A)$ in (2.5):

(4.2) $$A_R = A \cap (\mathfrak{R}_r(A) \times \mathfrak{R}_r(A)).$$

Hence it is clear that $A_S \subset A_\lambda \subset A_R$, $\lambda \in \mathbb{C} \cup \{\infty\}$.

If $\lambda \notin \sigma_\pi(A)$, then $\mathfrak{R}_c(A) = \mathfrak{R}_\lambda(A)$ and $A_S = A_\lambda$. However, if $\lambda \in \sigma_\pi(A)$, then $\mathfrak{R}_c(A) \subset \mathfrak{R}_\lambda(A)$ and $A_S \subset A_\lambda$ with strict inclusion. there is a reducing sum decomposition for A_λ of the form

$$A_\lambda = A_S \oplus J_\lambda(A), \tag{4.3}$$

where $J_\lambda(A)$ is a Jordan operator (if $\lambda \in \mathbb{C}$) or a Jordan relation (if $\lambda = \infty$), see Definition 4.1 below. Moreover, it will be shown that $J_\lambda(A)$ is spanned by the corresponding Jordan chains as in (2.3) or (2.4); see Theorems 4.4 and 4.5 below. Then in Theorem 4.6 below it will be shown that

$$A_R = A_S \oplus J_{\lambda_1}(A) \oplus \cdots \oplus J_{\lambda_l}(A) \oplus J_\infty(A), \tag{4.4}$$

where $\lambda_1, \ldots, \lambda_l \in \sigma_\pi(A)$ and $\infty \in \sigma_\pi(A)$ are the proper eigenvalues of A. If $\infty \notin \sigma_\pi(A)$, then the term $J_\infty(A)$ in (4.4) is absent.

In this section, the identity (4.3) will first be shown for the case $\lambda = 0$ in Lemma 4.3. The case when $\lambda \in \mathbb{C}$ will be obtained in Theorem 4.4 by a shift of the relation from Lemma 4.3. Likewise, the case when $\lambda = \infty$ will be obtained in Theorem 4.5 by an inversion of the relation from Lemma 4.3.

Definition 4.1. *A linear relation A in a finite-dimensional linear space $\mathfrak{H}$ is called a* Jordan operator *in $\mathfrak{H}$ (corresponding to $\lambda \in \mathbb{C}$), if $\operatorname{dom} A = \mathfrak{H}$, $\operatorname{mul} A = \{0\}$, and $\sigma_p(A) = \{\lambda\}$. The relation A is called a* Jordan relation *in $\mathfrak{H}$ (corresponding to ∞), if A^{-1} is a Jordan operator (corresponding to $0 \in \mathbb{C}$).*

Note that Jordan operators are essentially matrices and Jordan relations correspond to injective multi-valued operators. In particular, it is clear that a Jordan relation (corresponding to ∞) satisfies $\operatorname{ran} A = \mathfrak{H}$, $\ker A = \{0\}$, and $\sigma_p(A) = \{\infty\}$.

The construction of the Jordan operator $J_\lambda(A)$ for $\lambda = 0$ is based on an appropriate choice of a sequence of quotient spaces involving A. The sequence of quotient spaces $\mathfrak{V}_k(A)$ is defined by

$$\mathfrak{V}_1(A) = \frac{\ker A + \mathfrak{R}_c(A)}{\mathfrak{R}_c(A)}, \quad \mathfrak{V}_k(A) = \frac{\ker A^k + \mathfrak{R}_c(A)}{\ker A^{k-1} + \mathfrak{R}_c(A)}, \quad k \geq 2. \tag{4.5}$$

Indeed, since the denominator is contained in the numerator, each quotient space $\mathfrak{V}_k(A)$, $k \geq 1$, is well defined. Define the sequence $(d_k)_{k\geq 1}$ by

$$d_k := \dim \mathfrak{V}_k(A), \quad k \geq 1. \tag{4.6}$$

Note that the three conditions $0 \notin \sigma_\pi(A)$, $\mathfrak{R}_c(A) = \mathfrak{R}_0(A)$ and $d_k = 0$ for all $k \geq 1$ are all equivalent.

Now the case $0 \in \sigma_\pi(A)$ or, equivalently, $\mathfrak{R}_c(A) \subsetneq \mathfrak{R}_0(A)$, will be considered. Then the sequence in (4.6) is not trivial, although ultimately the entries are zero. To see this, observe that since the space $\mathfrak{H}$ is finite-dimensional, the number

$$v = \min\left\{k \in \mathbb{N} : \ker A^{k+1} + \mathfrak{R}_c(A) = \ker A^k + \mathfrak{R}_c(A)\right\} \tag{4.7}$$

is well defined.

Lemma 4.2. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ and assume $0 \in \sigma_\pi(A)$. Let $v \geq 1$ be given by (4.7), then for $k > v$ one has*

$$\ker A^k + \mathfrak{R}_c(A) = \ker A^d + \mathfrak{R}_c(A) \quad \textit{and, hence,} \quad d_k = 0.$$

Moreover, $d_1 \geq 1$.

Proof. In view of (4.7), by induction it is sufficient to show that $\ker A^k + \mathfrak{R}_c(A) = \ker A^{k+1} + \mathfrak{R}_c(A)$ for some natural number k implies that $\ker A^{k+1} + \mathfrak{R}_c(A) = \ker A^{k+2} + \mathfrak{R}_c(A)$. Let $x \in \ker A^{k+2} + \mathfrak{R}_c(A)$ and write $x = x_{k+2} + x_c$ with $x_{k+2} \in \ker A^{k+2}$ and $x_c \in \mathfrak{R}_c(A)$. Then there exist $x_1, \dots, x_{k+1}$ such that

$$(x_{k+2}, x_{k+1}), (x_{k+1}, x_k)(x_k, x_{k-1}), \dots, (x_1, 0) \in A.$$

As $x_{k+1} \in \ker A^{k+1} \subset \ker A^{k+1} + \mathfrak{R}_c(A) = \ker A^k + \mathfrak{R}_c(A)$ by assumption, there exists $x_c' \in \mathfrak{R}_c(A)$ such that $x_{k+1} - x_c' \in \ker A^k$ and there exist $x_1', \dots, x_{k-1}'$ such that

$$(x_{k+1} - x_c', x_{k-1}'), (x_{k-1}', x_{k-2}'), \dots, (x_1', 0) \in A.$$

As $x_c' \in \mathfrak{R}_c(A)$ there is $y_c' \in \mathfrak{R}_c(A)$ with $(y_c', x_c') \in A$ and one concludes

$$(x_{k+2} - y_c', x_{k+1} - x_c'), (x_{k+1} - x_c', x_{k-1}'), (x_{k-1}', x_{k-2}'), \dots, (x_1', 0) \in A$$

so that $x_{k+2} \in \ker A^{k+1} + \mathfrak{R}_c(A)$ follows. This shows $\ker A^{k+2} + \mathfrak{R}_c(A) \subset \ker A^{k+1} + \mathfrak{R}_c(A)$. The opposite inclusion follows from the fact that $\ker A^{k+1} \subset \ker A^{k+2}$.

To see that $d_1 \geq 1$, observe that $0 \in \sigma_\pi(A)$ implies $\mathfrak{R}_c(A) \subsetneq \ker A$. Hence, $\mathfrak{R}_c(A) \subsetneq \mathfrak{R}_c(A) + \ker A$, which proves the claim. □

Lemma 4.3. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ with $0 \in \sigma_\pi(A)$. Let $v \geq 1$ be given by (4.5), then the sequence $(d_k)_{k \geq 1}$ in (4.6) satisfies*

$$d_1 \geq d_2 \geq \cdots \geq d_v \geq 1 \quad \text{and} \quad d_k = 0, \quad k > v.$$

Moreover, there exist Jordan chains for A corresponding to the eigenvalue 0 of the following form:

$$\begin{array}{llllll} (x_v^i, x_{v-1}^i), & (x_{v-1}^i, x_{v-2}^i), & \dots, & (x_2^i, x_1^i), & (x_1^i, 0), & 1 \leq i \leq d_v, \\ & (x_{v-1}^i, x_{v-2}^i), & \dots, & (x_2^i, x_1^i), & (x_1^i, 0), & d_v + 1 \leq i \leq d_{v-1}, \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & (x_2^i, x_1^i), & (x_1^i, 0), & d_3 + 1 \leq i \leq d_2, \\ & & & & (x_1^i, 0), & d_2 + 1 \leq i \leq d_1, \end{array} \tag{4.8}$$

where $\{[x_k^1], \dots, [x_k^{d_k}]\}$ is a basis of $\mathfrak{V}_k(A)$ in (4.5), $1 \leq k \leq v$, and, consequently, the elements in the set $\{x_k^i : 1 \leq i \leq d_k,\ 1 \leq k \leq v\}$ are linearly independent in $\mathfrak{H}$. Then the linear space $\mathfrak{R}_0(A)$ has the direct sum decomposition

$$\mathfrak{R}_0(A) = \mathfrak{R}_c(A) \oplus \mathfrak{X}_0(A), \tag{4.9}$$

where the space $\mathfrak{X}_0(A)$ is given by

$$\mathfrak{X}_0(A) = \operatorname{span}\{x_k^i : 1 \leq i \leq d_k,\ 1 \leq k \leq v\}.$$

Furthermore, with respect to (4.9), the graph restriction of A to $\mathfrak{R}_0(A) \times \mathfrak{R}_0(A)$,

$$A_0 = A \cap (\mathfrak{R}_0(A) \times \mathfrak{R}_0(A))$$

has the reducing sum decomposition

$$A_0 = A_S \oplus J_0(A), \tag{4.10}$$

where the linear relation $J_0(A) = A \cap (\mathfrak{X}_0(A) \oplus \mathfrak{X}_0(A))$ admits the representation

$$\begin{aligned} J_0(A) = \operatorname{span}\{(x_k^i, x_{k-1}^i), \dots\dots, (x_2^i, x_1^i),\ (x_1^i, 0) : \\ d_{k+1} + 1 \leq i \leq d_k,\ 1 \leq k \leq v\}. \end{aligned} \tag{4.11}$$

In fact, $J_0(A)$ is a Jordan operator in $\mathfrak{X}_0(A)$ corresponding to $0 \in \mathbb{C}$ and the total dimension of $J_0(A)$ is

$$\dim J_0(A) = d_1 + d_2 + \ldots + d_{v-1} + d_v. \tag{4.12}$$

Proof. The main tool in the proof is the existence of linear operators $A_k : \mathfrak{V}_k(A) \to \mathfrak{V}_{k-1}(A)$, $2 \le k \le v$, which are injective. From this, it follows that the sequence $(d_k)_{k\ge 1}$ is nonincreasing. The Jordan chains for A will be constructed via suitably chosen bases in each of the quotient spaces $\mathfrak{V}_k(A)$, $1 \le k \le v$, beginning with $\mathfrak{V}_v(A)$ and working backwards to $\mathfrak{V}_1(A)$. This procedure is carried out in a number of steps.

Step 1: Let $1 \le k \le v$ and $[x] \in \mathfrak{V}_k(A)$. Then there exist $x_1, y_1 \in \mathfrak{H}$ such that

$$(x_1, y_1) \in A, \quad x_1 \in \ker A^k, \quad y_1 \in \ker A^{k-1} \quad \text{with} \quad [x_1] = [x].$$

To see this, observe that for $x \in [x]$ one has $x = x_1 + x_2$ with $x_1 \in \ker A^k$ and $x_2 \in \mathfrak{R}_c(A)$. Hence, there is some $y_1 \in \ker A^{k-1}$ such that $(x_1, y_1) \in A$. It follows from

$$x - x_1 \in \mathfrak{R}_c(A) \subset \ker A^{k-1} + \mathfrak{R}_c(A),$$

that $[x] = [x_1]$.

Step 2: Define the linear relation $A_k \subset \mathfrak{V}_k(A) \times \mathfrak{V}_{k-1}(A)$, $2 \le k \le v$, by:

$$\begin{aligned} A_k := \big\{ ([x],[y]) \in \mathfrak{V}_k(A) \times \mathfrak{V}_{k-1}(A) : \\ \exists\, (x', y') \in A \text{ with } [x'] = [x] \text{ and } [y'] = [y] \big\}. \end{aligned} \tag{4.13}$$

Since $\ker A^{k-1} \subset \ker A^{k-1} + \mathfrak{R}_c(A)$, it follows from Step 1 that A_k is defined on all of $\mathfrak{V}_k(A)$, $2 \le k \le v$.

