LINEAR RELATIONS AND THEIR SINGULAR CHAINS

THOMAS BERGER, HENK DE SNOO, CARSTEN TRUNK, AND HENRIK WINKLER

Dedicated to our friend Vladimir Derkach on the occasion of his seventieth birthday

ABSTRACT. Singular chain spaces for linear relations in linear spaces play a fundamental role in the decomposition of linear relations in finite-dimensional spaces. In this paper singular chains and singular chain spaces are discussed in detail for not necessarily finite-dimensional linear spaces. This leads to an identity characterizing a singular chain space in terms of root spaces. The so-called proper eigenvalues of a linear relation play an important role in the finite-dimensional case.

1. INTRODUCTION

Let A be a linear relation in a linear space $\mathfrak{H}$. In this paper the singular chain space $\mathfrak{R}_c(A)$ of A is considered in detail. In the nontrivial case, such subspaces can only occur for relations that are not graphs of linear operators. The singular chain space $\mathfrak{R}_c(A)$ plays a fundamental role in the decomposition of a linear relation A when the linear space $\mathfrak{H}$ is finite-dimensional; see [3], [6], and [12].

Linear relations in linear spaces go back to [1]; see also [2] and [7]. Let $\mathfrak{H}$ be a possibly infinite-dimensional linear space over $\mathbb{C}$. A linear relation A in $\mathfrak{H}$ is defined as a linear subspace of the product $\mathfrak{H} \times \mathfrak{H}$. The usual notions for a linear relation A in $\mathfrak{H}$ are defined as follows:

$$\begin{aligned} \operatorname{dom} A &= \{x \in \mathfrak{H} : \exists\, y \in \mathfrak{H} \text{ with } (x,y) \in A\}, \text{ domain}, \\ \ker A &= \{x \in \mathfrak{H} : (x,0) \in A\}, \text{kernel}, \\ \operatorname{ran} A &= \{y \in \mathfrak{H} : \exists\, x \in \mathfrak{H} \text{ with } (x,y) \in A\}, \text{ range}, \\ \operatorname{mul} A &= \{y \in \mathfrak{H} : (0,y) \in A\}, \text{ multivalued part}, \\ A^{-1} &= \{(y,x) \in \mathfrak{H} \times \mathfrak{H} : (x,y) \in A\}, \text{ inverse}. \end{aligned}$$

Furthermore, for a linear relation A and a complex number $\lambda \in \mathbb{C}$ one defines the following operations:

$$\begin{aligned} A - \lambda = A - \lambda I &= \{(x, y - \lambda x) \in \mathfrak{H} \times \mathfrak{H} : (x,y) \in A\}, \\ \lambda A &= \{(x, \lambda y) \in \mathfrak{H} \times \mathfrak{H} : (x,y) \in A\}. \end{aligned}$$

The product of linear relations A and B in $\mathfrak{H}$ is given by

$$AB = \{(x,y) \in \mathfrak{H} \times \mathfrak{H} : \exists\, z \in \mathfrak{H} \text{ with } (x,z) \in B, (z,y) \in A\},$$

and note that

$$A^k = AA^{k-1}, \quad k \geq 1, \quad \text{where} \quad A^0 = I.$$

2010 *Mathematics Subject Classification.* Primary 47A06, 15A18, 15A22; Secondary 15A04.
Key words and phrases. Linear relation, Jordan chain, singular chain, proper eigenvalue.

This paper is concerned with root spaces for a linear relation A in a linear space $\mathfrak{H}$ over $\mathbb{C}$. The *root spaces* $\mathfrak{R}_\lambda(A)$ of A for $\lambda \in \mathbb{C} \cup \{\infty\}$ are linear subspaces of $\mathfrak{H}$ defined by

$$\mathfrak{R}_\lambda(A) = \bigcup_{i=1}^{\infty} \ker\,(A-\lambda)^i, \quad \lambda \in \mathbb{C}, \quad \text{and} \quad \mathfrak{R}_\infty(A) = \bigcup_{i=1}^{\infty} \operatorname{mul} A^i. \tag{1.1}$$

Let $\lambda \in \mathbb{C}$, then $x \in \mathfrak{R}_\lambda(A)$ if and only if for some $n \in \mathbb{N}$ there exists a chain of elements of the form

$$(x_n, x_{n-1} + \lambda x_n), (x_{n-1}, x_{n-2} + \lambda x_{n-1}), \dots, (x_2, x_1 + \lambda x_2), (x_1, \lambda x_1) \in A, \tag{1.2}$$

and such that $x = x_n$, the "endpoint" of (1.2). The chain in (1.2) is said to be a *Jordan chain* for A corresponding to the eigenvalue $\lambda \in \mathbb{C}$; note that for all $1 \leq i \leq n$ one has $(x_i, 0) \in (A-\lambda)^i$. In fact, if $x \in \ker\,(A-\lambda)^n \setminus \ker\,(A-\lambda)^{n-1}$, then $x_1, \dots, x_{n-1}$ are linearly independent and satisfy $x_i \in \ker\,(A-\lambda)^i \setminus \ker\,(A-\lambda)^{i-1}$, $1 \leq i \leq n-1$. Likewise, $y \in \mathfrak{R}_\infty(A)$ if and only if for some $m \in \mathbb{N}$ there exists a chain of elements of the form

$$(0, y_1),\ (y_1, y_2),\ \dots\dots\dots, (y_{m-2}, y_{m-1}),\ (y_{m-1}, y_m) \in A, \tag{1.3}$$

and such that $y = y_m$, the "endpoint" of (1.3). The chain in (1.3) is said to be a *Jordan chain* for A corresponding to the eigenvalue ∞; note that for all $1 \leq i \leq m$ one has $(0, y_i) \in A^i$. In fact, if $y \in \operatorname{mul} A^m \setminus \operatorname{mul} A^{m-1}$, then $y_1, \dots, y_{m-1}$ are linearly independent and satisfy $y_j \in \operatorname{mul} A^j \setminus \operatorname{mul} A^{j-1}$, $1 \leq j \leq m-1$.

Definition 1.1. *Let A be a linear relation in $\mathfrak{H}$. Then the singular chain space $\mathfrak{R}_c(A)$ is defined by*

$$\mathfrak{R}_c(A) = \mathfrak{R}_0(A) \cap \mathfrak{R}_\infty(A). \tag{1.4}$$

Note that $u \in \mathfrak{R}_c(A)$ if and only if for some $k \in \mathbb{N}$ there is a chain of elements of the form

$$(0, u_k),\ (u_k, u_{k-1}),\ \dots\dots\dots, (u_2, u_1),\ (u_1, 0) \in A, \tag{1.5}$$

and such that $u = u_l$ for some $1 \leq l \leq k$. The chain in (1.5) is said to be a *singular chain* for A. It is clear from (1.5) that $\mathfrak{R}_c(A) \subset \operatorname{dom} A \cap \operatorname{ran} A$, and that $\mathfrak{R}_c(A) \neq \{0\}$ implies that $\ker A \cap \mathfrak{R}_c(A)$ and $\operatorname{mul} A \cap \mathfrak{R}_c(A)$ are non-trivial.

In this paper the singular chain space $\mathfrak{R}_c(A)$ will be characterized by means of different combinations of the root spaces. As a consequence, one sees that

$$\mathfrak{R}_c(A) \subset \mathfrak{R}_\lambda(A)$$

for all $\lambda \in \mathbb{C} \cup \{\infty\}$ and, hence, if $\mathfrak{R}_c(A)$ is non-trivial, all of $\mathbb{C} \cup \{\infty\}$ consists of eigenvalues of A. However, in this case, the eigenvalues $\lambda \in \mathbb{C} \cup \{\infty\}$ for which $\mathfrak{R}_c(A) \neq \mathfrak{R}_\lambda(A)$, i.e., the proper eigenvalues, deserve special attention. The main features are Theorem 4.3 and Theorem 4.4 for the singular chain spaces, and Theorem 5.2 for the proper eigenvalues. In the finite-dimensional case, Corollary 5.3, as a consequence of the last result, gives some fundamental results for the proper eigenvalues. For the convenience of the reader the present paper has been made self-contained by borrowing a few arguments from [12]. The various chain spaces are illustrated by the case of matrix pencils in Section 6. In a further publication [3] the results in the present paper will be used to give a detailed study of the structure of linear relations in finite-dimensional spaces, extending [12].

