An improved input-constrained funnel controller for nonlinear systems

Thomas Berger

***Abstract*— We present an improvement of a recent funnel controller design for uncertain nonlinear multi-input, multi-output systems modeled by higher order functional differential equations in the presence of input constraints. The objective is to guarantee the evolution of the tracking error within a performance funnel with prescribed desired shape for the case of inactive saturation. Compared to its precursor, controller complexity is significantly reduced, much fewer design parameters are involved and simulations exhibit a superior performance.**

***Index Terms*— adaptive control, functional differential equations, funnel control, input constraints, nonlinear systems.**

I. Introduction

The purpose of this technical note is to present an improvement of a controller design recently published by the author in this journal [1]. For an extensive literature survey on the topic we refer to the aforementioned work and do not repeat it here. In the following we briefly recall the problem statement.

We study funnel control for the class of nonlinear multi-input multi-output systems described by the r-th order functional differential equation

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t), u(t)\big), \\ y|_{[-h,0]} &= y^0 \in C^{r-1}([-h,0],\mathbb{R}^m), \end{aligned} \tag{1}$$

with continuous function f and operator T (with properties to be specified later) as well as bounded disturbance d and initial trajectory y^0 – all of these parameters are unknown and not available for controller design. Furthermore, the system is subject to input constraints

$$u(t) = \mathrm{sat}(v(t)) \tag{2}$$

with *known* saturation function sat and control function v provided by the to-be-designed controller.

The control objective is to ideally achieve a prescribed performance of the tracking error, that is $\|y(t) - y_{\mathrm{ref}}(t)\| < \psi(t)$ for some given reference signal y_{ref} and funnel function ψ. However, since we consider the input constraints to be *hard constraints*, a conflict of objectives arises: As explained in [1], it is not possible to simultaneously satisfy the input and output constraints for any given bounded reference signal. Instead, we consider the aforementioned output constraints to be *soft constraints*, i.e., they can be relaxed when needed in order to meet the input constraints. To achieve this, in [1] a mechanism to dynamically adjust the funnel function ψ has been presented. The idea is to widen the performance funnel described by ψ whenever the input saturation is active, so that no constraints are violated. When the saturation becomes inactive the function ψ reverts to its prescribed shape exponentially fast. Recently, in a series of papers [2], [3], [4], Trakas and Bechlioulis have developed a similar approach, however for different system classes. In particular, the most recent result in [4] requires the system to be input-to-state stable, which however is often not satisfied in practical applications, see e.g. the simple example in Section IV. While it is further shown in [4] that the input-to-state stability assumption can be relaxed to a bounded-input-bounded-state assumption on the internal dynamics (commonly required in funnel control, see e.g. [5]), boundedness of closed-loop signals can then only be guaranteed for those evolving in a certain (unknown) compact set. In the present work we focus on a different system class and do not require either of both assumptions.

While in [1] a chain of r interconnected funnel functions was used, the improved control design that we present here uses only one funnel function. Therefore, the number of dynamic equations involved in the controller design (and hence its complexity) is significantly reduced. Furthermore, in contrast to [1], much fewer controller design parameters are comprised and we are able to prove that the control function v is always bounded for bounded reference signals.

A. Nomenclature

In the following let $\mathbb{N}$ denote the natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{R}_{\geq 0} = [0,\infty)$. By $\|x\|$ we denote the Euclidean norm of $x \in \mathbb{R}^n$. For some interval $I \subseteq \mathbb{R}$, some $V \subseteq \mathbb{R}^m$ and $k \in \mathbb{N}$, $L^\infty(I,\mathbb{R}^n)$ $\big(L^\infty_{\mathrm{loc}}(I,\mathbb{R}^n)\big)$ is the Lebesgue space of measurable, (locally) essentially bounded functions $f\colon I \to \mathbb{R}^n$, $W^{k,\infty}(I,\mathbb{R}^n)$ is the Sobolev space of all functions $f : I \to \mathbb{R}^n$ with k-th order weak derivative $f^{(k)}$ and $f, f^{(1)}, \dots, f^{(k)} \in L^\infty(I,\mathbb{R}^n)$, and $C^k(V,\mathbb{R}^n)$ is the set of k-times continuously differentiable functions $f : V \to \mathbb{R}^n$, with $C(V,\mathbb{R}^n) := C^0(V,\mathbb{R}^n)$.

B. System Class

We recall the necessary definitions from [1].

Definition 1.1: For $n, q \in \mathbb{N}$ and $h \geq 0$ the set $\mathbb{T}^{n,q}_h$ denotes the class of operators $T\colon C([-h,\infty),\mathbb{R}^n) \to L^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0},\mathbb{R}^q)$ with the following properties.

(P1) T is causal, i.e., for all ζ, $\xi \in C([-h,\infty),\mathbb{R}^n)$ and all $t \geq 0$,

$$\zeta|_{[-h,t]} = \xi|_{[-h,t]} \quad \Longrightarrow \quad T(\zeta)|_{[0,t]} = T(\xi)|_{[0,t]}.$$

(P2) T is locally Lipschitz, i.e., for each $t \geq 0$ and all $\xi \in C([-h,t],\mathbb{R}^n)$, there exist positive constants $c_0, \delta, \tau > 0$ such that, for all $\zeta_1, \zeta_2 \in C([-h,\infty),\mathbb{R}^n)$ with $\zeta_i|_{[-h,t]} = \xi$ and $\|\zeta_i(s) - \xi(t)\| < \delta$ for all $s \in [t, t+\tau]$ and $i = 1, 2$, we have

$$\begin{aligned} &\operatorname*{ess\,sup}_{s\in[t,t+\tau]} \|T(\zeta_1)(s) - T(\zeta_2)(s)\| \\ &\qquad\qquad \leq c_0 \sup_{s\in[t,t+\tau]} \|\zeta_1(s) - \zeta_2(s)\|. \end{aligned}$$