Moreover, A_k is an operator. To prove this, let $([0],[y]) \in A_k$. Hence, there exists $(x', y') \in A$ with $y' \in \ker A^{k-1} + \mathfrak{R}_c(A)$, such that $[x'] = [0] \in \mathfrak{V}_k(A)$ and $[y'] = [y] \in \mathfrak{V}_{k-1}(A)$. In particular,

$$x' = x'_1 + x'_2 \quad \text{with} \quad x'_1 \in \ker A^{k-1} \quad \text{and} \quad x'_2 \in \mathfrak{R}_c(A).$$

Therefore, there exist $y'_1 \in \ker A^{k-2}$ with $(x'_1, y'_1) \in A$, and $y'_2 \in \mathfrak{R}_c(A)$ with $(x'_2, y'_2) \in A$. It follows that $(0, y' - y'_1 - y'_2) \in A$, thus $y'_3 := y' - y'_1 - y'_2 \in \mathfrak{R}_\infty(A) \cap \mathfrak{R}_0(A) = \mathfrak{R}_c(A)$. Note that

$$y' = y'_1 + y'_2 + y'_3 \quad \text{with} \quad y'_1 \in \ker A^{k-2} \quad \text{and} \quad y'_2 + y'_3 \in \mathfrak{R}_c(A),$$

which implies that $[y] = [y'] = [0] \in \mathfrak{V}_{k-1}(A)$.

Furthermore, A_k is injective. To prove this, let $([x],[0]) \in A_k$. Hence, there exists $(x', y') \in A$ with $[x] = [x'] \in \mathfrak{V}_k(A)$ and $[y'] = [0] \in \mathfrak{V}_{k-1}(A)$. In particular,

$$y' = y'_1 + y'_2 \quad \text{with} \quad y'_1 \in \ker A^{k-2} \quad \text{and} \quad y'_2 \in \mathfrak{R}_c(A).$$

Therefore, there exists $x'_2 \in \mathfrak{R}_c(A)$ with $(x'_2, y'_2) \in A$. Thus it follows from

$$(x' - x'_2, y'_1) \in A \quad \text{and} \quad y'_1 \in \ker A^{k-2},$$

that $x' - x'_2 \in \ker A^{k-1}$. In other words, $x' \in \ker A^{k-1} + \mathfrak{R}_c(A)$, which implies that $[x] = [x'] = [0] \in \mathfrak{V}_k(A)$.

Step 3: The construction of the Jordan chains for A, corresponding to the eigenvalue $0 \in \sigma_\pi(A)$, is associated with the quotient spaces $\mathfrak{V}_v(A), \ldots, \mathfrak{V}_1(A)$, where $v \ge 1$. Since $\dim \mathfrak{V}_v(A) = d_v$, let

$$\{[x_v^1], \ldots, [x_v^{d_v}]\} \tag{4.14}$$

be a basis of $\mathfrak{V}_v(A)$.

First assume that $v = 1$. Then by Step 1 each $[x_1^i]$ has a representative x_1^i such that $(x_1^i, 0) \in A$, $1 \le i \le d_1$. This agrees with the statement of the lemma.

Next assume that $v \ge 2$. Then there are $v - 1$ mappings:

$$A_v : \mathfrak{V}_v(A) \to \mathfrak{V}_{v-1}(A),\ A_{v-1} : \mathfrak{V}_{v-1}(A) \to \mathfrak{V}_{v-2}(A),\ \dots,\ A_2 : \mathfrak{V}_2(A) \to \mathfrak{V}_1(A),$$

of the form (4.13). By Step 1, without loss of generality one may assume that the choice (4.14) is such that there are elements x_{v-1}^i so that for $1 \le i \le d_v$:

$$(x_v^i, x_{v-1}^i) \in A, \quad x_v^i \in \ker A^v, \quad x_{v-1}^i \in \ker A^{v-1}. \tag{4.15}$$

As $x_{v-1}^i \in \ker A^{v-1}$, $1 \le i \le d_v$, there exists $x_{v-2}^i \in \ker A^{v-2}$ with $(x_{v-1}^i, x_{v-2}^i) \in A$. In addition, by the definition of A_v, one has

$$A_v[x_v^i] = [x_{v-1}^i], \quad 1 \le i \le d_v,$$

Since $A_v : \mathfrak{V}_v(A) \to \mathfrak{V}_{v-1}(A)$ is injective by Step 2, the elements $[x_{v-1}^i]$, $1 \le i \le d_v$, are linearly independent in the space $\mathfrak{V}_{v-1}(A)$. Now choose $[x_{v-1}^{d_v+1}], \dots, [x_{v-1}^{d_{v-1}}] \in \mathfrak{V}_{v-1}(A)$ such that

$$\{[x_{v-1}^1], \dots, [x_{v-1}^{d_v}], [x_{v-1}^{d_v+1}], \dots, [x_{v-1}^{d_{v-1}}]\} \tag{4.16}$$

forms a basis of $\mathfrak{V}_{v-1}(A)$. Hence, by Step 1, without loss of generality one may assume that $x_{v-1}^{d_v+1}, \dots, x_{v-1}^{d_{v-1}}$ are chosen such that there exist elements x_{v-2}^i so that for $d_v + 1 \le i \le d_{v-1}$:

$$(x_{v-1}^i, x_{v-2}^i) \in A, \quad x_{v-1}^i \in \ker A^{v-1},\ x_{v-2}^i \in \ker A^{v-2}.$$

Furthermore, via the basis in (4.16) for $\mathfrak{V}_{v-1}(A)$, one finds elements $x_{v-2}^i \in \ker A^{v-2}$ such that $(x_{v-1}^i, x_{v-2}^i) \in A$ for $1 \le i \le d_{v-1}$.

Repeating this procedure a number of times, one finally arrives at a basis of $\mathfrak{V}_1(A)$ of the form $\{[x_1^1], \dots, [x_1^{d_1}]\}$ with elements $x_1^i \in \ker A$ such that

$$(x_1^i, 0) \in A, \quad 1 \le i \le d_1.$$

Step 4: The elements in $\mathfrak{X}_0(A) = \left\{x_k^i : 1 \le i \le d_k,\ 1 \le k \le v\right\}$ are linearly independent. In fact, this follows in the same way as in Step 4 of the proof of Theorem 3.2, when one replaces d by v, $\mathfrak{K}_d(A)$ by $\mathfrak{V}_v$, and B_k by d_k.

Step 5: The direct sum decomposition in (4.9) holds. In order to see that $\mathfrak{X}_0(A) \cap \mathfrak{R}_c(A) = \{0\}$, one uses a similar argument in Step 4: if $y \in \mathfrak{R}_c(A)$, then $[y] = 0$ in all spaces $\mathfrak{V}_k$, and so $y \in \mathfrak{X}_0(A) \cap \mathfrak{R}_c(A)$ implies $y = 0$.

It is clear that $\mathfrak{R}_c(A) \oplus \mathfrak{X}_0(A) \subset \mathfrak{R}_0(A)$. To see that equality holds, note that the definition of v gives $\mathfrak{R}_0(A) = \ker A^v + \mathfrak{R}_c(A)$, while $\mathfrak{R}_c(A) = \ker A^0 + \mathfrak{R}_c(A)$. Hence, it follows from (4.5) that

$$\dim \frac{\mathfrak{R}_0(A)}{\mathfrak{R}_c(A)} = \dim \frac{\ker A^v + \mathfrak{R}_c(A)}{\ker A^0 + \mathfrak{R}_c(A)} = \sum_{k=1}^{v} \dim \mathfrak{V}_k = \dim \mathfrak{X}_0(A),$$

which completes the argument. Thus (4.9) has been shown.

Step 6: Define the linear relation $J_0(A)$ by (4.11). Then it is clear that $J_0(A) \subset A_0$ and it follows from (4.9) that $\operatorname{dom} J_0(A) = \mathfrak{X}_0(A)$. Since the sum (4.9) is direct, one sees that the sum (4.10) is direct.

It is clear that $A_S \oplus J_0(A) \subset A_0$. To see the reverse inclusion, let $(x,y) \in A_0$. Since $x \in \mathfrak{R}_0(A)$, it follows from (4.9) that $x = x_1 + x_2$ with $x_1 \in \mathfrak{X}_0(A) = \operatorname{dom} J_0(A)$ and $x_2 \in \mathfrak{R}_c(A)$. Hence there exist elements $y_1 \in \operatorname{ran} J_0(A)$ and $y_2 \in \mathfrak{R}_c(A)$ such that $(x_1,y_1) \in J_0(A)$ and $(x_2,y_2) \in A_S$. With $(x,y) \in A$ this gives $(0, y-y_1-y_2) \in A$, hence

$$y - y_1 - y_2 \in \operatorname{mul} A \cap \mathfrak{R}_0(A) \subset \mathfrak{R}_c(A),$$

so that $(0, y-y_1-y_2) \in A_S$. It follows that

$$(x,y) = (x_2,y_2) + (0, y-y_1-y_2) + (x_1,y_1) \in A_S \oplus J_0(A),$$

thus $A_0 \subset A_S \oplus J_0(A)$. Hence the identity (4.10) has been shown. The identity $J_0(A) = A \cap (\mathfrak{X}_0(A) \oplus \mathfrak{X}_0(A))$ is obvious from (4.10).

By construction, $J_0(A)$ is an operator in $\mathfrak{X}_0(A)$. Assume that $(x, \lambda x) \in J_0(A)$ for some $\lambda \neq 0$. Then one sees

$$x \in \mathfrak{R}_\lambda(A) \cap \mathfrak{R}_0(A) = \mathfrak{R}_c(A)$$

by (2.8). Hence $x \in \mathfrak{R}_c(A) \cap \mathfrak{X}_0(A) = \{0\}$ by Step 4 and $\sigma_p(J_0(A)) = \{0\}$ is proved. In particular, $J_0(A)$ is a Jordan operator in $\mathfrak{X}_0(A)$ corresponding to $0 \in \mathbb{C}$.

Step 7: It remains to show (4.12), which directly follows from (4.8). □

Before stating Theorem 4.4, some properties of the shifted relation $A - \lambda$, $\lambda \in \mathbb{C}$, will be discussed. Fix $\lambda \in \mathbb{C}$. Then one has the obvious identities

$$\mathfrak{R}_\lambda(A) = \mathfrak{R}_0(A-\lambda), \quad \mathfrak{R}_\infty(A-\lambda) = \mathfrak{R}_\infty(A). \tag{4.17}$$

It is clear from (2.6) and (4.17) that

$$\mathfrak{R}_c(A-\lambda) = \mathfrak{R}_\lambda(A) \cap \mathfrak{R}_\infty(A),$$

which, invoking (2.8), gives

$$\mathfrak{R}_c(A) = \mathfrak{R}_c(A-\lambda). \tag{4.18}$$

It is clear that $\lambda \in \sigma_p(A)$ if and only if $0 \in \sigma_p(A-\lambda)$. This equivalence can be refined:

$$\lambda \in \sigma_\pi(A) \quad \Leftrightarrow \quad 0 \in \sigma_\pi(A-\lambda);$$

cf. (2.10) and (4.18). Hence, one obtains for the total root space (see (2.11)) that

$$\mathfrak{R}_r(A) = \mathfrak{R}_r(A-\lambda).$$

As a consquence of (4.18) and (3.1) the relation $A_S - \lambda$ is given by

$$\begin{aligned} A \cap (\mathfrak{R}_c(A) \times \mathfrak{R}_c(A)) - \lambda &= (A-\lambda) \cap (\mathfrak{R}_c(A) \times \mathfrak{R}_c(A)) \\ &= (A-\lambda) \cap (\mathfrak{R}_c(A-\lambda) \times \mathfrak{R}_c(A-\lambda)), \end{aligned}$$

which leads to the identity

$$A_S - \lambda = (A-\lambda)_S. \tag{4.19}$$

Similarly, according to (4.17) and (4.1), the relation $A_\lambda - \lambda$ is given by

$$\begin{aligned} A \cap (\mathfrak{R}_\lambda(A) \times \mathfrak{R}_\lambda(A)) - \lambda &= (A-\lambda) \cap (\mathfrak{R}_\lambda(A) \times \mathfrak{R}_\lambda(A)) \\ &= (A-\lambda) \cap (\mathfrak{R}_0(A-\lambda) \times \mathfrak{R}_0(A-\lambda)), \end{aligned}$$

which leads to the identity

$$A_\lambda - \lambda = (A-\lambda)_0. \tag{4.20}$$

For the case $\lambda \in \sigma_\pi(A) \cap \mathbb{C}$ it will be shown that the linear relation A_λ, given in (4.1), has the reducing sum decomposition $A_\lambda = A_S \oplus J_\lambda(A)$ involving the completely singular part A_S and a Jordan operator $J_\lambda(A)$. The sequence of quotient spaces $\mathfrak{Z}_k(A,\lambda)$ is defined by

$$\begin{aligned} \mathfrak{Z}_1(A,\lambda) &:= \frac{\ker(A-\lambda) + \mathfrak{R}_c(A)}{\mathfrak{R}_c(A)}, \\ \mathfrak{Z}_k(A,\lambda) &:= \frac{\ker(A-\lambda)^k + \mathfrak{R}_c(A)}{\ker(A-\lambda)^{k-1} + \mathfrak{R}_c(A)}, \quad k \geq 2. \end{aligned} \tag{4.21}$$