2. Some transformation results involving chains

This section contains some formal transformation results concerning chains related to a linear relation A in a linear space $\mathfrak{H}$.

Lemma 2.1. *Let A be a linear relation in $\mathfrak{H}$ and let $\lambda \in \mathbb{C}$. Then the chain*

$$(0,x_1),(x_1,x_2),\dots,(x_{n-2},x_{n-1}),(x_{n-1},x_n) \quad \textit{in} \quad A-\lambda, \tag{2.1}$$

is transformed by

$$z_m := \sum_{i=1}^{m} \binom{n-i-1}{n-m-1} (-\lambda)^{m-i} x_i, \quad m=1,\dots,n-1, \tag{2.2}$$

into a chain

$$(0,z_1),(z_1,z_2),\dots,,(z_{n-2},z_{n-1}),(z_{n-1},x_n) \quad \textit{in} \quad A. \tag{2.3}$$

Moreover,

$$\operatorname{span}\{z_1,\dots,z_{n-1}\} = \operatorname{span}\{x_1,\dots,x_{n-1}\}. \tag{2.4}$$

Proof. For later use, note that the assumption (2.1) is equivalent to

$$(0,x_1),(x_1,x_2+\lambda x_1),\dots,(x_{n-2},x_{n-1}+\lambda x_{n-2}),(x_{n-1},x_n+\lambda x_{n-1}) \quad \text{in} \quad A. \tag{2.5}$$

It follows from (2.2) that $z_1 = x_1$ and that $z_{n-1} = \sum_{i=1}^{n-1}(-\lambda)^{n-1-i}x_i$. Hence, it is clear that $(0,z_1)\in A$, and, by working backwards in (2.5), one also sees that

$$(z_{n-1},x_n) = (x_{n-1} - \lambda x_{n-2} + \lambda^2 x_{n-3} + \cdots + (-\lambda)^{n-2}x_1, x_n) \in A.$$

Thus it remains to verify that $(z_m, z_{m+1}) \in A$ for $1 \le m \le n-2$. For this purpose, define the elements $\tilde{z}_m$, $1 \le m \le n-2$, by

$$\tilde{z}_m := \sum_{i=1}^{m} \binom{n-i-1}{n-m-1} (-\lambda)^{m-i}(x_{i+1} + \lambda x_i),$$

and it follows from (2.5) that $(z_m, \tilde{z}_m) \in A$, $1 \le m \le n-2$. A calculation shows for $1 \le m \le n-2$ that

$$\begin{aligned}
\tilde{z}_m &= \sum_{i=1}^{m} \binom{n-i-1}{n-m-1} (-\lambda)^{m-i} x_{i+1} - \sum_{i=1}^{m} \binom{n-i-1}{n-m-1} (-\lambda)^{m-i+1} x_i \\
&= \sum_{i=2}^{m+1} \binom{n-(i-1)-1}{n-m-1} (-\lambda)^{m-i+1} x_i - \sum_{i=1}^{m} \binom{n-i-1}{n-m-1} (-\lambda)^{m-i+1} x_i \\
&= \sum_{i=2}^{m} \left[\binom{n-(i-1)-1}{n-m-1} - \binom{n-i-1}{n-m-1} \right] (-\lambda)^{m-i+1} x_i \\
&\qquad\qquad + x_{m+1} - \binom{n-2}{n-m-1} (-\lambda)^m x_1 \\
&= \sum_{i=2}^{m} \binom{n-i-1}{n-(m+1)-1} (-\lambda)^{m-i+1} x_i + x_{m+1} - \binom{n-2}{n-m-1} (-\lambda)^m x_1 \\
&= \sum_{i=1}^{m+1} \binom{n-i-1}{n-(m+1)-1} (-\lambda)^{m-i+1} x_i - c x_1, \quad c = \binom{n-1}{n-m-1} (-\lambda)^m.
\end{aligned}$$

As a consequence of this calculation and (2.2) one sees that $\tilde{z}_m = z_{m+1} - cx_1$ with the constant c as indicated. One concludes that

$$(z_m, z_{m+1}) = (z_m, \tilde{z}_m) + c(0, x_1) \in A$$

for $1 \leq m \leq n-2$. Hence, the proof of (2.3) is complete for the case $n \geq 3$. The remaining case is trivial.

Finally, observe that the $(n-1) \times (n-1)$ matrix of the transformation in (2.2) is triangular with 1's on the diagonal. Hence, the equality (2.4) holds. □

Lemma 2.2. *Let A be a linear relation in $\mathfrak{H}$ and let $\lambda \in \mathbb{C}$. Then the chain*

$$(x_0, x_1), (x_1, x_2), \dots, (x_{k-1}, x_k), (x_k, 0) \quad \text{in} \quad A - \lambda \tag{2.6}$$

is transformed by

$$z_m := \sum_{i=0}^{m} \binom{m}{i} \lambda^{m-i} x_i, \quad m \geq 0, \tag{2.7}$$

with $x_i = 0$ for $i > k$, into a chain

$$(z_0, z_1), (z_1, z_2), (z_2, z_3), \dots \quad \text{in} \quad A. \tag{2.8}$$

Proof. For later use, note that the assumption (2.6) with the additional convention $x_i = 0$ for $i > k$ is equivalent to

$$(x_i, \lambda x_i + x_{i+1}) \in A, \quad i \geq 0. \tag{2.9}$$

It follows from (2.7) that $z_0 = x_0$, $z_1 = \lambda x_0 + x_1$. Hence it is clear that $(z_0, z_1) \in A$. Thus it remains to verify $(z_m, z_{m+1}) \in A$ for $m \geq 1$. For this purpose, define the elements $\tilde{z}_m$, $m \geq 1$, by

$$\tilde{z}_m := \sum_{i=0}^{m} \binom{m}{i} \lambda^{m-i} (\lambda x_i + x_{i+1}),$$

and it follows from (2.9) that $(z_m, \tilde{z}_m) \in A$ for $m \geq 1$. A calculation shows for $m \geq 1$ that

$$\begin{aligned}
\tilde{z}_m &= \sum_{i=0}^{m} \binom{m}{i} \lambda^{m+1-i} x_i + \sum_{i=0}^{m} \binom{m}{i} \lambda^{m-i} x_{i+1} \\
&= \sum_{i=0}^{m} \binom{m}{i} \lambda^{m+1-i} x_i + \sum_{i=1}^{m+1} \binom{m}{i-1} \lambda^{m+1-i} x_i \\
&= \lambda^{m+1} x_0 + \sum_{i=1}^{m} \left[\binom{m}{i} + \binom{m}{i-1} \right] \lambda^{m+1-i} x_i + x_{m+1} \\
&= \lambda^{m+1} x_0 + \sum_{i=1}^{m} \binom{m+1}{i} \lambda^{m+1-i} x_i + x_{m+1} \\
&= \sum_{i=0}^{m+1} \binom{m+1}{i} \lambda^{m+1-i} x_i.
\end{aligned}$$

As a consequence of this calculation and (2.7) one sees that $\tilde{z}_m = z_{m+1}$. One concludes that $(z_m, z_{m+1}) \in A$ for $m \geq 1$. Hence, the proof of (2.8) is complete. □

Remark 2.3. The transformation result in Lemma 2.2 can be written in the following way. For any $s \geq 0$ one has

$$[z_0, \dots, z_s]^\top = C_{s,k}(\lambda)\,[x_0, \dots, x_k]^\top, \tag{2.10}$$

where $C_{s,k}(\lambda)$ is an $(s+1) \times (k+1)$ matrix:

$$C_{s,k}(\lambda) = (c_{m,i})_{m=0,i=0}^{s,k}, \tag{2.11}$$

whose coefficients $c_{m,i}$ are given by

$$c_{m,i} = \binom{m}{i} \lambda^{m-i} \text{ if } 0 \leq i \leq m \leq s, \text{ and } c_{m,i} = 0 \text{ if } m < i \leq k.$$