Since the denominator is included in the numerator, each quotient space $\mathfrak{Z}_k(A)$, $k \geq 1$, is well defined. The *Weyr characteristic* of A with respect to the quotient spaces (4.21) is defined as the sequence $(W_k(\lambda))_{k\geq 1}$, where

$$W_k(\lambda) := \dim \mathfrak{Z}_k(A,\lambda). \tag{4.22}$$

One sees from (4.21) and (4.18) that

$$\mathfrak{Z}_k(A,\lambda) = \mathfrak{V}_k(A-\lambda), \quad W_k(\lambda) = \dim \mathfrak{V}_k(A-\lambda), \quad \lambda \in \mathbb{C}. \tag{4.23}$$

Since $\mathfrak{H}$ is finite-dimensional, the number

$$s(\lambda) = \min\left\{k \in \mathbb{N} : \ker(A-\lambda)^{k+1} + \mathfrak{R}_c(A) = \ker(A-\lambda)^k + \mathfrak{R}_c(A)\right\} \tag{4.24}$$

is well defined, as follows from (4.7) (with A replaced by $A-\lambda$) and (4.18). The following theorem will be proved by means of Lemma 4.3 via a shift.

Theorem 4.4. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ with $\lambda \in \sigma_\pi(A) \cap \mathbb{C}$, and let $s(\lambda) \geq 1$ be given by (4.24). Then the Weyr characteristic $(W_k(\lambda))_{k\geq 1}$ in (4.22) satisfies*

$$W_1(\lambda) \geq W_2(\lambda) \geq \cdots \geq W_{s(\lambda)}(\lambda) \geq 1 \quad \text{and} \quad W_k(\lambda) = 0, \quad k > s(\lambda).$$

Moreover, there exist Jordan chains for A corresponding to λ of the form:

$$\begin{array}{rl} (x^i_{s(\lambda)}, x^i_{s(\lambda)-1} + \lambda x^i_{s(\lambda)}), \dots, (x^i_2, x^i_1 + \lambda x^i_2),\ (x^i_1, \lambda x^i_1), & 1 \leq i \leq W_{s(\lambda)}(\lambda), \\ (x^i_{s(\lambda)-1}, x^i_{s(\lambda)-2} + \lambda x^i_{s(\lambda)-1}), \dots, (x^i_1, \lambda x^i_1), & W_{s(\lambda)}(\lambda) + 1 \leq i \\ & \leq W_{s(\lambda)-1}(\lambda), \\ \ddots \qquad \vdots & \vdots \\ (x^i_2, x^i_1 + \lambda x^i_2),\ (x^i_1, \lambda x^i_1), & W_3(\lambda) + 1 \leq i \leq W_2(\lambda), \\ (x^i_1, \lambda x^i_1), & W_2(\lambda) + 1 \leq i \leq W_1(\lambda) \end{array}$$

where $\{[x^1_k], \dots, [x^{W_k(\lambda)}_k]\}$ is a basis of $\mathfrak{Z}_k(A,\lambda)$ in (4.21), $1 \leq k \leq s(\lambda)$, and, consequently, the elements in the set $\left\{x^i_k : 1 \leq i \leq W_k(\lambda),\ 1 \leq k \leq s(\lambda)\right\}$ are linearly independent in $\mathfrak{H}$. Then the linear space $\mathfrak{R}_\lambda(A)$ has the direct sum decomposition

$$\mathfrak{R}_\lambda(A) = \mathfrak{R}_c(A) \oplus \mathfrak{X}_\lambda(A), \tag{4.25}$$

where the linear space $\mathfrak{X}_\lambda(A)$ is given by

$$\mathfrak{X}_\lambda(A) = \operatorname{span}\left\{x^i_k : 1 \leq i \leq W_k(\lambda),\ 1 \leq k \leq s(\lambda)\right\}. \tag{4.26}$$

Furthermore, with respect to (4.25), the graph restriction of A to $\mathfrak{R}_\lambda(A) \times \mathfrak{R}_\lambda(A)$,

$$A_\lambda = A \cap (\mathfrak{R}_\lambda(A) \times \mathfrak{R}_\lambda(A))$$

has the reducing sum decomposition

$$A_\lambda = A_S \oplus J_\lambda(A), \tag{4.27}$$

where the linear relation $J_\lambda(A) = A \cap (\mathfrak{X}_\lambda(A) \times \mathfrak{X}_\lambda(A))$ *admits the representation*

$$\begin{aligned} J_\lambda(A) = \operatorname{span}\big\{&(x_k^i, x_{k-1}^i + \lambda x_k^i), \ldots\ldots, (x_2^i, x_1^i + \lambda x_2^i),\ (x_1^i, \lambda x_1^i) : \\ &W_{k+1}(\lambda) + 1 \le i \le W_k(\lambda),\ 1 \le k \le s(\lambda)\big\}, \end{aligned} \tag{4.28}$$

In fact, $J_\lambda(A)$ *is a Jordan operator in* $\mathfrak{X}_\lambda(A)$ *corresponding to* λ *and the total dimension of* $J_\lambda(A)$ *is*

$$\dim J_\lambda(A) = W_1(\lambda) + W_2(\lambda) + \ldots + W_{s(\lambda)}(\lambda). \tag{4.29}$$

Proof. The assumption $\lambda \in \sigma_\pi(A) \cap \mathbb{C}$ implies that $0 \in \sigma_\pi(A - \lambda)$. Now apply Lemma 4.3 where A, v, and d_k are replaced by $A - \lambda$, $s(\lambda)$, and $W_k(\lambda)$. Then $s(\lambda) \ge 1$ and it is clear that $(W_k(\lambda))_{k \ge 1}$ in (4.22) is nonincreasing with $W_k(\lambda) = 0$ for $k > s(\lambda)$, as follows from (4.23).

Moreover, with the Jordan chains in Lemma 4.3 interpreted for $A - \lambda$ at the eigenvalue 0, the present Jordan chains for A at the eigenvalue λ follow. For this purpose recall that

$$(u_n, u_{n-1} + \lambda u_n), \ldots, (u_2, u_1 + \lambda u_2),\ (u_1, \lambda u_1)$$

is a (Jordan) chain for A at λ if and only if

$$(u_n, u_{n-1}),\ (u_{n-1}, u_{n-2}), \ldots, (u_2, u_1),\ (u_1, 0)$$

is a (Jordan) chain for $A - \lambda$ at 0. The statement about the basis of $\mathfrak{Z}_k(A, \lambda)$ follows from (4.22).

According to Lemma 4.3, $\mathfrak{R}_0(A - \lambda)$ has the direct sum decomposition

$$\mathfrak{R}_0(A - \lambda) = \mathfrak{R}_c(A - \lambda) \oplus \mathfrak{X}_0(A - \lambda), \tag{4.30}$$

and, with respect to (4.30), the linear relation $(A - \lambda)_0$ has the reducing sum decomposition

$$(A - \lambda)_0 = (A - \lambda)_S \oplus J_0(A - \lambda). \tag{4.31}$$

Using (4.17) and (4.18) in (4.30) gives the direct sum decomposition (4.25), where $\mathfrak{X}_\lambda(A) = \mathfrak{X}_0(A - \lambda)$. Likewise, using (4.20), (4.19) in (4.31), one obtains

$$A_\lambda - \lambda = (A_S - \lambda) \oplus J_0(A - \lambda),$$

or, in other words, the reducing sum decomposition (4.27), where $J_\lambda(A) = J_0(A - \lambda) + \lambda$. Recall from Lemma 4.3 that

$$\mathfrak{X}_0(A - \lambda) = \operatorname{span}\left\{x_k^i : 1 \le i \le W_k(\lambda),\ 1 \le k \le s(\lambda)\right\},$$

which gives (4.26). Likewise, one has from Lemma 4.3 that

$$J_0(A - \lambda) = (A - \lambda) \cap (\mathfrak{X}_0(A - \lambda) \oplus \mathfrak{X}_0(A - \lambda)),$$

which leads to

$$J_\lambda(A) = A \cap (\mathfrak{X}_\lambda(A) \times \mathfrak{X}_\lambda(A)),$$

as stated in the theorem. Similarly,

$$\begin{aligned} J_0(A - \lambda) = \operatorname{span}\big\{&(x_k^i, x_{k-1}^i), \ldots\ldots, (x_2^i, x_1^i),\ (x_1^i, 0) : \\ &W_{k+1}(\lambda) + 1 \le i \le W_k(\lambda),\ 1 \le k \le s(\lambda)\big\}, \end{aligned}$$

which gives (4.28). Finally, from Lemma 4.3 one obtains

$$\operatorname{dom} J_0(A - \lambda) = \mathfrak{X}_0(A - \lambda) \quad \text{and} \quad \sigma_p(J_0(A - \lambda)) = \{0\},$$

which shows that $J_\lambda(A)$ is a Jordan operator in $\mathfrak{X}_\lambda(A)$ corresponding to λ. The formula (4.29) directly follows from the representation (4.28). □

Before stating Theorem 4.5, some general properties of the inverse A^{-1} of a linear relation A will be discussed. It is easy to see that for $\lambda \in \mathbb{C}$ one has:

$$\ker(A-\lambda) = \ker\left(A^{-1} - \frac{1}{\lambda}\right), \quad \lambda \neq 0, \quad \text{and} \quad \ker A = \operatorname{mul} A^{-1}, \tag{4.32}$$

and it is therefore clear that

$$\begin{aligned} \lambda \in \sigma_p(A) \quad &\iff \quad \frac{1}{\lambda} \in \sigma_p(A^{-1}), \quad \lambda \neq 0, \\ 0 \in \sigma_p(A) \quad &\iff \quad \infty \in \sigma_p(A^{-1}). \end{aligned} \tag{4.33}$$

In a more general context, one also has

$$\mathfrak{R}_\lambda(A) = \mathfrak{R}_{\frac{1}{\lambda}}(A^{-1}), \quad \lambda \neq 0, \quad \text{and} \quad \mathfrak{R}_0(A) = \mathfrak{R}_\infty(A^{-1}), \tag{4.34}$$

see [5, Lem. 3.2]. It is clear from (4.34) that

$$\mathfrak{R}_c(A) = \mathfrak{R}_c(A^{-1}). \tag{4.35}$$

Moreover, the equivalences in (4.33), together with (4.34) and (4.35), lead to

$$\begin{aligned} \lambda \in \sigma_\pi(A) \quad &\iff \quad \frac{1}{\lambda} \in \sigma_\pi(A^{-1}), \quad \lambda \neq 0, \\ 0 \in \sigma_\pi(A) \quad &\iff \quad \infty \in \sigma_\pi(A^{-1}). \end{aligned}$$

Thus, it follows from the definition of the total root space (see (2.11)) that

$$\mathfrak{R}_r(A) = \mathfrak{R}_r(A^{-1}). \tag{4.36}$$

As a consequence of (4.35) and (3.1), the completely singular part $(A^{-1})_S$ of A^{-1} is given by

$$(A^{-1})_S = A^{-1} \cap (\mathfrak{R}_c(A^{-1}) \times \mathfrak{R}_c(A^{-1})) = A^{-1} \cap (\mathfrak{R}_c(A) \times \mathfrak{R}_c(A)),$$

which leads to the identity

$$((A^{-1})_S)^{-1} = A \cap (\mathfrak{R}_c(A) \times \mathfrak{R}_c(A)) = A_S. \tag{4.37}$$

Similarly, according to (4.34) and (4.1), the relation $(A^{-1})_0$ is given by

$$(A^{-1})_0 = A^{-1} \cap (\mathfrak{R}_0(A^{-1}) \times \mathfrak{R}_0(A^{-1})) = A^{-1} \cap (\mathfrak{R}_\infty(A) \times \mathfrak{R}_\infty(A)),$$

which leads to the identity

$$((A^{-1})_0)^{-1} = A \cap (\mathfrak{R}_\infty(A) \times \mathfrak{R}_\infty(A)) = A_\infty. \tag{4.38}$$

For the case $\infty \in \sigma_\pi(A)$ it will be shown below that the linear relation A_∞ has the reducing sum decomposition $A_\infty = A_S \oplus J_\infty(A)$ involving the completely singular part A_S and a Jordan relation $J_\infty(A)$. The sequence of quotient spaces $\mathfrak{W}_k(A)$ is defined by

$$\mathfrak{W}_1(A) := \frac{\operatorname{mul} A + \mathfrak{R}_c(A)}{\mathfrak{R}_c(A)}, \quad \mathfrak{W}_k(A) := \frac{\operatorname{mul} A^k + \mathfrak{R}_c(A)}{\operatorname{mul} A^{k-1} + \mathfrak{R}_c(A)}, \quad k \geq 2. \tag{4.39}$$

Since the denominator is included in the numerator, each quotient space $\mathfrak{W}_k(A)$, $k \geq 1$, is well defined. The *Weyr characteristic* of A with respect to the quotient spaces (4.39) is defined as the sequence $(A_k)_{k\geq 1}$, where

$$A_k := \dim \mathfrak{W}_k(A), \quad k \geq 2. \tag{4.40}$$

One sees from (4.32) and (4.35) that

$$\mathfrak{W}_k(A) = \mathfrak{V}_k(A^{-1}), \quad A_k = \dim \mathfrak{W}_k(A) = \dim \mathfrak{V}_k(A^{-1}). \tag{4.41}$$

Since $\mathfrak{H}$ is finite-dimensional, the number

$$\aleph = \min\left\{k \in \mathbb{N} : \ \operatorname{mul} A^{k+1} + \mathfrak{R}_c(A) = \operatorname{mul} A^k + \mathfrak{R}_c(A)\right\} \tag{4.42}$$

is well defined, as follows from (4.7) (with A replaced by A^{-1}) and (4.35). The following theorem will be proved by means of Lemma 4.3 via an inversion.

Theorem 4.5. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ with $\infty \in \sigma_\pi(A)$, and let $\aleph \geq 1$ be given by (4.42). Then the Weyr characteristic $(A_k)_{k\geq 1}$ in (4.40) satisfies*

$$A_1 \geq A_2 \geq \cdots \geq A_\aleph \geq 1 \quad \text{and} \quad A_k = 0, \quad k > \aleph.$$

Moreover, there exist Jordan chains for A corresponding to the eigenvalue ∞ of the form

$$\begin{array}{ll}
(0,x_1^i),\ (x_1^i,x_2^i),\ \ldots,(x_{\aleph-2}^i,x_{\aleph-1}^i),\ (x_{\aleph-1}^i,x_\aleph^i), & 1 \leq i \leq A_\aleph,\\
(0,x_1^i),\ (x_1^i,x_2^i),\ \ldots,(x_{\aleph-2}^i,x_{\aleph-1}^i), & A_\aleph+1 \leq i \leq A_{\aleph-1},\\
\quad\vdots \qquad\quad \vdots \qquad \iddots & \qquad\vdots\\
(0,x_1^i),\ (x_1^i,x_2^i), & A_3+1 \leq i \leq A_2,\\
(0,x_1^i), & A_2+1 \leq i \leq A_1,
\end{array}$$

where $\{[x_k^1],\ldots,[x_k^{A_k}]\}$ is a basis of $\mathfrak{W}_k(A)$ in (4.39), $1 \leq k \leq \aleph$, and, consequently, the elements in the set $\left\{x_k^i : 1 \leq i \leq A_k,\ 1 \leq k \leq \aleph\right\}$ are linearly independent in $\mathfrak{H}$. Then the linear space $\mathfrak{R}_\infty(A)$ has the direct sum decomposition

$$\mathfrak{R}_\infty(A) = \mathfrak{R}_c(A) \oplus \mathfrak{X}_\infty(A), \tag{4.43}$$

where the linear space $\mathfrak{X}_\infty(A)$ is given by

$$\mathfrak{X}_\infty(A) = \operatorname{span}\left\{x_k^i : 1 \leq i \leq A_k,\ 1 \leq k \leq \aleph\right\}. \tag{4.44}$$

Furthermore, with respect to (4.43), the graph restriction of A to $\mathfrak{R}_\infty(A) \times \mathfrak{R}_\infty(A)$,

$$A_\infty = A \cap (\mathfrak{R}_\infty(A) \times \mathfrak{R}_\infty(A))$$

has the reducing sum decomposition

$$A_\infty = A_S \oplus J_\infty(A), \tag{4.45}$$

where the linear relation $J_\infty(A) = A \cap (\mathfrak{X}_\infty(A) \times \mathfrak{X}_\infty(A))$ is given by

$$\begin{aligned} J_\infty(A) = \operatorname{span}\big\{&(0,x_1^i),\ (x_1^i,x_2^i),\ \ldots\ldots,(x_{k-1}^i,x_k^i) : \\ & A_{k+1}+1 \leq i \leq A_k,\ 1 \leq k \leq \aleph\big\}. \end{aligned} \tag{4.46}$$

In fact, $J_\infty(A)$ is a Jordan relation in $\mathfrak{X}_\infty(A)$ corresponding to ∞ and the total dimension of $J_\infty(A)$ is

$$\dim J_\infty(A) = A_1 + A_2 + \ldots + A_\aleph. \tag{4.47}$$

Proof. The assumption $\infty \in \sigma_\pi(A)$ implies that $0 \in \sigma_\pi(A^{-1})$. Now apply Lemma 4.3 where A, v, and d_k are replaced by A^{-1}, $\aleph$, and A_k. Then $\aleph \geq 1$ and it is clear that $(A_k)_{k\geq 1}$ in (4.40) is nonincreasing with $A_k = 0$ for $k > \aleph$, as follows from (4.41).

Moreover, with the Jordan chains in Lemma 4.3 interpreted for A^{-1} at the eigenvalue 0, the present Jordan chains for A at the eigenvalue ∞ follow. For this purpose recall that

$$(u_n, u_{n-1}),\ (u_{n-1}, u_{n-2}),\ \dots\dots\dots,(u_2, u_1),\ (u_1, 0)$$

is a Jordan chain for A at 0 if and only if

$$(0, u_1),\ (u_1, u_2),\ \dots\dots\dots,(u_{n-2}, u_{n-1},\ (u_{n-1}, u_n)$$

is a Jordan chain for A^{-1} at ∞. The statement about the basis of $\mathfrak{W}_k(A)$ follows from (4.41).

According to Lemma 4.3, $\mathfrak{R}_0(A^{-1})$ has the direct sum decomposition

$$\mathfrak{R}_0(A^{-1}) = \mathfrak{R}_c(A^{-1}) \oplus \mathfrak{X}_0(A^{-1}), \tag{4.48}$$

and, with respect to (4.48), the linear relation $(A^{-1})_0$ has the reducing sum decomposition

$$(A^{-1})_0 = (A^{-1})_S \oplus J_0(A^{-1}). \tag{4.49}$$

Using (4.34) and (4.35) in (4.48) gives the direct sum decomposition (4.43), where $\mathfrak{X}_\infty(A) = \mathfrak{X}_0(A^{-1})$. Likewise, using (4.37), (4.38), and taking inverses in (4.49), one obtains the reducing sum decomposition (4.45), where $J_\infty(A) = (J_0(A^{-1}))^{-1}$. Recall from Lemma 4.3 that

$$\mathfrak{X}_0(A^{-1}) = \operatorname{span}\left\{x_k^i : 1 \le i \le A_k,\ 1 \le k \le \aleph\right\},$$

which gives (4.44). Likewise, one has from Lemma 4.3

$$J_0(A^{-1}) = A^{-1} \cap (\mathfrak{X}_0(A^{-1}) \oplus \mathfrak{X}_0(A^{-1})),$$

which, taking inverses, leads to

$$J_\infty(A) = A \cap (\mathfrak{X}_\infty(A) \times \mathfrak{X}_\infty(A)),$$

as stated in the theorem. Similarly,

$$\begin{aligned} J_0(A^{-1}) = \operatorname{span}\big\{&(x_k^i, x_{k-1}^i),\ \dots\dots,(x_2^i, x_1^i),\ (x_1^i, 0) : \\ &A_{k+1} + 1 \le i \le A_k,\ 1 \le k \le \aleph\big\}. \end{aligned}$$

Taking the inverse of the linear relations on both sides of the above equation gives (4.46). Note that $J_0(A^{-1})$ is an operator and that $\ker J_\infty(A) = \operatorname{mul} J_0(A^{-1}) = \{0\}$; hence the relation $J_\infty(A)$ is injective. Finally, recall from Lemma 4.3 that

$$\operatorname{dom} J_0(A^{-1}) = \mathfrak{X}_0(A^{-1}) \quad \text{and} \quad \sigma_p(J_0(A^{-1})) = \{0\},$$

by which $J_\infty(A)$ is a Jordan relation in $\mathfrak{X}_\infty(A)$ corresponding to ∞. The formula (4.47) directly follows from the representation (4.46). □

Theorems 4.4 and 4.5 will now be combined to show the main result of this section: a reducing sum decomposition of the root part.

Theorem 4.6. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ and let $\sigma_\pi(A) \setminus \{\infty\} = \{\lambda_1, \dots, \lambda_l\}$. For all $\lambda \in \sigma_\pi(A) \setminus \{\infty\}$ let $J_\lambda(A)$ be the Jordan operator in $\mathfrak{X}_\lambda(A)$ corresponding to λ such that $A_\lambda = A_S \oplus J_\lambda(A)$ is the reducing sum decomposition as in Theorem 4.4 and, if $\infty \in \sigma_\pi(A)$, let $J_\infty(A)$ be the Jordan relation in $\mathfrak{X}_\infty(A)$ such that $A_\infty = A_S \oplus J_\infty(A)$ is the reducing sum decomposition as in Theorem 4.5. Then $\mathfrak{R}_r(A)$ has the direct sum decomposition*

$$\mathfrak{R}_r(A) = \mathfrak{R}_c(A) \oplus \mathfrak{X}_{\lambda_1}(A) \oplus \cdots \oplus \mathfrak{X}_{\lambda_l}(A) \oplus \mathfrak{X}_\infty(A). \tag{4.50}$$

Furthermore, with respect to (4.50)*, the linear relation* A_R *has the reducing sum decomposition*

$$A_R = A_S \oplus J_{\lambda_1}(A) \oplus \cdots \oplus J_{\lambda_l}(A) \oplus J_\infty(A). \tag{4.51}$$

Furthermore, one has the equalities

$$\mathfrak{R}_r(A) = \operatorname{dom} A_R + \operatorname{ran} J_\infty(A) = \operatorname{ran} A_R + \operatorname{dom} J_0(A). \tag{4.52}$$

If $\infty \notin \sigma_\pi(A)$*, then the space* $\mathfrak{X}_\infty(A)$ *and the linear relation* $J_\infty(A)$ *are absent.*

Proof. First, it will be shown that the identity (4.50) holds and that the sum is direct. Recall from Theorem 4.4 that $\mathfrak{R}_{\lambda_i}(A) = \mathfrak{R}_c(A) \oplus \mathfrak{X}_{\lambda_i}(A)$ for $1 \le i \le l$, and $\mathfrak{R}_\lambda(A) = \mathfrak{R}_c(A)$ for $\lambda \notin \sigma_\pi(A)$ by (2.10). Hence, the following identity is clear:

$$\mathfrak{R}_r(A) = \mathfrak{R}_c(A) + \mathfrak{X}_{\lambda_1}(A) + \cdots + \mathfrak{X}_{\lambda_l}(A) + \mathfrak{X}_\infty(A).$$

To see that the sum on the right-hand side is direct, let the elements $x_c \in \mathfrak{R}_c(A)$, $x_i \in \mathfrak{X}_{\lambda_i}(A)$, $1 \le i \le l$, and $x_\infty \in \mathfrak{X}_\infty(A)$ be such that

$$x_c + x_1 + \cdots + x_l + x_\infty = 0.$$

Then it follows from [5, Cor. 4.5] that

$$x_i \in \mathfrak{R}_{\lambda_i}(A) \cap \left(\mathfrak{R}_c(A) + \mathfrak{R}_\infty(A) + \sum_{j=1, j\neq i}^{l} \mathfrak{R}_{\lambda_j}(A) \right) = \mathfrak{R}_c(A), \quad 1 \le i \le l,$$

hence $x_i \in \mathfrak{X}_{\lambda_i}(A) \cap \mathfrak{R}_c(A) = \{0\}$. In a similar way one obtains $x_\infty = 0$ and concludes that $x_c = 0$. Thus the sum in (4.50) is direct.