Thus, for $s \geq k$ one has

$$C_{s,k}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & \dots & \dots & 0 & 0 \\ \lambda & 1 & 0 & \dots & \dots & 0 & 0 \\ \lambda^2 & \binom{2}{1}\lambda & 1 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda^{k-1} & \binom{k-1}{1}\lambda^{k-2} & \binom{k-1}{2}\lambda^{k-3} & \dots & \dots & 1 & 0 \\ \lambda^k & \binom{k}{1}\lambda^{k-1} & \binom{k}{2}\lambda^{k-2} & \dots & \dots & \binom{k}{k-1}\lambda & 1 \\ \lambda^{k+1} & \binom{k+1}{1}\lambda^{k} & \binom{k+1}{2}\lambda^{k-1} & \dots & \dots & \binom{k+1}{k-1}\lambda^2 & \binom{k+1}{k}\lambda \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \lambda^{s-1} & \binom{s-1}{1}\lambda^{s-2} & \binom{s-1}{2}\lambda^{s-3} & \dots & \dots & \binom{s-1}{k-1}\lambda^{s-k} & \binom{s-1}{k}\lambda^{s-k-1} \\ \lambda^{s} & \binom{s}{1}\lambda^{s-1} & \binom{s}{2}\lambda^{s-2} & \dots & \dots & \binom{s}{k-1}\lambda^{s-k+1} & \binom{s}{k}\lambda^{s-k} \end{pmatrix}.$$

3. Identities for root spaces

Let A be a linear relation in a linear space $\mathfrak{H}$. There are a number of useful algebraic identities for the root spaces in (1.1). First of all there is a simple identity involving a shift of the parameter. It is clear from the definition in (1.1) that for all $\lambda, \mu \in \mathbb{C}$:

$$\mathfrak{R}_\lambda(A) = \mathfrak{R}_{\lambda-\mu}(A - \mu). \tag{3.1}$$

In particular, one has for all $\lambda \in \mathbb{C}$:

$$\mathfrak{R}_0(A - \lambda) = \mathfrak{R}_\lambda(A). \tag{3.2}$$

The following lemma shows some invariance properties for the root space at ∞. These results go back to [12, Lemma 2.3]; the proof is included for completeness.

Lemma 3.1. *Let A be a linear relation in $\mathfrak{H}$. Then for all $\lambda \in \mathbb{C}$*

$$\mathfrak{R}_\infty(A) = \mathfrak{R}_\infty(A - \lambda), \tag{3.3}$$

and for all $\lambda \in \mathbb{C} \setminus \{0\}$

$$\mathfrak{R}_\infty(A) = \mathfrak{R}_\infty(\lambda A). \tag{3.4}$$

Proof. First, it will be shown that $\mathfrak{R}_\infty(A - \lambda) \subset \mathfrak{R}_\infty(A)$ for any $\lambda \in \mathbb{C}$. To see this, let $x \in \mathfrak{R}_\infty(A - \lambda)$. Then $x = x_n$ for some chain of the form (2.1). By Lemma 2.1 it follows that $x = z_n$ for some chain of the form (2.3). In other words, one concludes that $x \in \mathfrak{R}_\infty(A)$. Thus it follows that $\mathfrak{R}_\infty(A - \lambda) \subset \mathfrak{R}_\infty(A)$ for any $\lambda \in \mathbb{C}$. Now it is clear that

$$\mathfrak{R}_\infty(A) = \mathfrak{R}_\infty(A - \lambda + \lambda) \subset \mathfrak{R}_\infty(A - \lambda),$$

where the inclusion is obtained by applying the earlier observation (with A replaced by $A-\lambda$ and λ replaced by $-\lambda$). Therefore, (3.3) has been shown.

Next it will be shown that $\mathfrak{R}_\infty(A) \subset \mathfrak{R}_\infty(\lambda A)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. To see this, let $x \in \mathfrak{R}_\infty(A)$. Then $x = x_n$ for some chain of the form

$$(0, x_1), (x_1, x_2), \dots, (x_{n-1}, x_n) \in A.$$

Then, clearly,

$$(0, \lambda x_1), (\lambda x_1, \lambda^2 x_2), \dots\dots, (\lambda^{n-1} x_{n-1}, \lambda^n x_n) \in \lambda A,$$

and this shows that $\lambda^n x_n \in \mathfrak{R}_\infty(\lambda A)$, so that also $x_n \in \mathfrak{R}_\infty(\lambda A)$. Thus it follows that $\mathfrak{R}_\infty(A) \subset \mathfrak{R}_\infty(\lambda A)$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Now observe that

$$\mathfrak{R}_\infty(\lambda A) \subset \mathfrak{R}_\infty\left(\frac{1}{\lambda}\lambda A\right) = \mathfrak{R}_\infty(A), \quad \lambda \in \mathbb{C} \setminus \{0\},$$

where the inclusion is obtained by applying the earlier observation (with A replaced by λA and λ replaced by $1/\lambda$). Therefore, (3.4) has been shown. □

The following simple identity, for a linear relation A in $\mathfrak{H}$ and $\lambda \in \mathbb{C} \setminus \{0\}$,

$$(A-\lambda)^{-1} = -\frac{1}{\lambda} - \frac{1}{\lambda^2}\left(A^{-1} - \frac{1}{\lambda}\right)^{-1}, \tag{3.5}$$

is easily verified; cf. [2]. By taking multivalued parts in this identity, and noting that multivalued parts are shift-invariant, one sees that

$$\ker\,(A-\lambda) = \ker\,\left(A^{-1} - \tfrac{1}{\lambda}\right), \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

This last identity between the kernels can be further extended to the root spaces; cf. [12, Proposition 2.4]. The simple proof is included as an illustration of Lemma 3.1.

Lemma 3.2. *Let A be a linear relation in $\mathfrak{H}$. Then for all $\lambda \in \mathbb{C} \setminus \{0\}$:*

$$\mathfrak{R}_\lambda(A) = \mathfrak{R}_{\lambda^{-1}}(A^{-1}). \tag{3.6}$$

Moreover,

$$\mathfrak{R}_0(A) = \mathfrak{R}_\infty(A^{-1}), \quad \mathfrak{R}_\infty(A) = \mathfrak{R}_0(A^{-1}). \tag{3.7}$$

Proof. The identities in (3.7) follow directly from the definition and $\ker\, A = \operatorname{mul} A^{-1}$. For $\lambda \in \mathbb{C} \setminus \{0\}$ it is a consequence of (3.3), (3.4), and (3.5) that

$$\mathfrak{R}_\infty((A-\lambda)^{-1}) = \mathfrak{R}_\infty\left(\left(A^{-1} - \frac{1}{\lambda}\right)^{-1}\right). \tag{3.8}$$

Hence, for $\lambda \in \mathbb{C} \setminus \{0\}$, one sees from this

$$\begin{aligned} \mathfrak{R}_\lambda(A) &\overset{(3.2)}{=} \mathfrak{R}_0(A-\lambda) \overset{(3.7)}{=} \mathfrak{R}_\infty((A-\lambda)^{-1}) \\ &\overset{(3.8)}{=} \mathfrak{R}_\infty\left(\left(A^{-1} - \frac{1}{\lambda}\right)^{-1}\right) \overset{(3.7)}{=} \mathfrak{R}_0\left(A^{-1} - \frac{1}{\lambda}\right) \overset{(3.2)}{=} \mathfrak{R}_{\lambda^{-1}}(A^{-1}), \end{aligned}$$

which gives (3.6). □

4. Identities for singular chain spaces

Let A be a linear relation in a linear space $\mathfrak{H}$. Recall from Definition 1.1 that $\mathfrak{R}_c(A) = \mathfrak{R}_0(A) \cap \mathfrak{R}_\infty(A)$. In Lemma 3.1 it has been shown that $\mathfrak{R}_\infty(A)$ is shift-invariant. In fact, it can be shown that also the space $\mathfrak{R}_c(A)$ in Definition 1.1 is invariant under translations of the relation A.