Next, it will be shown that the identity (4.51) holds and that the sum is direct. Observe that it follows from (4.50) that the sum

$$A_S \oplus J_{\lambda_1}(A) \oplus \cdots \oplus J_{\lambda_l}(A) \oplus J_\infty(A)$$

is direct. From Theorem 4.4 and Theorem 4.5 one finds that

$$A_S \oplus J_{\lambda_1}(A) \oplus \cdots \oplus J_{\lambda_l}(A) \oplus J_\infty(A) \subset A_R.$$

In order to show the reverse inclusion, let $(x, y) \in A_R$. Then $x \in \mathfrak{R}_r(A)$ and, by (4.50), there exist $x_c \in \mathfrak{R}_c(A)$, $x_i \in \operatorname{dom} J_{\lambda_i}(A)$, $1 \le i \le l$, and $x_\infty \in \operatorname{ran} J_\infty(A)$ such that

$$x = x_c + x_1 + \cdots + x_l + x_\infty.$$

Hence, there exist $y_c \in \mathfrak{R}_c(A)$ with $(x_c, y_c) \in A$ and $y_i \in \operatorname{ran} J_{\lambda_i}(A)$ with $(x_i, y_i) \in A$ for $1 \le i \le l$ so that

$$(x, y) = (x_c, y_c) + (x_1, y_1) + \cdots + (x_l, y_l) + (x_\infty, \tilde{y} := y - y_c - y_1 - \cdots - y_l) \in A.$$

In particular, one has

$$(x_c, y_c) \in A_S, \quad (x_i, y_i) \in J_{\lambda_i}, \ 1 \le i \le l, \quad \text{and} \quad (x_\infty, \tilde{y}) \in A.$$

As $x_\infty \in \mathfrak{R}_\infty(A)$ it follows that $\tilde{y} \in \mathfrak{R}_\infty(A)$. By (4.43) there exist $\tilde{y}_\infty \in \mathfrak{X}_\infty(A)$ and $\hat{y} \in \mathfrak{R}_c(A)$ such that $\tilde{y} = \tilde{y}_\infty + \hat{y}$, and $\tilde{x}_\infty \in \mathfrak{X}_\infty(A)$ such that $(\tilde{x}_\infty, \tilde{y}_\infty) \in J_\infty(A) \subset A$. Then

$$(x, y) = (x_c, y_c) + (x_1, y_1) + \cdots + (x_l, y_l) + (\tilde{x}_\infty, \tilde{y}_\infty) + (x_\infty - \tilde{x}_\infty, \hat{y}) \in A$$

and one sees that the last term belongs to A. But since $x_\infty - \tilde{x}_\infty \in \mathfrak{X}_\infty(A)$ and $\hat{y} \in \mathfrak{R}_c(A)$ it follows that $x_\infty - \tilde{x}_\infty \in \mathfrak{R}_c(A) \cap \mathfrak{X}_\infty(A) = \{0\}$, thus $x_\infty = \tilde{x}_\infty$. Therefore, (x, y) belongs to the right-hand side of (4.51).

Finally, equation (4.52) directly follows from Theorems 3.2, 4.4 and 4.5. □

Recall from (2.15) that the restriction of A to $\mathfrak{H}(A) = \operatorname{dom} A + \operatorname{ran} A$ is formally the same relation but in a possibly smaller space. In particular the space $\mathfrak{R}_r(A)$ is contained in $\mathfrak{H}(A)$:

$$\mathfrak{R}_r(A) \subset \operatorname{dom} A + \operatorname{ran} A.$$

As a consequence of Theorem 4.6 it is possible to characterize when equality holds. The case of strict inclusion will be considered in detail in Section 5.

Corollary 4.7. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) $\operatorname{dom} A + \operatorname{ran} A = \mathfrak{R}_r(A)$,
(ii) $\operatorname{dom} A \subset \mathfrak{R}_r(A)$,
(iii) $\operatorname{ran} A \subset \mathfrak{R}_r(A)$.

Proof. (i) ⇒ (iii) is clear.

(iii) ⇒ (i): Since $\operatorname{ran} A \subset \mathfrak{R}_r(A)$, it suffices to show that $\operatorname{dom} A \subset \mathfrak{R}_r(A)$. Assume that $x \in \operatorname{dom} A$, then $(x, y) \in A$ for some $y \in \operatorname{ran} A \subset \mathfrak{R}_r(A)$. Then $y = y_r + y_0$ with $y_r \in \operatorname{ran} A_R$ and $y_0 \in \operatorname{dom} J_0(A)$ according to (4.52), hence $(x_r, y_r) \in A_R$ for some $x_r \in \mathfrak{R}_r(A)$. The decomposition

$$(x, y) = (x_r, y_r) + (x - x_r, y_0)$$

shows that $(x - x_r, y_0) \in A$. Due to $y_0 \in \operatorname{dom} J_0(A) \subset \ker A^i$ for some i one finds $x - x_r \in \ker A^{i+1} \subset \mathfrak{R}_r(A)$. Hence, it follows that $x \in \mathfrak{R}_r(A)$.

(ii) ⇔ (iii): This is due to the symmetry when A is replaced by A^{-1}; cf. (4.36). □

5. The multishift part of a linear relation

Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$. In this section it will be shown that there exists a linear subspace $\mathfrak{R}_m(A) \subseteq \mathfrak{H}$ spanned by entries of linearly independent shift chains such that the restriction of A to $\mathfrak{R}_m(A) \times \mathfrak{R}_m(A)$ is a multishift.

The construction of the shift chains in A is based on an appropriate choice of a sequence of quotient spaces. The sequence of quotient spaces $\mathfrak{M}_k(A)$ is defined by

$$\mathfrak{M}_0(A) := \frac{\operatorname{dom} A + \operatorname{ran} A}{\operatorname{ran} A + \mathfrak{R}_r(A)}, \qquad \mathfrak{M}_k(A) := \frac{\operatorname{ran} A^k + \mathfrak{R}_r(A)}{\operatorname{ran} A^{k+1} + \mathfrak{R}_r(A)}, \quad k \geq 1. \tag{5.1}$$

Indeed, as the denominator is included in the numerator, each quotient space $\mathfrak{M}_k(A)$, $k \geq 0$, is well defined. The *Weyr characteristic* of A with respect to (5.1) is defined as the sequence $(C_k)_{k \geq 0}$, where

$$C_k := \dim \mathfrak{M}_k(A), \quad k \geq 0. \tag{5.2}$$

If $\operatorname{dom} A + \operatorname{ran} A = \mathfrak{R}_r(A)$, then by Corollary 4.7 this condition is equivalent to $\operatorname{ran} A \subset \mathfrak{R}_r(A)$. This implies that $\operatorname{ran} A^k \subset \mathfrak{R}_r(A)$ for all $k \geq 1$, so that $C_k = 0$ for all $k \geq 0$. In this case one may define $\mathfrak{R}_m(A) := \{0\}$ and then the restriction A_M is given by $A_M = A \cap (\mathfrak{R}_m(A) \times \mathfrak{R}_m(A)) = \{0, 0\}$.

Now consider the case that the inclusion $\mathfrak{R}_r(A) \subset \operatorname{dom} A + \operatorname{ran} A$ is strict. Then the sequence in (5.2) is not trivial, although ultimately the entries are zero. To see this, observe that since the linear space $\mathfrak{H}$ is finite-dimensional, the number

$$m = \min \{ k \in \mathbb{N} : \ \operatorname{ran} A^{k+1} + \mathfrak{R}_r(A) = \operatorname{ran} A^k + \mathfrak{R}_r(A) \} \tag{5.3}$$

is well defined.

Lemma 5.1. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ and assume that the inclusion $\mathfrak{R}_r(A) \subset \operatorname{dom} A + \operatorname{ran} A$ is strict. Then the Weyr characteristic $(C_k)_{k \geq 0}$ is nontrivial. In fact, with m given by* (5.3)*, one has*

$$\operatorname{ran} A^k + \mathfrak{R}_r(A) = \operatorname{ran} A^{k+1} + \mathfrak{R}_r(A), \quad k \geq m, \tag{5.4}$$

and it follows that $C_k = 0$, $k \geq m$. Moreover,

$$\operatorname{ran} A^k \subset \mathfrak{R}_r(A), \quad k \geq m, \tag{5.5}$$

which, in particular, implies that $m \geq 2$ and, consequently, $C_1 \geq 1$.

Proof. The proof is divided into a number of steps.

Step 1: First observe that

$$\mathfrak{R}_r(A) + \operatorname{ran} A^k = \mathfrak{R}_0(A) + \operatorname{ran} A^k, \quad k \geq 1. \tag{5.6}$$

To prove (5.6) it suffices to show that the left-hand side is contained in the right-hand side. Recall from (4.50), that

$$\mathfrak{R}_r(A) = \mathfrak{R}_{\neq 0}(A) \oplus \mathfrak{X}_0(A) \quad \text{where} \quad \mathfrak{R}_{\neq 0}(A) = \mathfrak{R}_c(A) \oplus \sum_{\lambda \in \sigma_\pi \setminus \{0\}} \mathfrak{X}_\lambda(A).$$

Fix $k \geq 1$. Since $\mathfrak{R}_c(A)$ is spanned by entries of singular chains (2.7), it is clear that $\mathfrak{R}_c(A) \subset \operatorname{ran} A^k$. Moreover, for $\lambda \in \sigma_\pi(A) \setminus \{0, \infty\}$ the Jordan operator $J_\lambda(A)$ defined in Theorem 4.4 is a bijection in $\mathfrak{X}_\lambda(A)$, from which it follows that $\operatorname{ran} J_\lambda(A) = \mathfrak{X}_\lambda(A)$. Hence,

$$\mathfrak{X}_\lambda(A) = \operatorname{ran}(J_\lambda(A))^k \subset \operatorname{ran} A^k,$$

where the last inclusion follows from $J_\lambda(A) \subset A$. For the Jordan relation $J_\infty(A)$ defined in Theorem 4.5 it is immediate that $\operatorname{ran} J_\infty(A) = \mathfrak{X}_\infty(A)$, so that

$$\mathfrak{X}_\infty(A) = \operatorname{ran}(J_\infty(A))^k \subset \operatorname{ran} A^k,$$

where the last inclusion is due to $J_\infty(A) \subset A$. Therefore one finds that

$$\mathfrak{R}_{\neq 0}(A) \subset \operatorname{ran} A^k, \quad k \geq 1.$$

Hence (5.6) has been shown.

Step 2: In order to prove (5.4), by (5.6) it is sufficient to show that

$$\operatorname{ran} A^k + \mathfrak{R}_0(A) = \operatorname{ran} A^{k+1} + \mathfrak{R}_0(A), \quad k \geq m. \tag{5.7}$$

By induction it suffices to show that $\operatorname{ran} A^k + \mathfrak{R}_0(A) = \operatorname{ran} A^{k+1} + \mathfrak{R}_0(A)$ for some $k \geq m$ implies that $\operatorname{ran} A^{k+1} + \mathfrak{R}_0(A) = \operatorname{ran} A^{k+2} + \mathfrak{R}_0(A)$. It will be shown that

$$\operatorname{ran} A^{k+1} + \mathfrak{R}_0(A) \subset \operatorname{ran} A^{k+2} + \mathfrak{R}_0(A), \tag{5.8}$$

since the converse inclusion follows from the inclusion $\operatorname{ran} A^{k+2} \subset \operatorname{ran} A^{k+1}$. Let $x \in \operatorname{ran} A^{k+1} + \mathfrak{R}_0(A)$. Then $x = x_r + x_0$ for some $x_r \in \operatorname{ran} A^{k+1}$ and $x_0 \in \mathfrak{R}_0(A)$. Furthermore, there exists $x_k \in \operatorname{ran} A^k$ such that $(x_k, x_r) \in A$. By assumption one has

$$x_k = x_{k+1} + z_0 \quad \text{with} \quad x_{k+1} \in \operatorname{ran} A^{k+1} \quad \text{and} \quad z_0 \in \mathfrak{R}_0(A).$$

and hence there exists $y_0 \in \mathfrak{N}_0(A)$ with $(z_0, y_0) \in A$. With $(x_{k+1} + z_0, x_r) \in A$ it follows that $(x_{k+1}, x - x_0 - y_0) \in A$ and $x - x_0 - y_0 \in \operatorname{ran} A^{k+2}$. Hence one obtains $x \in \operatorname{ran} A^{k+2} + \mathfrak{N}_0(A)$. This shows (5.8).