Lemma 4.1. *Let A be a linear relation in $\mathfrak{H}$. For any $\lambda \in \mathbb{C}$ one has*

$$\mathfrak{R}_c(A) = \mathfrak{R}_c(A - \lambda). \tag{4.1}$$

Proof. It will be shown that $\mathfrak{R}_c(A-\lambda) \subset \mathfrak{R}_c(A)$, $\lambda \in \mathbb{C}$. To see this last inclusion, let $x \in \mathfrak{R}_c(A-\lambda)$. Then the element x is some entry of an element in a chain of the form

$$(0, x_1), (x_1, x_2), \dots, (x_{s-1}, x_s), (x_s, 0) \ \text{ in } \ A - \lambda.$$

By Lemma 2.1 the transformation (2.2) produces a chain which satisfies

$$(0, z_1), (z_1, z_2), \dots, (z_{s-1}, z_s), (z_s, 0) \in A,$$

and, in addition,

$$\operatorname{span}\{x_1, \dots, x_s\} = \operatorname{span}\{z_1, \dots, z_s\}.$$

By definition, each $z_i \in \mathfrak{R}_c(A)$ and, hence, $x \in \mathfrak{R}_c(A)$. Therefore it follows that $\mathfrak{R}_c(A-\lambda) \subset \mathfrak{R}_c(A)$, $\lambda \in \mathbb{C}$. Now it is clear that

$$\mathfrak{R}_c(A) = \mathfrak{R}_c(A - \lambda + \lambda) \subset \mathfrak{R}_c(A - \lambda),$$

where the inclusion is obtained by applying the earlier observation (with A replaced by $A-\lambda$ and λ replaced by $-\lambda$). Therefore, (4.1) has been shown. □

By Lemma 4.1, if $\mathfrak{R}_c(A) = \{0\}$, one sees that for $x \in \ker\,(A-\lambda)^n$ the chain in (1.2) with $x = x_n$ is uniquely determined. Likewise, for $y \in \operatorname{mul} A^m$ the chain in (1.3) with $y = y_m$ is uniquely determined if $\mathfrak{R}_c(A) = \{0\}$. Furthermore, one is able to characterize $\mathfrak{R}_c(A)$ in a different way, involving the root space at $\lambda \in \mathbb{C}$ rather than the one at $\lambda = 0$.

Proposition 4.2. *Let A be a linear relation in $\mathfrak{H}$. For all $\lambda \in \mathbb{C}$ one has*

$$\mathfrak{R}_c(A) = \mathfrak{R}_\lambda(A) \cap \mathfrak{R}_\infty(A). \tag{4.2}$$

Proof. Observe that by (4.1) one has that

$$\mathfrak{R}_c(A) = \mathfrak{R}_c(A-\lambda) = \mathfrak{R}_0(A-\lambda) \cap \mathfrak{R}_\infty(A-\lambda) = \mathfrak{R}_\lambda(A) \cap \mathfrak{R}_\infty(A),$$

where the last equality was obtained by using the identities (3.2) and (3.3). □

It was shown in [12] that $\mathfrak{R}_c(A)$ is non-trivial if and only if $\mathfrak{R}_\lambda(A) \cap \mathfrak{R}_\mu(A)$ is non-trivial. In fact, Proposition 4.2 is the stepping stone for the following general result, namely, the calculation of the intersection of $\mathfrak{R}_\lambda(A)$ and $\mathfrak{R}_\mu(A)$.

Theorem 4.3. *Let A be a linear relation in $\mathfrak{H}$. Then one has for $\lambda, \mu \in \mathbb{C} \cup \{\infty\}$ with $\lambda \neq \mu$ that*

$$\mathfrak{R}_c(A) = \mathfrak{R}_\lambda(A) \cap \mathfrak{R}_\mu(A).$$

Proof. Due to Proposition 4.2 it suffices to consider the case where $\lambda, \mu \in \mathbb{C}$ and $\lambda \neq \mu$. With these restrictions define the linear relation $\tilde{A}$ by

$$\tilde{A} = (A-\lambda)^{-1} - (\mu - \lambda)^{-1}, \tag{4.3}$$

so that the following identities are clear

$$\begin{aligned}\mathfrak{R}_\infty(\tilde{A}) &\overset{(4.3)}{=} \mathfrak{R}_\infty((A-\lambda)^{-1}-(\mu-\lambda)^{-1}) \\ &\overset{(3.3)}{=} \mathfrak{R}_\infty((A-\lambda)^{-1}) \overset{(3.7)}{=} \mathfrak{R}_0(A-\lambda) \overset{(3.2)}{=} \mathfrak{R}_\lambda(A).\end{aligned} \tag{4.4}$$

By Proposition 4.2 one may write $\mathfrak{R}_c(\tilde{A}) = \mathfrak{R}_0(\tilde{A}) \cap \mathfrak{R}_\infty(\tilde{A})$ as

$$\mathfrak{R}_c(\tilde{A}) = \mathfrak{R}_{-(\mu-\lambda)^{-1}}(\tilde{A}) \cap \mathfrak{R}_\infty(\tilde{A}), \tag{4.5}$$

since $\lambda \neq \mu$. Now observe that

$$\begin{aligned}\mathfrak{R}_{-(\mu-\lambda)^{-1}}(\tilde{A}) &\overset{(4.3)}{=} \mathfrak{R}_{-(\mu-\lambda)^{-1}}((A-\lambda)^{-1}-(\mu-\lambda)^{-1}) \\ &\overset{(3.2)}{=} \mathfrak{R}_0((A-\lambda)^{-1}) \overset{(3.7)}{=} \mathfrak{R}_\infty(A-\lambda) \overset{(3.3)}{=} \mathfrak{R}_\infty(A).\end{aligned} \tag{4.6}$$

Combining (4.5) with (4.4) and (4.6) leads to

$$\mathfrak{R}_c(\tilde{A}) = \mathfrak{R}_\lambda(A) \cap \mathfrak{R}_\infty(A) = \mathfrak{R}_c(A), \tag{4.7}$$

by Proposition 4.2. Moreover, observe that

$$\begin{aligned}\mathfrak{R}_0(\tilde{A}) &\overset{(4.3)}{=} \mathfrak{R}_0((A-\lambda)^{-1}-(\mu-\lambda)^{-1}) \\ &\overset{(3.2)}{=} \mathfrak{R}_{(\mu-\lambda)^{-1}}((A-\lambda)^{-1}) \overset{(3.6)}{=} \mathfrak{R}_{\mu-\lambda}(A-\lambda) \overset{(3.1)}{=} \mathfrak{R}_\mu(A).\end{aligned} \tag{4.8}$$

Consequently, a combination of (4.7), the definition of $\mathfrak{R}_c(\tilde{A})$, (4.4), and (4.8) leads to

$$\mathfrak{R}_c(A) = \mathfrak{R}_c(\tilde{A}) = \mathfrak{R}_0(\tilde{A}) \cap \mathfrak{R}_\infty(\tilde{A}) = \mathfrak{R}_\lambda(A) \cap \mathfrak{R}_\mu(A). \qquad \square$$

The following theorem is the main result in this section. It will lead to a final extension of Theorem 4.3.

Theorem 4.4. *Let A be a linear relation in $\mathfrak{H}$ and let the sets*

$$\{\lambda_1, \dots, \lambda_l\} \subseteq \mathbb{C} \quad \textit{and} \quad \{\mu\} \subseteq \mathbb{C} \cup \{\infty\} \tag{4.9}$$

be disjoint. Assume that

$$x^r \in \mathfrak{R}_{\lambda_r}(A), \quad r = 1, \dots, l, \quad \textit{and} \quad \sum_{r=1}^{l} x^r \in \mathfrak{R}_\mu(A). \tag{4.10}$$

Then $x^r \in \mathfrak{R}_c(A)$ for $r = 1, \dots, l$.

Proof. Assume the conditions in (4.9) and let the elements x^r, $1 \leq r \leq l$, satisfy the conditions in (4.10). The proof will be given in two steps: in the first step the case with $\mu = \infty$ will be considered and in the second step the case with $\mu \in \mathbb{C}$ will be considered by a reduction to Step 1.

Step 1: Assume that $\mu = \infty$. Then (4.10) reads

$$x^r \in \mathfrak{R}_{\lambda_r}(A), \quad r = 1, \dots, l, \quad \text{and} \quad \sum_{r=1}^{l} x^r \in \mathfrak{R}_\infty(A). \tag{4.11}$$

By the first condition in (4.11): $x^r \in \mathfrak{R}_{\lambda_r}(A)$, there exist $k_r \in \mathbb{N}$ and chains

$$(x_0^r, x_1^r), \dots, (x_{k_r-1}^r, x_{k_r}^r), (x_{k_r}^r, 0) \in A - \lambda_r, \quad r = 1, \dots, l,$$

with $x_0^r = x^r$. Set $x_i^r = 0$ for $i > k_r$, then an application of Lemma 2.2 shows that for all $m \geq 0$ one has with

$$z_m^r = \sum_{i=0}^{m} \binom{m}{i} \lambda_r^{m-i} x_i^r,$$

that $z_0^r = x_0^r = x^r$ and for all $m \geq 0$

$$(z_m^r, z_{m+1}^r) \in A.$$

It is helpful to introduce the notation

$$w_m = \sum_{r=1}^{l} z_m^r, \quad m \geq 0, \tag{4.12}$$

which leads to

$$w_0 = \sum_{r=1}^{l} x^r \in \mathfrak{R}_\infty(A), \quad (w_m, w_{m+1}) \in A.$$

Therefore, it is clear that $w_m \in \mathfrak{R}_\infty(A)$ for all $m \in \mathbb{N} \cup \{0\}$.

The statement $x^r \in \mathfrak{R}_\infty(A)$ for $r = 1, \dots, l$, will now be retrieved from (4.12) for suitably many indices m. For this purpose, recall that with the $(s+1) \times (k_r+1)$ matrix $C_{s,k_r}(\lambda_r)$, as defined in (2.11), one may write for any $s \geq 0$:

$$[z_0^r, \dots, z_s^r] = [x_0^r, \dots, x_{k_r}^r]\, C_{s,k_r}(\lambda_r)^\top. \tag{4.13}$$

In the present context, choose $s \geq 0$ as

$$s + 1 = \sum_{r=1}^{l} (k_r + 1),$$

so that, in particular, each $k_i \leq s$. Observe that, due to Remark 2.3, the matrix

$$W = (C_{s,k_1}(\lambda_1), \, \cdots, C_{s,k_l}(\lambda_l)) \tag{4.14}$$

is, in fact, an $(s+1) \times (s+1)$ confluent Vandermonde matrix. This matrix W is invertible, since it is well-known that

$$\det W = \prod_{1 \leq i < j \leq l} (\lambda_i - \lambda_j)^{(k_i+1)(k_j+1)}. \tag{4.15}$$

A short proof of (4.15) in a more general setting can be found in [9]. By means of (4.12), (4.13), and (4.14), one may write:

$$\begin{aligned} [w_0, \dots, w_s] &= [z_0^1 + \cdots + z_0^l, \dots, z_s^1 + \cdots + z_s^l] \\ &= \left[x_0^1, \dots, x_{k_1}^1, \dots, x_0^l, \dots, x_{k_l}^l\right] W^\top, \end{aligned} \tag{4.16}$$

and it therefore follows from the invertibility of W that all vectors

$$x_0^1, \dots, x_{k_1}^1, \dots, x_0^l, \dots, x_{k_l}^l$$

in (4.16) are linear combinations of $w_0, \dots, w_s$ and, hence, they belong to $\mathfrak{R}_\infty(A)$. Moreover, by Proposition 4.2 one finds that

$$x^r = x_0^r \in \mathfrak{R}_{\lambda_r}(A) \cap \mathfrak{R}_\infty(A) = \mathfrak{R}_c(A), \quad r = 1, \dots, l.$$

Step 2: Assume that $\mu \in \mathbb{C}$. To reduce this case to the situation in Step 1 the linear relation $\tilde{A}$ is introduced by

$$\tilde{A} = (A - \mu)^{-1}. \tag{4.17}$$

By the assumptions in (4.9), it is clear that the sets

$$\left\{\frac{1}{\lambda_1-\mu},\cdots,\frac{1}{\lambda_l-\mu}\right\} \quad \text{and} \quad \{\infty\} \tag{4.18}$$

are disjoint. In addition, one sees that

$$\mathfrak{R}_\mu(A) \overset{(3.2)}{=} \mathfrak{R}_0(A-\mu) \overset{(3.7)}{=} \mathfrak{R}_\infty((A-\mu)^{-1}) \overset{(4.17)}{=} \mathfrak{R}_\infty(\tilde{A}),$$

and for $1 \le r \le l$

$$\mathfrak{R}_{\lambda_r}(A) \overset{(3.1)}{=} \mathfrak{R}_{\lambda_r-\mu}(A-\mu) \overset{(3.6)}{=} \mathfrak{R}_{(\lambda_r-\mu)^{-1}}((A-\mu)^{-1}) \overset{(4.17)}{=} \mathfrak{R}_{(\lambda_r-\mu)^{-1}}(\tilde{A}).$$

Therefore, by the assumptions in (4.10), one obtains

$$x^r \in \mathfrak{R}_{(\lambda_r-\mu)^{-1}}(\tilde{A}), \quad r=1,\dots,l, \quad \text{and} \quad \sum_{r=1}^{l} x^r \in \mathfrak{R}_\infty(\tilde{A}). \tag{4.19}$$

Hence, according to (4.18) and (4.19), Step 1 may be applied when the linear relation A and the scalars λ_r, $r=1,\dots,l$, are replaced by

$$\tilde{A} \quad \text{and} \quad \frac{1}{\lambda_r-\mu}, \; r=1,\dots,l.$$

This leads to the conclusion that $x^r \in \mathfrak{R}_c(\tilde{A})$ for $r=1,\dots,l$. Finally, observe that

$$\begin{aligned} \mathfrak{R}_c(\tilde{A}) &\overset{(4.17)}{=} \mathfrak{R}_c((A-\mu)^{-1}) = \mathfrak{R}_0((A-\mu)^{-1}) \cap \mathfrak{R}_\infty((A-\mu)^{-1}) \\ &\overset{(3.7)}{=} \mathfrak{R}_0(A-\mu) \cap \mathfrak{R}_\infty(A-\mu) = \mathfrak{R}_c(A-\mu) \overset{(4.1)}{=} \mathfrak{R}_c(A). \end{aligned} \tag{4.20}$$

Thus it follows that $x^r \in \mathfrak{R}_c(A)$ for $r=1,\dots,l$. □

There is a special case of Theorem 4.4 worth mentioning:

$$x^r \in \mathfrak{R}_{\lambda_r}(A), \; r=1,\dots,l \quad \text{and} \quad \sum_{r=1}^{l} x^r = 0 \quad \Rightarrow \quad x^r \in \mathfrak{R}_c(A), \; r=1,\dots,l.$$

To see this, observe that $\sum_{r=1}^{l} x^r \in \mathfrak{R}_\mu(A)$ for any $\mu \in \mathbb{C}\cup\{\infty\}$ and, in particular, for any $\mu \in \mathbb{C}\cup\{\infty\}$ that does not belong to $\{\lambda_1,\dots,\lambda_l\} \subseteq \mathbb{C}$.

Theorem 4.4 leads to Corollary 4.5 below, which is an extension Theorem 4.3.