Step 3: To prove (5.5), it suffices to show that $\operatorname{ran} A^m \subset \mathfrak{R}_r(A)$, since $\operatorname{ran} A^k \subset \operatorname{ran} A^m$ for $k \geq m$. Let $x \in \operatorname{ran} A^m$, then $x = x_1 + x_1^0$ with $x_1 \in \operatorname{ran} A^{m+1}$ and $x_1^0 \in \mathfrak{N}_0(A)$ by (5.7). Hence, there exists $y_1 \in \operatorname{ran} A^m$ with $(y_1, x_1) \in A$. Again by (5.7), $y_1 = x_2 + x_2^0$ with $x_2 \in \operatorname{ran} A^{m+1}$ and $x_2^0 \in \mathfrak{N}_0(A)$, and there is some $x_2^1 \in \mathfrak{N}_0(A)$ such that $(x_2^0, x_2^1) \in A$. With $(x_2 + x_2^0, x_1) = (y_1, x_1) \in A$ it follows that $(x_2, x_1 - x_2^1) \in A$.

Next observe that there exists $y_2 \in \operatorname{ran} A^m$ with $(y_2, x_2) \in A$, and by (5.7), $y_2 = x_3 + x_3^0$ with $x_3 \in \operatorname{ran} A^{m+1}$ and $x_3^0 \in \mathfrak{N}_0(A)$. Moreover, there are $x_3^1 \in \mathfrak{N}_0(A)$ and $x_3^2 \in \mathfrak{N}_0(A)$ such that $(x_3^0, x_3^1) \in A$ and $(x_3^1, x_3^2) \in A$. With $(x_3 + x_3^0, x_2) = (y_2, x_2) \in A$ it follows that $(x_3, x_2 - x_3^1) \in A$, and

$$(x_3, x_2 - x_3^1), \quad (x_2 - x_3^1, x_1 - x_2^1 - x_3^2)$$

form a chain in A. A continuation of this argument leads to a chain of the form

$$(z_n, z_{n-1}), \dots, (z_3, z_2), (z_2, z_1) \in A,$$

where each z_k is of the form $z_k = x_k + z_k^0$ with $x_k \in \operatorname{ran} A^{m+1}$ and $z_k^0 \in \mathfrak{N}_0(A)$. Let $l \geq 2$ be the smallest index such that $z_1, \dots, z_{l-1}$ are linearly independent and $z_l \in \operatorname{span}\{z_1, \dots, z_{l-1}\}$. Let $B = \operatorname{span}\{(z_k, z_{k+1}) : 1 \leq k \leq l-1\}$. Then B is an everywhere defined linear operator in $\operatorname{span}\{z_1, \dots, z_{l-1}\} = \mathfrak{R}_r(B)$ with $B \subset A^{-1}$. By means of (4.36) one has that $\mathfrak{R}_r(B) \subset \mathfrak{R}_r(A^{-1}) = \mathfrak{R}_r(A)$, thus $z_1 \in \mathfrak{R}_r(A)$. Hence, $x = x_1 + x_1^0 = z_1 - z_1^0 + x_1^0 \in \mathfrak{R}_r(A)$ follows.

Step 4: If $m = 1$, then it follows from (5.5) that $\operatorname{ran} A \subset \mathfrak{R}_r(A)$. By Corollary 4.7 this contradicts the assumption that the inclusion $\mathfrak{R}_r(A) \subset \operatorname{dom} A + \operatorname{ran} A$ is strict. Thus $m \geq 2$ and, in particular, $\operatorname{ran} A^2 + \mathfrak{R}_r(A)$ is a proper subset of $\operatorname{ran} A + \mathfrak{R}_r(A)$, which implies that $C_1 \geq 1$. □

Theorem 5.2. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ and assume that the inclusion $\mathfrak{R}_r(A) \subset \operatorname{dom} A + \operatorname{ran} A$ is strict, so that m in* (5.3) *satisfies $m \geq 2$. Then the Weyr characteristic $(C_k)_{k \geq 0}$ in* (5.2) *satisfies*

$$C_0 = C_1 \geq \cdots \geq C_{m-1} \geq 1 \quad \text{and} \quad C_k = 0, \quad k \geq m.$$

Moreover, there exist shift chains for A of the following form

(5.9)
$$\begin{array}{ll}
(x_0^i, x_1^i),\, (x_1^i, x_2^i),\, \dots, (x_{m-3}^i, x_{m-2}^i),\; (x_{m-2}^i, x_{m-1}^i), & 1 \leq i \leq C_{m-1}, \\
(x_0^i, x_1^i),\, (x_1^i, x_2^i),\, \dots, (x_{m-3}^i, x_{m-2}^i), & C_{m-1} + 1 \leq i \leq C_{m-2}, \\
\quad\vdots \qquad\quad \vdots \qquad \iddots & \qquad\vdots \\
(x_0^i, x_1^i),\, (x_1^i, x_2^i), & C_3 + 1 \leq i \leq C_2, \\
(x_0^i, x_1^i), & C_2 + 1 \leq i \leq C_1,
\end{array}$$

where $\{[x_k^1], \dots, [x_k^{C_k}]\}$ is a basis of $\mathfrak{M}_k(A)$, $0 \leq k \leq m-1$. The elements in $\left\{x_k^i : 1 \leq i \leq C_k,\ 0 \leq k \leq m-1\right\}$ are linearly independent in $\mathfrak{H}$. Then $\mathfrak{H}(A) = \operatorname{dom} A + \operatorname{ran} A$ has the direct sum decomposition

$$\mathfrak{H}(A) = \mathfrak{R}_r(A) \oplus \mathfrak{R}_m(A), \tag{5.10}$$

where

$$\mathfrak{R}_m(A) := \operatorname{span}\left\{x_k^i : 1 \leq i \leq C_k,\ 0 \leq k \leq m-1\right\}. \tag{5.11}$$

Furthermore, with respect to the decomposition (5.10)*, the relation A has the reducing sum decomposition*

$$A = A_R \oplus A_M, \tag{5.12}$$

where $A_M := A \cap (\mathfrak{R}_m(A) \times \mathfrak{R}_m(A))$ admits the representation

$$A_M = \operatorname{span}\big\{(x_0^i, x_1^i),\ (x_1^i, x_2^i),\ \ldots\ldots, (x_{k-1}^i, x_k^i) : \\ C_{k+1} + 1 \le i \le C_k,\ 1 \le k \le m-1\big\}. \tag{5.13}$$

In fact, A_M is a multishift and the total dimension of A_M is

$$\dim A_M = C_1 + C_2 + \ldots + C_{m-2} + C_{m-1}. \tag{5.14}$$

Proof. Throughout the proof the identity (5.6) will be used. The proof is carried out in several steps.

Step 1: For $1 \le k \le m-1$, define the linear relations

$$\hat{B}_k := \{([y],[x]) \in \mathfrak{M}_k(A) \times \mathfrak{M}_{k-1}(A) : \exists\,(x',y') \in A \text{ with } [x'] = [x] \text{ and } [y'] = [y]\}.$$

It is shown that $\hat{B}_k : \mathfrak{M}_k(A) \to \mathfrak{M}_{k-1}(A)$ are injective operators, i.e., $\operatorname{dom} \hat{B}_k = \mathfrak{M}_k(A)$ and $\ker \hat{B}_k = \operatorname{mul} \hat{B}_k = \{[0]\}$ for $1 \le k \le m-1$. Moreover, $\operatorname{ran} \hat{B}_1 = \mathfrak{M}_0(A)$.

To see that $\operatorname{dom} \hat{B}_k = \mathfrak{M}_k(A)$ let $[y] \in \mathfrak{M}_k(A)$, then $y = y_1 + y_2$ with $y_1 \in \operatorname{ran} A^k$ and $y_2 \in \mathfrak{R}_0(A)$, and there exists $x \in \operatorname{ran} A^{k-1}$ ($x \in \operatorname{dom} A$ if $k = 1$) such that $(x, y_1) \in A$. Since $y - y_1 = y_2 \in \mathfrak{R}_0(A)$ one has $[y] = [y_1]$ and as further $[x] \in \mathfrak{M}_{k-1}(A)$ it follows $([y],[x]) = ([y_1],[x]) \in \hat{B}_k$.

To see that $\operatorname{mul} \hat{B}_k = \{[0]\}$ let $([0],[x]) \in \hat{B}_k$. Then there exist $y' \in [0]$ and $x' \in [x]$ such that $(x',y') \in A$. Hence there exist $y_1 \in \operatorname{ran} A^{k+1}$ and $y_2 \in \mathfrak{R}_0(A)$ with $y' = y_1 + y_2$. Therefore, $(x_1, y_1) \in A$ with some $x_1 \in \operatorname{ran} A^k$. It follows that $(x' - x_1, y_2) \in A$, and since $y_2 \in \mathfrak{R}_0(A)$ one obtains $x' - x_1 \in \mathfrak{R}_0(A)$. Consequently, $[x] = [x_1]$ and $x_1 \in \operatorname{ran} A^k + \mathfrak{R}_0(A)$ which gives $[x] = [0] \in \mathfrak{M}_{k-1}(A)$.

To see that $\ker \hat{B}_k = \{[0]\}$ let $([y],[0]) \in \hat{B}_k$. Then there exists $x' \in [0]$ and $y' \in [y]$ such that $(x',y') \in A$. Hence there exist $x_1 \in \operatorname{ran} A^k$ and $x_2 \in \mathfrak{R}_0(A)$ with $x' = x_1 + x_2$. Furthermore, there exists $y_2 \in \mathfrak{R}_0(A)$ with $(x_2, y_2) \in A$. It follows that $(x_1, y' - y_2) \in A$, which together with $x_1 \in \operatorname{ran} A^k$ implies that $y' - y_2 \in \operatorname{ran} A^{k+1}$. Consequently, $[y] = [y_2]$ and $y_2 \in \operatorname{ran} A^{k+1} + \mathfrak{R}_0(A)$ which gives $[y] = [0] \in \mathfrak{M}_k(A)$.

To see that $\operatorname{ran} \hat{B}_1 = \mathfrak{M}_0(A)$ let $[x] \in \mathfrak{M}_0(A)$. Then $x = x_1 + x_2$ with $x_1 \in \operatorname{dom} A$ and $x_2 \in \operatorname{ran} A$, and there exists y_1 with $(x_1, y_1) \in A$ so that $([y_1],[x_1]) \in \hat{B}_1$. Since $x - x_1 \in \operatorname{ran} A$ it follows that $[x] = [x_1]$ and the statement of Step 1 is shown.

The properties of $\hat{B}_k$ imply that

$$C_0 = C_1, \quad C_{k-1} \ge C_k, \quad 2 \le k \le m.$$

Step 2: Let $\{[x_{m-1}^1], \ldots, [x_{m-1}^{C_{m-1}}]\}$ be a basis of $\mathfrak{M}_{m-1}(A)$. Then, for $i = 1, \ldots, C_{m-1}$, $x_{m-1}^i = x_0^i + \tilde{x}_{m-1}^i$ with $x_0^i \in \mathfrak{R}_0(A)$ and $\tilde{x}_{m-1}^i \in \operatorname{ran} A^{m-1}$. Therefore, there are elements $x_{m-2}^i \in \operatorname{ran} A^{m-2}$ with $(x_{m-2}^i, \tilde{x}_{m-1}^i) \in A$, and $[x_{m-1}^i] = [\tilde{x}_{m-1}^i]$ for $i = 1, \ldots, C_{m-1}$, thus it is shown that $\{[\tilde{x}_{m-1}^1], \ldots, [\tilde{x}_{m-1}^{C_{m-1}}]\}$ is a basis of $\mathfrak{M}_{m-1}(A)$ and $[x_{m-2}^i] = \hat{B}_{m-1}[\tilde{x}_{m-1}^i]$ for $i = 1, \ldots, C_{m-1}$. Since $\hat{B}_{m-1}$ is injective by Step 1, the elements $[x_{m-2}^1], \ldots, [x_{m-2}^{C_{m-1}}] \in \mathfrak{M}_{m-2}(A)$ are linearly independent. Now choose additional linearly independent elements $[x_{m-2}^i] \in \mathfrak{M}_{m-2}(A)$, $C_{m-1} + 1 \le i \le C_{m-2}$ (note that this range is empty if $C_{m-1} = C_{m-2}$), with $x_{m-2}^i \in \operatorname{ran} A^{m-2}$ (this can be achieved by substracting appropriate elements from

$\mathfrak{N}_0(A)$ without changing the equivalence class) such that $\{[x_{m-2}^1],\dots,[x_{m-2}^{C_{m-2}}]\}$ forms a basis of $\mathfrak{M}_{m-2}(A)$.