Corollary 4.5. *Let A be a linear relation in $\mathfrak{H}$ and let*

$$\{\lambda_1,\dots,\lambda_l\} \quad \textit{and} \quad \{\mu_1,\dots,\mu_m\} \tag{4.21}$$

be two disjoint subsets of $\mathbb{C}\cup\{\infty\}$. Then

$$\mathfrak{R}_c(A) = \left(\sum_{i=1}^{l} \mathfrak{R}_{\lambda_i}(A)\right) \bigcap \left(\sum_{j=1}^{m} \mathfrak{R}_{\mu_j}(A)\right). \tag{4.22}$$

Proof. It is clear from Theorem 4.3 that the left-hand side of (4.22) is contained in the right-hand side. Therefore it suffices to show that the right-hand side of (4.22) is contained in the left-hand side. Let x be an element of the space on the right-hand side of (4.22), which means that x can be written as

$$x = \sum_{i=1}^{l} y_i = \sum_{j=1}^{m} z_j \quad \text{with} \quad y_i \in \mathfrak{R}_{\lambda_i}(A), \; z_j \in \mathfrak{R}_{\mu_j}(A).$$

In the sets in (4.21) at most one element equals ∞. If this is the case, let it be λ_1 without loss of generality. In all cases, it is clear that

$$\sum_{j=1}^{m} z_j - \sum_{i=2}^{l} y_i = y_1 \in \mathfrak{R}_{\lambda_1}(A).$$

Therefore, by Theorem 4.4 it follows that $z_j \in \mathfrak{R}_c(A)$, $j = 1, \dots, m$, and $y_i \in \mathfrak{R}_c(A)$, $i = 2, \dots, l$, and consequently, also $y_1 \in \mathfrak{R}_c(A)$. Thus one sees that $x \in \mathfrak{R}_c(A)$. One concludes that the right-hand side of (4.22) is contained in the left-hand side. Thus the identity (4.22) has been proved. □

5. The proper point spectrum of a linear relation

The number $\lambda \in \mathbb{C}$ is an *eigenvalue* of A, if $\ker\,(A - \lambda) \neq \{0\}$, and ∞ is an *eigenvalue* of A, if $\operatorname{mul} A \neq \{0\}$. The usual *point spectrum* $\sigma_p(A)$ is the set of all eigenvalues $\lambda \in \mathbb{C} \cup \{\infty\}$ of the relation A:

$$\sigma_p(A) = \{\lambda \in \mathbb{C} \cup \{\infty\} : \lambda \text{ is an eigenvalue of } A\}. \tag{5.1}$$

It follows from (4.2) that

$$\mathfrak{R}_c(A) \subset \mathfrak{R}_\lambda(A), \quad \lambda \in \mathbb{C} \cup \{\infty\}. \tag{5.2}$$

Thus, by (5.2), the following implication is trivial:

$$\mathfrak{R}_c(A) \neq \{0\} \quad \Longrightarrow \quad \sigma_p(A) = \mathbb{C} \cup \{\infty\}. \tag{5.3}$$

Note that the present geometric treatment takes care of [12, Proposition 3.2, Corollary 3.3, Corollary 3.4]. The following definition is based on the inclusion (5.2).

Definition 5.1. *Let A be a linear relation in a linear space $\mathfrak{H}$. The proper point spectrum $\sigma_\pi(A)$ is a subset of the point spectrum $\sigma_p(A)$, defined for $\lambda \in \mathbb{C} \cup \{\infty\}$ by*

$$\lambda \in \sigma_\pi(A) \quad \Longleftrightarrow \quad \mathfrak{R}_\lambda(A) \setminus \mathfrak{R}_c(A) \neq \emptyset. \tag{5.4}$$

The elements in $\sigma_\pi(A)$ are called the proper eigenvalues of A.

Note that if $\mathfrak{R}_c(A) = \{0\}$, then $\sigma_\pi(A) = \sigma_p(A)$. The following result is a direct consequence of Corollary 4.5.

Theorem 5.2. *Let A be a linear relation in a linear space $\mathfrak{H}$. Assume that there exists $k \in \mathbb{N}$ such that $\lambda_1, \dots, \lambda_k \in \sigma_\pi(A)$ are pairwise distinct proper eigenvalues and let*

$$x_i \in \mathfrak{R}_{\lambda_i}(A) \setminus \mathfrak{R}_c(A), \quad i = 1, \dots, k.$$

Then the elements $x_1, \dots, x_k$ are linearly independent in $\mathfrak{H}$.

Proof. Seeking a contradiction, assume that the elements $x_1, \dots, x_k$ are linearly dependent. In fact, assume without loss of generality that

$$\sum_{i=1}^{k} c_i x_i = 0 \quad \text{with} \quad c_i \in \mathbb{C},\ i = 1, \dots, k,$$

and that $c_k \neq 0$. Then, clearly,

$$c_k x_k = -\sum_{i=1}^{k-1} c_i x_i \in \mathfrak{R}_{\lambda_k}(A) \cap \operatorname{span}\,\{\, \mathfrak{R}_{\lambda_i}(A) : \, i = 1, \dots, k-1 \,\}\,.$$

Since $c_k \neq 0$, this implies, by Corollary 4.5, that $x_k \in \mathfrak{R}_c(A)$, a contradiction. Hence $x_1, \ldots, x_k$ are linearly independent. □

If the linear space $\mathfrak{H}$ is finite-dimensional, then there is a bound for the number of proper eigenvalues in $\sigma_\pi(A)$, as follows from Theorem 5.2.

Corollary 5.3. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$. Then*

$$|\sigma_\pi(A)| \leq \dim \mathfrak{H}, \tag{5.5}$$

so that, in particular, $\sigma_\pi(A)$ consists of finitely many elements.

The following result is a simple consequence of Theorem 5.2 and Corollary 5.3; it goes back to [12, Lemma 4.5, Theorem 5.1].

Corollary 5.4. *Let A be a linear relation in a finite-dimensional space $\mathfrak{H}$ and assume that $\mathfrak{R}_c(A) = \{0\}$. Let $\lambda_i \in \sigma_p(A)$ be pairwise distinct for $i = 1, \ldots, k$, and let $x_i \in \mathfrak{R}_{\lambda_i}(A)$ be nontrivial. Then the elements $x_1, \ldots, x_k$ are linearly independent and, consequently, $|\sigma_p(A)| \leq \dim \mathfrak{H}$.*

The implication in (5.3) can be reversed if the space is finite-dimensional; cf. [12, Theorem 4.4]. A proof is included for completeness.

Proposition 5.5. *Let A be a linear relation in $\mathfrak{H}$ and let $\mathfrak{H}$ be finite-dimensional. Then*

$$\mathfrak{R}_c(A) \neq \{0\} \quad \Longleftrightarrow \quad \sigma_p(A) = \mathbb{C} \cup \{\infty\}. \tag{5.6}$$

Proof. ($\Rightarrow$) This is (5.3).

($\Leftarrow$) Assume that $\sigma_p(A) = \mathbb{C} \cup \{\infty\}$. By assumption, $\mathfrak{H}$ is finite-dimensional, say, $\dim \mathfrak{H} = m$. Therefore, let $\lambda_1, \ldots, \lambda_{m+1}$ in $\mathbb{C}$ be different eigenvalues of A and let $(x_i, \lambda_i x_i) \in A$ with nontrivial $x_i \in \mathfrak{H}$, $1 \leq i \leq m+1$. Then $x_1, \ldots, x_{m+1}$ are linearly dependent and there exist $c_i \in \mathbb{C}$ such that

$$\sum_{i=1}^{m+1} c_i x_i = 0, \quad \sum_{i=1}^{m+1} |c_i| > 0.$$

Clearly, $c_i x_i \in \mathfrak{R}_{\lambda_i}(A)$ and one may choose $\mu \in \mathbb{C}$ with $\mu \neq \lambda_i$ for $i = 1, \ldots, m+1$ so that $\sum_{i=1}^{m+1} c_i x_i = 0 \in \mathfrak{R}_\mu(A)$. Then Theorem 4.4 implies that $c_i x_i \in \mathfrak{R}_c(A)$ for all $i = 1, \ldots, m+1$, hence $\mathfrak{R}_c(A) \neq \{0\}$. □

6. Matrix pencils and linear relations

Linear relations naturally appear in the study of matrix pencils. Let E and F be matrices in $\mathbb{C}^{m \times d}$ and consider the associated matrix pencil $sE - F$, which is a polynomial matrix in $\mathbb{C}[s]^{m \times d}$. Then it is natural to consider the corresponding operator range

$$\mathcal{A} = \operatorname{ran} \begin{pmatrix} E \\ F \end{pmatrix}, \tag{6.1}$$

so that $\mathcal{A}$ is a linear relation in the space $\mathfrak{H} = \mathbb{C}^m$. There is a close connection in terms of the root spaces and singular chain spaces of $\mathcal{A}$ and the pencil $sE - F$, which will be described in the following by means of the *Kronecker canonical form* for linear matrix pencils, see e.g. [4, 5, 8]. For this purpose, define for $k \in \mathbb{N}$ the matrices N_k, K_k, and L_k by

$$N_k := \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in \mathbb{C}^{k\times k},$$

and

$$K_k := \begin{pmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \quad L_k := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{pmatrix} \in \mathbb{C}^{(k-1)\times k}.$$

Likewise, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$ with absolute value $|\alpha| = \sum_{i=1}^{l} \alpha_i$, define

$$\begin{aligned} &N_\alpha := \operatorname{diag}(N_{\alpha_1}, \ldots, N_{\alpha_l}) \in \mathbb{C}^{|\alpha|\times|\alpha|}, \\ &K_\alpha := \operatorname{diag}(K_{\alpha_1}, \ldots, K_{\alpha_l}), \quad L_\alpha := \operatorname{diag}(L_{\alpha_1}, \ldots, L_{\alpha_l}) \in \mathbb{C}^{(|\alpha|-l)\times|\alpha|}. \end{aligned}$$

See, e.g., [6] for a discussion of the case that some or all of the entries of α are equal to one.

According to Kronecker [10], there exist invertible matrices $W \in \mathbb{C}^{m\times m}$ and $T \in \mathbb{C}^{d\times d}$ such that

$$W(sE - F)T = \begin{pmatrix} sI_{n_0} - A_0 & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\varepsilon - L_\varepsilon & 0 \\ 0 & 0 & 0 & sK_\eta^\top - L_\eta^\top \end{pmatrix}, \tag{6.2}$$

for some $A_0 \in \mathbb{C}^{n_0\times n_0}$ in Jordan canonical form, and multi-indices α, ε, η, with lengths n_α, n_ε, n_η, respectively, ordered non-decreasingly. For details, see [8, Chapter XII] or [11].

The following theorem shows how the singular chain space and the root spaces together with the proper eigenvalues can be read off from the Kronecker canonical form. In this sense the Kronecker canonical form can be seen as a canonical form for the linear relation in (6.1). Vice versa, this provides a simple geometric interpretation for the four parts in the Kronecker canonical form.

Theorem 6.1. *Let E and F be matrices in $\mathbb{C}^{m\times d}$ with Kronecker canonical form* (6.2) *and let the linear relation $\mathcal{A}$ be defined by* (6.1). *Then the following statements hold:*

(i) *the singular chain space $\mathfrak{R}_c(\mathcal{A})$ is given by*

$$\mathfrak{R}_c(\mathcal{A}) = W^{-1}\left(\{0\}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\varepsilon|-n_\varepsilon} \times \{0\}^{|\eta|}\right);$$

(ii) *the root space $\mathfrak{R}_\infty(\mathcal{A})$ is given by*

$$\mathfrak{R}_\infty(\mathcal{A}) = W^{-1}\left(\{0\}^{n_0} \times \mathbb{C}^{|\alpha|} \times \mathbb{C}^{|\varepsilon|-n_\varepsilon} \times \{0\}^{|\eta|}\right);$$

(iii) *the root space $\mathfrak{R}_\lambda(\mathcal{A})$ for some $\lambda \in \mathbb{C}$ is given by*

$$\mathfrak{R}_\lambda(\mathcal{A}) = W^{-1}\left(\mathfrak{R}_\lambda(A_0) \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\varepsilon|-n_\varepsilon} \times \{0\}^{|\eta|}\right),$$

where $\mathfrak{R}_\lambda(A_0)$ is the root space for $\lambda \in \mathbb{C}$ of the matrix A_0 in (6.2)*;*

(iv) *the proper eigenvalues of* $\mathcal{A}$ *are given by*

$$\sigma_\pi(\mathcal{A}) = \begin{cases} \sigma_p(A_0) \cup \{\infty\}, & \text{if } |\alpha| \neq 0, \\ \sigma_p(A_0), & \text{if } |\alpha| = 0, \end{cases}$$

where $\sigma_p(A_0)$ *is the point spectrum of the matrix* A_0 *in* (6.2).

Proof. Statements (i) and (ii) are from [6, Theorem 4.5].

In order to show (iii), let $x \in \mathfrak{R}_\lambda(\mathcal{A}) \setminus \{0\}$. Then there exists a chain of the form (1.2) with $x = x_n$. By (6.1) there exist $z_1, \ldots, z_n \in \mathbb{C}^d$ such that

$$\begin{array}{rcl} (x_1, \lambda x_1) & = & (Ez_1, Fz_1), \\ (x_2, x_1 + \lambda x_2) & = & (Ez_2, Fz_2), \\ \cdots & \cdots & \cdots \\ (x_{n-1}, x_{n-2} + \lambda x_{n-1}) & = & (Ez_{n-1}, Fz_{n-1}), \\ (x_n, x_{n-1} + \lambda x_n) & = & (Ez_n, Fz_n). \end{array} \tag{6.3}$$

For $i \in \{1, \ldots, n\}$ define $y_i = T^{-1} z_i$. Partitioning y_i according to the decomposition (6.2):

$$y_i = (y_{i,1}^\top, \ldots, y_{i,4}^\top)^\top \quad \text{with} \quad y_{i,1} \in \mathbb{C}^{n_0},\ y_{i,2} \in \mathbb{C}^{|\alpha|},\ y_{i,3} \in \mathbb{C}^{|\varepsilon|},\ y_{i,4} \in \mathbb{C}^{|\eta| - n_\eta},$$

one obtains from the first equation in (6.3) that

$$\begin{aligned} Wx_1 &= WEz_1 = (WET)T^{-1}z_1 = \begin{pmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\varepsilon & 0 \\ 0 & 0 & 0 & K_\eta^\top \end{pmatrix} \begin{pmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ y_{1,4} \end{pmatrix}, \\ \lambda W x_1 &= WFz_1 = (WFT)T^{-1}z_1 = \begin{pmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\varepsilon & 0 \\ 0 & 0 & 0 & L_\eta^\top \end{pmatrix} \begin{pmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ y_{1,4} \end{pmatrix}. \end{aligned}$$

Therefore, each of the four components leads to an equation, thus

$$\begin{aligned} (A_0 - \lambda I_{n_0}) y_{1,1} = 0, &\quad (I_{|\alpha|} - \lambda N_\alpha) y_{1,2} = 0, \\ (L_\varepsilon - \lambda K_\varepsilon) y_{1,3} = 0, &\quad (L_\eta^\top - \lambda K_\eta^\top) y_{1,4} = 0. \end{aligned}$$

Clearly, if $y_{1,1} \neq 0$, λ is an eigenvalue of A_0 with eigenvector $y_{1,1}$. Moreover, $I_{|\alpha|} - \lambda N_\alpha$ is an invertible matrix and, invoking [6, Lemma 4.1], one obtains

$$y_{1,2} = 0 \quad \text{and} \quad y_{1,4} = 0,$$

hence,

$$Wx_1 = \begin{pmatrix} y_{1,1} \\ 0 \\ K_\varepsilon y_{1,3} \\ 0 \end{pmatrix}. \tag{6.4}$$

The second equation in (6.3) gives

$$\begin{aligned} Wx_2 &= WEz_2 = (WET)T^{-1}z_2 = \begin{pmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\varepsilon & 0 \\ 0 & 0 & 0 & K_\eta^\top \end{pmatrix} \begin{pmatrix} y_{2,1} \\ y_{2,2} \\ y_{2,3} \\ y_{2,4} \end{pmatrix}, \\ Wx_1 + \lambda Wx_2 &= WFz_2 = (WFT)T^{-1}z_2 = \begin{pmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\varepsilon & 0 \\ 0 & 0 & 0 & L_\eta^\top \end{pmatrix} \begin{pmatrix} y_{2,1} \\ y_{2,2} \\ y_{2,3} \\ y_{2,4} \end{pmatrix}. \end{aligned}$$