To continue in an inductive way, assume that, for some $2 \le k \le m-2$, $\{[x_k^1],\dots,[x_k^{C_k}]\}$ is a basis of $\mathfrak{M}_k(A)$ such that $x_k^1,\dots,x_k^{C_k} \in \operatorname{ran} A^k$. Then there exist $x_{k-1}^i \in \operatorname{ran} A^{k-1}$ such that $(x_{k-1}^i, x_k^i) \in A$ for $i = 1,\dots,C_k$. Therefore, $[x_{k-1}^i] = \hat{B}_k[\tilde{x}_k^i]$ for $i = 1,\dots,C_{m-1}$ and, since $\hat{B}_k$ is injective by Step 1, the elements $[x_{k-1}^1],\dots,[x_k^{C_k}] \in \mathfrak{M}_{k-1}(A)$ are linearly independent. Choose additional linearly independent elements $[\tilde{x}_{k-1}^i] \in \mathfrak{M}_{k-1}(A)$ for $C_k + 1 \le i \le C_{k-1}$ with $x_{k-1}^i \in \operatorname{ran} A^{k-1}$ such that $\{[\tilde{x}_{k-1}^1],\dots,[\tilde{x}_{k-1}^{C_{k-1}}]\}$ is a basis of $\mathfrak{M}_{k-1}(A)$.

This procedure continues until one arrives at a basis $\{[x_1^1],\dots,[x_1^{C_1}]\}$ of $\mathfrak{M}_1(A)$ with $x_1^1,\dots,x_1^{C_1} \in \operatorname{ran} A$. Then there are elements x_0^i with $(x_0^i, x_1^i) \in A$ for $i = 1,\dots,C_1 = C_0$. Since $[x_0^i] = \hat{B}_1[\tilde{x}_1^i]$ for $i = 1,\dots,C_0$ and $\hat{B}_1 : \mathfrak{M}_1(A) \to \mathfrak{M}_0(A)$ is bijective by Step 1, $\{[x_0^1],\dots,[x_0^{C_0}]\}$ is a basis of $\mathfrak{M}_0(A)$. In the end, the shift chains as in the statement of the theorem have been constructed.

Step 3: It follows from the construction in Step 2 that $\{[x_k^1],\dots,[x_k^{C_k}]\}$ is a basis of $\mathfrak{M}_k(A)$, $0 \le k \le m-1$. To see that the elements $\left\{x_k^i : 1 \le i \le C_k,\ 0 \le k \le m-1\right\}$ are linearly independent in $\mathfrak{H}$, assume that

$$\sum_{k=0}^{m-1} \sum_{i=1}^{C_k} c_k^i x_k^i = 0. \tag{5.15}$$

By (5.6) $\sum_{k=1}^{m-1} \sum_{i=1}^{C_k} c_k^i x_k^i \in \operatorname{ran} A + \mathfrak{N}_0(A)$, so that by taking equivalence classes in (5.15) with respect to $\mathfrak{M}_0(A)$, one obtains

$$\sum_{i=1}^{C_0} c_0^i [x_0^i] = 0 \in \mathfrak{M}_0(A),$$

which implies that $c_0^i = 0$ for $1 \le i \le C_0$. Note that therefore the assumption (5.15) is reduced to

$$\sum_{k=1}^{m-1} \sum_{i=1}^{C_k} c_k^i x_k^i = 0.$$

Now form equivalence classes in $\mathfrak{M}_1(A)$ and proceed in a similar way. Then ultimately it follows that $c_k^i = 0$ for all the coefficients, which proves the claim.

Step 4: To show (5.10), first observe that it is clear that $\mathfrak{R}_r(A) + \mathfrak{R}_m(A) \subset \mathfrak{H}(A)$. To see that equality holds, it will be shown that

$$\dim \frac{\operatorname{dom} A + \operatorname{ran} A}{\mathfrak{R}_r(A)} = \sum_{k=0}^{m-1} C_k. \tag{5.16}$$

To this end, observe the following identity

$$\begin{aligned}\dim \frac{\operatorname{dom} A + \operatorname{ran} A}{\mathfrak{R}_r(A)} &= \dim \frac{\operatorname{dom} A + \operatorname{ran} A}{\operatorname{ran} A + \mathfrak{R}_r(A)} \\ &\quad + \sum_{k=1}^{m-1} \dim \frac{\operatorname{ran} A^k + \mathfrak{R}_r(A)}{\operatorname{ran} A^{k+1} + \mathfrak{R}_r(A)} + \dim \frac{\operatorname{ran} A^m + \mathfrak{R}_r(A)}{\mathfrak{R}_r(A)},\end{aligned}$$

where for the last term one has by Lemma 5.1 that

$$\dim \frac{\operatorname{ran} A^m + \mathfrak{R}_r(A)}{\mathfrak{R}_r(A)} = \dim \frac{\mathfrak{R}_r(A)}{\mathfrak{R}_r(A)} = 0.$$

This gives (5.16). To see that the sum (5.10) is direct, let $x \in \mathfrak{R}_r(A) \cap \mathfrak{R}_m(A)$, then x is of the form as the left hand side of equation (5.15). Since $x \in \mathfrak{R}_r(A)$ one further has that $[x] = 0 \in \mathfrak{M}_k(A)$ for all $0 \le k \le m-1$. Then, similar to Step 3, it follows that $x = 0$.

Step 5: It will be shown that

$$\operatorname{ran} A \cap \mathfrak{R}_m(A) = \operatorname{span}\left\{x_k^i : 1 \le i \le C_k,\ 1 \le k \le m-1\right\}. \tag{5.17}$$

It is clear by construction of $\mathfrak{R}_m(A)$ that the right-hand side is contained in the left hand side, so it suffices to show that

$$\dim\big(A \cap \mathfrak{R}_m(A)\big) = \sum_{k=1}^{m-1} C_k.$$

By (5.10) it follows that

$$\dim \operatorname{ran} A = \dim\big(\operatorname{ran} A \cap \mathfrak{R}_r(A)\big) + \dim\big(\operatorname{ran} A \cap \mathfrak{R}_m(A)\big),$$

so it remains to show that

$$\dim \frac{\operatorname{ran} A}{\operatorname{ran} A \cap \mathfrak{R}_r(A)} = \sum_{k=1}^{m-1} C_k. \tag{5.18}$$

To this end, observe that

$$\dim \frac{\operatorname{dom} A + \operatorname{ran} A}{\mathfrak{R}_r(A)} = \dim \frac{\operatorname{dom} A + \operatorname{ran} A}{\operatorname{ran} A + \mathfrak{R}_r(A)} + \dim \frac{\operatorname{ran} A + \mathfrak{R}_r(A)}{\mathfrak{R}_r(A)}. \tag{5.19}$$

By (5.16) the left-hand side of the above equation equals $\sum_{k=0}^{m-1} C_k$ and the first term on the right-hand side is C_0. Hence,

$$\sum_{k=1}^{m-1} C_k = \dim \frac{\operatorname{ran} A + \mathfrak{R}_r(A)}{\mathfrak{R}_r(A)} = \dim \frac{\operatorname{ran} A}{\operatorname{ran} A \cap \mathfrak{R}_r(A)},$$

where the last equality is due to [17, Lem 2.2]. This proves (5.18).

Step 6: For (5.12) it suffices to show that $A \subset A_R \,\widehat{+}\, A_M$ and that the sum is direct. Let $(x, y) \in A$, so that $y \in \operatorname{ran} A$. Then by (5.10) one has $y = y_r + y_m$ with $y_r \in \operatorname{ran} A \cap \mathfrak{R}_r(A)$ and $y_m \in \operatorname{ran} A \cap \mathfrak{R}_m(A)$. Therefore, invoking (5.17), there are $x_r \in \mathfrak{R}_r(A)$ and $x_m \in \mathfrak{R}_m(A)$ such that $(x_r, y_r) \in A_R$ and $(x_m, y_m) \in A_M$. It follows that $(x - x_r - x_m, 0) \in A$, thus $x - x_r - x_m \in \mathfrak{R}_0(A)$ and $(x - x_r - x_m, 0) \in A_R$. Therefore, one obtains that $(x - x_m, y - y_m) \in A_R$ and hence

$$(x, y) = (x - x_m, y - y_m) + (x_m, y_m) \in A_R \,\widehat{+}\, A_M.$$

That the sum (5.12) is direct follows from (5.10).

Step 7: It will be shown that (5.13) holds. It is clear that the right-hand side is contained in the left-hand side. For the converse inclusion, let $(x, y) \in A \cap (\mathfrak{R}_m(A) \times \mathfrak{R}_m(A))$. Since $y \in \operatorname{ran} A \cap \mathfrak{R}_m(A)$, by (5.17) there is some $x_m \in \mathfrak{R}_m(A)$ with $(x_m, y) \in A_M$. It follows that $x - x_m \in \mathfrak{R}_m(A) \cap \mathfrak{R}_0(A) = \{0\}$ by (5.10), thus $x = x_m$ and (5.13) is shown. It is a direct consequence of (5.13) that A_M is an operator (i.e., $\operatorname{mul} A_M = \{0\}$), and that $\sigma_p(A_M) = \emptyset$, thus A_M is a multishift.

Step 8: It remains to show (5.14), which directly follows from (5.9). □

The following is a consequence of Theorem 5.2. It is implicitly contained in [22].

Corollary 5.3. *Let A be a linear relation in a finite-dimensional linear space $\mathfrak{H}$. Then the following statements are equivalent:*

(i) *A is a multishift (i.e., $\sigma_p(A) = \emptyset$);*
(ii) *there exists a basis for $\mathfrak{H}$ of the form* (5.11) *and A is given by the right-hand side of* (5.13).

6. Main result: Jordan-like decomposition

In this section the main result of this note will be stated. Any linear relation in a finite-dimensional space admits a reducing sum decomposition into a completely singular relation, a Jordan relation, and a multishift. This is obtained by a combination of Theorems 3.2, 4.6, and 5.2. Furthermore, it will be shown that the chain structure of singular chains, Jordan chains, and shift chains of any such decomposition is uniquely determined by A and given by its Weyr characteristics. Moreover, it turns out that the resulting decomposition of A is a unique representative of the equivalence class of with respect to the notion of strict equivalence (see Definition 6.5).

Theorem 6.1. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$. Then there exist linear relations $A_S, J_{\lambda_1}(A), \dots, J_{\lambda_l}(A), J_\infty(A), A_M$, all contained in A, where $\{\lambda_1, \dots, \lambda_l\} = \sigma_\pi(A) \cap \mathbb{C}$, such that*

$$A = A_S \oplus J_{\lambda_1}(A) \oplus \cdots \oplus J_{\lambda_l}(A) \oplus J_\infty(A) \oplus A_M, \tag{6.1}$$

is a reducing sum decomposition of A with respect to

$$\mathfrak{H}(A) = \mathfrak{R}_c(A) \oplus \mathfrak{X}_{\lambda_1}(A) \oplus \cdots \oplus \mathfrak{X}_{\lambda_l}(A) \oplus \mathfrak{X}_\infty(A) \oplus \mathfrak{R}_m(A) \tag{6.2}$$

with the spaces defined in Theorems 3.2, 4.6, *and* 5.2. *Furthermore,*

(a) *A_S is completely singular in $\mathfrak{R}_c(A)$;*
(b) *$J_{\lambda_i}(A)$ is a Jordan operator in $\mathfrak{X}_{\lambda_i}(A)$ corresponding to λ_i for $1 \le i \le l$;*
(c) *$J_\infty(A)$ is a Jordan relation in $\mathfrak{X}_\infty(A)$;*
(d) *A_M is a multishift in $\mathfrak{R}_m(A)$.*

Any of the linear relations in (6.1) *may be absent, if the corresponding space in* (6.2) *is trivial.*

Remark 6.2. The following special cases may serve to illustrate Theorem 6.1.

(a) Consider the case of a trivial singular chain subspace $\mathfrak{R}_c(A) = \{0\}$. Then the completely singular part is absent and the treatment in Section 4 becomes simpler. In this case the quotient spaces $\mathfrak{Z}_k(A, \lambda)$ in (4.21) are given by

$$\ker(A-\lambda),\ \frac{\ker(A-\lambda)^2}{\ker(A-\lambda)},\ \frac{\ker(A-\lambda)^3}{\ker(A-\lambda)^2},\ \cdots, \tag{6.3}$$

whereas the quotient spaces $\mathfrak{W}_k(A)$ in (4.39) are given by

$$\operatorname{mul} A,\ \frac{\operatorname{mul} A^2}{\operatorname{mul} A},\ \frac{\operatorname{mul} A^3}{\operatorname{mul} A^2},\ \cdots. \tag{6.4}$$

Recall that $\mathfrak{R}_c(A) = \{0\}$ implies that $\sigma_\pi(A) = \sigma_p(A)$. Hence, if $\lambda \notin \sigma_p(A)$, then the Weyr characteristic corresponding to (6.3) is the null sequence and, similarly, if $\infty \notin \sigma_p(A)$, then the Weyr characteristic corresponding to (6.4) is the null sequence.

(b) Consider the case of a trivial multivalued part $\operatorname{mul} A = \{0\}$. Then certainly $\mathfrak{R}_c(A) = \{0\}$ (and the comments of (a) apply), but additionally the quotient spaces in (6.4) are trivial. The Weyr characteristic for (6.3) then essentially coincides with that considered in [25] for linear operators.

(c) Consider the case of $\operatorname{dom} A = \mathfrak{H}$. Then A_M equals the zero space. Assume that $A_M \neq \{(0,0)\}$. As (6.1) is a reducing sum decomposition, one can assume, for simplicity, $A = A_M$. As $A = A_M$ is a multishift, it has no eigenvalues and for every for every pair $(x,y) \in A$ the entries x, y are linearly independent. Let $(x_1, x_2) \in A$. As $\operatorname{dom} A = \mathfrak{H}$, it follows that $x_2 \in \operatorname{dom} A$, hence there exists x_3 with $(x_2, x_3) \in A$. Now, $\{x_1, x_2\}$ is linearly independent but $\{x_1, x_2, x_3\}$ might be linearly independent or not. If it is linearly independent, then there exists x_4 with $(x_3, x_4) \in A$. Again, $\{x_1, x_2, x_3, x_4\}$ is linearly independent or not. If it is linearly independent, then there exists x_5 with $(x_4, x_5) \in A$. This can be continued. Finally, as $\operatorname{dom} A = \mathfrak{H}$ is finite dimensional, this procedure shows that there is a smallest natural number m, $2 \leq m \leq \dim \mathfrak{H}$, with the properties

$$\begin{array}{ll} \{x_1, \dots, x_m\} & \text{are linearly independent,} \\ \{x_1, \dots, x_{m+1}\} & \text{are not linearly independent,} \\ (x_i, x_{i+1}) \in A & \text{for } i = 1, \dots, m. \end{array}$$

Therefore, one has $x_{m+1} = \sum_{i=1}^m \alpha_i x_i$ for some $\alpha_i \in \mathbb{C}$, $i = 1, \dots, m$. Set $M := \operatorname{span}\{x_1, \dots, x_m\}$ and define the matrix

$$T := \begin{bmatrix} 0 & \cdots & 0 & \alpha_1 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & 1 & \alpha_m \end{bmatrix}.$$

Let $\lambda \in \mathbb{C}$ be an eigenvalue of T with eigenvector $\beta = (\beta_1, \dots, \beta_m)^\top \in \mathbb{C}^m \setminus \{0\}$, i.e., $T\beta = \lambda\beta$. Then $x := \sum_{i=1}^m \beta_i x_i \in M$ is nontrivial and since $(x_i, x_{i+1}) \in A$ for $i = 1, \dots, m$ one finds that for

$$z := \beta_1 x_2 + \ldots + \beta_{m-1} x_m + \beta_m x_{m+1}$$

one has $(x, z) \in A$. Observe that by $x_{m+1} = \sum_{i=1}^m \alpha_i x_i$ it follows

$$z = \alpha_1 \beta_m x_1 + (\alpha_2 \beta_m + \beta_1) x_2 + \ldots + (\alpha_m \beta_m + \beta_{m-1}) x_m$$

and since

$$\begin{pmatrix} \alpha_1 \beta_m \\ \alpha_2 \beta_m + \beta_1 \\ \vdots \\ \alpha_m \beta_m + \beta_{m-1} \end{pmatrix} = T \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = \lambda \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix},$$

one has that

$$z = \lambda \beta_1 x_1 + \ldots + \lambda \beta_m x_m = \lambda x$$

so that $(x, \lambda x) \in A$ and hence A has an eigenvalue, a contradiction. Therefore, the quotient spaces in (5.1) are trivial, and the corresponding Weyr characteristic given by (5.2) is the null sequence. That is, the multishift part is absent.

(d) As a consequence of (a)–(c), the classical result of the Jordan canonical form for linear operators A in a finite-dimensional space $\mathfrak{H}$ is covered by Theorem 6.1. Since in particular $\operatorname{dom} A = \mathfrak{H}$ and $\operatorname{mul} A = \{0\}$, the Weyr characteristic of A is that corresponding to the spaces (6.3) and A has the reducing sum decomposition

$$A = J_{\lambda_1}(A) \oplus \cdots \oplus J_{\lambda_l}(A),$$

with Jordan operators $J_{\lambda_i}(A)$ whose structure coincides with that of classical Jordan blocks according to the representation (4.28).

In order to justify calling (6.1) a Jordan-like decomposition for linear relations it needs to exhibit a certain uniqueness. Recall that for the fixed decomposition of $\mathfrak{H}(A)$ in (6.2), any reducing sum decomposition is intrinsically unique, cf. Section 2. Moreover, the Jordan-like decomposition is uniquely determined by the Weyr characteristic of A; in particular, if any two linear relations have the same Weyr characteristic, then they have the same Jordan-like decomposition. To see this, recall that for a linear relation A in a finite-dimensional linear space $\mathfrak{H}$ the Weyr characteristic corresponding to the sequence of quotient spaces

(a) in (3.2) is given by the sequence $B := (B_k)_{k\geq 1}$ in (3.3);
(b) in (4.21) is given by the sequece $W(\lambda) := (W_k(\lambda))_{k\geq 1}$ in (4.22);
(c) in (4.39) is given by the sequence $A = (A_k)_{k\geq 1}$ in (4.40);
(d) in (5.1) is given by the sequence $C := (C_k)_{k\geq 0}$ in (5.2).

Note that each of these Weyr characteristics is a finitely supported nonincreasing sequence, which may be the null sequence. The Weyr characteristics corresponding to all different proper complex eigenvalues $\{\lambda_1, \ldots, \lambda_l\} = \sigma_\pi(A) \cap \mathbb{C}$ will be collected in a single sequence:

$$W := (W(\lambda_1), W(\lambda_2), \ldots, W(\lambda_l)). \tag{6.5}$$

Definition 6.3. *Let A be a linear relation in a finite-dimensional linear space $\mathfrak{H}$. The collection of the sequences*

$$(B, W, A, C), \tag{6.6}$$

given by (3.3)*,* (6.5)*,* (4.40)*, and* (5.2)*, is called the* Weyr characteristic *of the linear relation A.*

The Jordan-like decomposition (6.1) of a linear relation A in Theorem 6.1 is completely determined by the Weyr characteristic of A, which follows from the construction in Theorems 3.2, 4.6, and 5.2. Moreover, one has the following result.

Proposition 6.4. *Any two linear relations in a finite-dimensional space $\mathfrak{H}$ with the same Weyr characteristic have the same reducing sum decomposition* (6.1) *with respect to the same subspace decomposition* (6.2)*.*

In the remainder of this section consider finitely supported nonincreasing sequences

$$\big((B_k)_{k\geq 1}, (W_k^1)_{k\geq 1}, \ldots, (W_k^l)_{k\geq 1}, (A_k)_{k\geq 1}, (C_k)_{k\geq 0}\big), \tag{6.7}$$

where any of the sequences may be a null sequence. Then, by the above results, it is possible to construct a linear relation (given by the Jordan-like decomposition (6.1)) which has (6.7) as Weyr characteristic. But to which extent is this relation unique? To answer this question one introduces the following notion of strict equivalence.

Definition 6.5. *The linear relations S_1 and S_2 in a finite-dimensional space $\mathfrak{H}$ are said to be* strictly equivalent *if there exists an invertible matrix T such that*

$$(x,y) \in S_2 \quad \Longleftrightarrow \quad (T^{-1}x, T^{-1}y) \in S_1 \tag{6.8}$$

or, what is the same,

$$S_2 = TS_1T^{-1}. \tag{6.9}$$

Note that (6.9) is understood in the sense of multiplication of linear relations. The following theorem is taken from [13].

Theorem 6.6. *Two linear relations in a finite-dimensional space $\mathfrak{H}$ are strictly equivalent if and only if their Weyr characteristics coincide.*

As a direct consequence of Proposition 6.4 and Theorem 6.6, it follows that the Jordan-like decomposition (6.1) is a unique representative of the equivalence classes with respect to strict equivalence. Furthermore, one has the following result.

Theorem 6.7. *For any given finitely supported nonincreasing sequences* (6.7) (*and a finite-dimensional space $\mathfrak{H}$ with sufficiently large dimension*) *there exists, up to strict equivalence, exactly one linear relation A in $\mathfrak{H}$ with Weyr characteristic* (6.7)*.*

Remark 6.8. The Jordan-like decomposition of linear relations derived in Theorem 6.1 resolves a "non-uniqueness issue" of the decomposition from [22]. A componentwise direct sum decomposition of a linear relation A into a completely singular relation, a Jordan part and a multishift was derived in [22]. However, this decomposition does not exhibit uniqueness as the following example shows: For linearly independent elements x_1, x_2, x_3 in a finite-dimensional linear space $\mathfrak{H}$ define

$$A_1 := \operatorname{span}\{(0,x_1),(x_1,x_2),(x_2,0)\}, \quad A_2 := \{(x_1,x_3)\}$$

and $A := A_1 \oplus A_2$. Obviously, $\mathfrak{R}_c(A) = \operatorname{span}\{x_1,x_2\}$ and $\mathfrak{R}_r(A) = \operatorname{span}\{x_1,x_2,x_3\}$. Clearly, A_1 is completely singular and A_2 is a multishift, but $A_1 \oplus A_2$ is not a reducing sum decomposition. Furthermore, there is an alternative decomposition of A into

$$A = A_1 \oplus A_3, \quad A_3 = \operatorname{span}\{(0, x_3 - x_2)\},$$

where A_3 consists of a Jordan chain at ∞. Both decompositions are possible in the framework of [22]. On the other hand, the Jordan-like decomposition (6.1) in Theorem 6.1 is a reducing sum decomposition, and hence unique for the fixed decomposition of $\mathfrak{H}(A)$ in (6.2).

It should also be stressed that in the present paper the reducing sum decompositions are derived in the setting of linear spaces; no further structure (such as an inner product) is required.

Remark 6.9. The presentation of some of the material in [22] was inspired by the results in [18, 19]. In the setting of (what is now called) almost Pontryagin spaces, Kaltenbäck and Woracek considered selfadjoint extensions of symmetric relations with defect numbers $(1,1)$; one of the requirements was the existence of a shift chain relative to the isotropic part of the almost Pontryagin space. The multishifts in [22] were introduced with the work of Kaltenbäck and Woracek in mind. Shifts have also been considered in the context of Pontryagin spaces; see for instance [11], where references to further work can be found.

Remark 6.10. Let $E, F \in \mathbb{C}^{n\times m}$ be matrices and let $sE - F$ be the corresponding matrix pencil. Associated with E and F are two linear relations

$$E^{-1}F = \{(x,y) \in \mathbb{C}^m \times \mathbb{C}^m : Fx = Ey\} \quad \text{and} \quad FE^{-1} = \{(Ex, Fx) : x \in \mathbb{C}^m\}, \tag{6.10}$$

which were already studied in [3, 4]. Usually, $E^{-1}F$ is called the *kernel* representation and FE^{-1} the *range* representation (see also [9]). Matrix pencils have a canonical form, the so-called Kronecker canonical form [6, 12, 20]. There is a deep connection between the range and the kernel representation and the corresponding matrix pencil. This was already utilized in [8, 14, 15, 16, 21]. A complete set of invariants for the Kronecker canonical form are four multi-indices: the finite and infinite elementary divisors, the column and the row minimal indices. These quantities measure the sizes of the different blocks in the Kronecker canonical form. Moreover, two matrix pencils are strictly equivalent if and only if all the four indices coincide [12]. They are completely determined by the so-called Wong sequences [7], which are certain sequences of subspaces; the geometric approach in [7] is, in its spirit, close to the approach in the present paper (although not quotient spaces have been used). The relationship between linear relations and matrix pencils is investigated in [9, 13] and it will be continued in upcoming papers.

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