Thanks to (6.4), the first, second, and fourth components of the identity

$$(Wx_1 + \lambda Wx_2) - \lambda Wx_2 = Wx_1$$

lead to the equations

$$(A_0 - \lambda I_{n_0})y_{2,1} = y_{1,1}, \quad (I_{|\alpha|} - \lambda N_\alpha)y_{2,2} = 0, \quad (L_\eta^\top - \lambda K_\eta^\top)y_{2,4} = 0.$$

In the same way as above, one sees that

$$y_{2,2} = 0 \quad \text{and} \quad y_{2,4} = 0$$

and, hence,

$$Wx_2 = \begin{pmatrix} y_{2,1} \\ 0 \\ K_\varepsilon y_{2,3} \\ 0 \end{pmatrix}.$$

The third equation in (6.3) and, in fact, all the remaining equations in (6.3) are of the same form as the second one. Hence, proceeding in this way, one obtains for $i = 1, \dots, n$:

$$Wx_i = \begin{pmatrix} y_{i,1} \\ 0 \\ K_\varepsilon y_{i,3} \\ 0 \end{pmatrix} \quad \text{and} \quad (A_0 - \lambda I_{n_0})y_{i,1} = y_{i-1,1},,$$

where $y_{0,1} := 0$. Therefore, one has $y_{i,1} \in \mathfrak{R}_\lambda(A_0)$ for $i = 1, \dots, n$ and this shows

$$\mathfrak{R}_\lambda(\mathcal{A}) \subseteq W^{-1}\left(\mathfrak{R}_\lambda(A_0) \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\varepsilon| - n_\varepsilon} \times \{0\}^{|\eta|}\right). \tag{6.5}$$

The reverse inclusion remains to be shown:

$$W^{-1}\left(\mathfrak{R}_\lambda(A_0) \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\varepsilon| - n_\varepsilon} \times \{0\}^{|\eta|}\right) \subset \mathfrak{R}_\lambda(\mathcal{A}). \tag{6.6}$$

If λ is not an eigenvalue of the matrix A_0, then $\mathfrak{R}_\lambda(A_0) = \{0\}$ and (i) together with Proposition 4.2 imply

$$W^{-1}\left(\{0\}^{n_0} \times \{0\}^{|\alpha|} \times \mathbb{C}^{|\varepsilon| - n_\varepsilon} \times \{0\}^{|\eta|}\right) = \mathfrak{R}_c(\mathcal{A}) \subset \mathfrak{R}_\lambda(\mathcal{A}).$$

so that (6.6) holds in this case. Next assume that $\lambda \in \sigma_p(A_0)$ and it clearly suffices to show

$$W^{-1}\left(\mathfrak{R}_\lambda(A_0) \times \{0\}^{|\alpha|} \times \{0\}^{|\varepsilon| - n_\varepsilon} \times \{0\}^{|\eta|}\right) \subset \mathfrak{R}_\lambda(\mathcal{A}). \tag{6.7}$$

The matrix A_0 is in Jordan canonical form and it is no restriction to assume that the first Jordan block in A_0 is of the form $J_n(\lambda) = \lambda I_n + N_n$ for some $n \in \mathbb{N}$. Denote by e_i, $i = 1, \dots, n_0$, the standard unit vectors in $\mathbb{C}^{n_0}$. Then, for $i = 1, \dots, n$,

$$A_0 e_i = \begin{pmatrix} J_n(\lambda) & 0 \\ 0 & * \end{pmatrix} e_i = \begin{cases} \lambda e_i, & \text{if } i = n, \\ e_{i+1} + \lambda e_i, & \text{if } i = 1, \dots, n-1, \end{cases} \tag{6.8}$$

where the symbol $*$ stands for possibly more Jordan blocks. Set

$$x_i := W^{-1} \begin{pmatrix} e_i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad i = 1, \dots, n.$$

Together with (6.8) this implies that for $i = 1, \dots, n-1$

$$\begin{aligned} \begin{pmatrix} x_i \\ x_{i+1} + \lambda x_i \end{pmatrix} &= \begin{pmatrix} W^{-1} \begin{pmatrix} I_{n_0} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\varepsilon & 0 \\ 0 & 0 & 0 & K_\eta^\top \end{pmatrix} \\ W^{-1} \begin{pmatrix} A_0 & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\varepsilon & 0 \\ 0 & 0 & 0 & L_\eta^\top \end{pmatrix} \end{pmatrix} \begin{pmatrix} e_i \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} F \\ G \end{pmatrix} T \begin{pmatrix} e_i \\ 0 \\ 0 \\ 0 \end{pmatrix} \in \operatorname{ran} \begin{pmatrix} F \\ G \end{pmatrix}, \end{aligned}$$

which shows with (6.1) that $(x_i, x_{i+1} + \lambda x_i) \in \mathcal{A}$. A similar equation with (6.8) for $i = n$ shows that $(x_n, \lambda x_n) \in \mathcal{A}$, and therefore

$$(x_1, x_2 + \lambda x_1), (x_2, x_3 + \lambda x_3), \dots, (x_{n-1}, x_n + \lambda x_{n-1}), (x_n, \lambda x_n) \in \mathcal{A}.$$

The same arguments work for the remaining Jordan blocks of A_0 and (6.7) and, hence, (6.6) is proved. The inclusions (6.6) and (6.5) confirm (iii).

Statement (iv) follows from the representations of the singular chain space in (i), the root space $\mathfrak{R}_\infty(\mathcal{A})$ in (ii), the root space $\mathfrak{R}_\lambda(\mathcal{A})$ in (iii), and Definition 5.1. □

References

[1] R. Arens, Operational calculus of linear relations, *Pacific J. Math.* 11 (1961), 9–23.

[2] J. Behrndt, S. Hassi, and H.S.V. de Snoo, *Boundary value problems, Weyl functions, and differential operators*, Monographs in Mathematics, Vol. 108, Birkhäuser, 2020.

[3] T. Berger, H.S.V. de Snoo, C. Trunk, and H. Winkler, A Jordan-like decomposition for linear relations in finite-dimensional spaces, in preparation.

[4] T. Berger and S. Trenn, The quasi-Kronecker form for matrix pencils, *SIAM J. Matrix Anal. Appl.* 33 (2012), 336–368.

[5] T. Berger and S. Trenn, Addition to "The quasi-Kronecker form for matrix pencils", *SIAM J. Matrix Anal. Appl.* 34 (2013), 94–101.

[6] T. Berger, C. Trunk, and H. Winkler, Linear relations and the Kronecker canonical form, *Lin. Alg. Appl.* 488 (2016), 13–44.

[7] R.W. Cross, *Multivalued linear operators*, Marcel Dekker, New York, 1998.

[8] F. Gantmacher, *The theory of matrices. Vol. I & II*, Chelsea, New York, 1959.

[9] T.T. Ha and J.A. Gibson, A note on the determinant of a functional confluent Vandermonde matrix and controllability, *Lin. Alg. Appl.* 30 (1980), 69–75.

[10] L. Kronecker, Algebraische Reduction der Schaaren bilinearer Formen, *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin* 1890, 1225–1237.

[11] P. Kunkel and V. Mehrmann, *Differential-algebraic equations. Analysis and numerical solution*, EMS Publishing House, Zürich 2006.

[12] A. Sandovici, H.S.V. de Snoo, and H. Winkler, The structure of linear relations in Euclidean spaces, *Lin. Alg. Appl.* 397 (2005), 141–169.

[13] A. Sandovici, H.S.V. de Snoo, and H. Winkler, Ascent, descent, nullity, defect, and related notions for linear relations in linear spaces, *Lin. Alg. Appl.* 423 (2007), 456–497.

Institut für Mathematik, Universität Paderborn, Warburger Strasse 100, 33098 Paderborn, Germany

Email address: thomas.berger@math.upb.de

Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, P.O. Box 407, 9700 AK Groningen, Nederland

Email address: h.s.v.de.snoo@rug.nl

Institut für Mathematik, Technische Universität Ilmenau, Weimarer Strasse 25, 98693 Ilmenau, Germany

Email address: carsten.trunk@tu-ilmenau.de

Institut für Mathematik, Technische Universität Ilmenau, Weimarer Strasse 25, 98693 Ilmenau, Germany

Email address: henrik.winkler@tu-ilmenau.de