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(P3) T locally maps bounded functions to bounded functions, i.e., for all $\tau > 0$ and all $c_1 > 0$, there exists $c_2 > 0$ such that, for all $\zeta \in C([-h,\tau],\mathbb{R}^n)$,

$$\sup_{t\in[-h,\tau]} \|\zeta(t)\| \le c_1 \quad\Longrightarrow\quad \operatorname*{ess\,sup}_{t\in[0,\tau]} \|T(\zeta)(t)\| \le c_2.$$

Next we recall the sector bound property of f and T.

(P4) For all $y^0 \in C^{r-1}([-h,0],\mathbb{R}^m)$ there exist $M_1,\dots,M_{r+1} \in C(\mathbb{R}_{\ge 0}\times\mathbb{R}^p\times\mathbb{R}^m,\mathbb{R}_{\ge 0})$ which are bounded in t, such that for all $t \ge 0$, all $(d,v)\in\mathbb{R}^p\times\mathbb{R}^m$ and all $\zeta_1,\dots,\zeta_r \in C([-h,t],\mathbb{R}^m)$ with $\zeta_i|_{[-h,0]} = (y^0)^{(i-1)}$ for $i=1,\dots,r$ we have:

$$\|f(d,T(\zeta_1,\dots,\zeta_r)(t),v)\| \le M_1(t,d,v) \\ +M_2(t,d,v)\|\zeta_1|_{[-h,t]}\|_\infty + \ldots + M_{r+1}(t,d,v)\|\zeta_r|_{[-h,t]}\|_\infty$$

Note that the functions M_i in (P4) depend on the initial history y^0 in (1). Furthermore, compared to [1] we additionally assume that each M_i is bounded in t, which is required to show that the control signal v in (2), generated by the controller, is bounded.

Next we recall the system class from [1]. We stress that the high-gain property of system (1) required in earlier approaches, see e.g. [5], is not needed here; this is also different from [4]. It is not even required that f depends on u; however, in this case it is possible that the tracking error grows unbounded.

Definition 1.2: For $m,r \in \mathbb{N}$ we say that system (1) belongs to the system class $\mathcal{N}^{m,r}$, written $(d,f,T)\in\mathcal{N}^{m,r}$, if $d\in L^\infty(\mathbb{R}_{\ge 0},\mathbb{R}^p)$, $f\in C(\mathbb{R}^p\times\mathbb{R}^q\times\mathbb{R}^m,\mathbb{R}^m)$, $T\in\mathbb{T}_h^{rm,q}$ for some $p,q\in\mathbb{N}$, $h\ge 0$ and (f,T) satisfy property (P4).

As shown in [1, Rem. 1.3] the class $\mathcal{N}^{m,r}$ contains a large class of systems in well-known state-space form. The saturation function is required to satisfy the following property.

(P5) $\mathrm{sat}\in C(\mathbb{R}^m,\mathbb{R}^m)$ is bounded and there exists $\theta > 0$ such that for all $v\in\mathbb{R}^m$ with $\|v\|\le\theta$ we have $\mathrm{sat}(v) = v$.

We stress that the input saturation function sat must be known to the controller and it can be viewed as a design parameter, chosen according to the specific requirements of the application at hand. The above property (P5) allows for a large variety of possible saturations, apart from the standard saturation $\mathrm{sat}_i(v) = v_i$ for $|v_i|\le M$ and $\mathrm{sat}_i(v) = \mathrm{sgn}(v_i)M$ for $|v_i| > M$ for all $i=1,\dots,m$.

C. Control objective

The objective is to design a dynamic output derivative feedback strategy such that, ideally, for any reference signal $y_{\mathrm{ref}}\in W^{r,\infty}(\mathbb{R}_{\ge 0},\mathbb{R}^m)$ the tracking error $e = y - y_{\mathrm{ref}}$ evolves within a performance funnel

$$\mathcal{F}_\psi := \{\,(t,e)\in\mathbb{R}_{\ge 0}\times\mathbb{R}^m \mid \|e\| < \psi(t)\,\},$$

see Fig. 1, which has a desired shape of the form $\psi_{\mathrm{des}}(t) = ae^{-bt}+c$ whenever the saturation in (2) is not active, i.e., $\mathrm{sat}(v(t)) = v(t)$, and the actual funnel boundary $\psi(t)$ is allowed to deviate from this shape and become larger when the saturation is active. The specific value of $\psi(t)$ should be determined by a dynamic part of the control law.

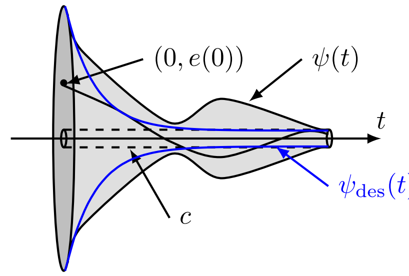

Fig. 1: Error evolution in a funnel $\mathcal{F}_\psi$ with boundary $\psi(t)$ and desired shape $\psi_{\mathrm{des}}(t)$.

In contrast to classical funnel control [5], [6], [7], the funnel boundary ψ is not fully prescribed, but widened when necessary in order to meet the input constraints. Nevertheless, the desired "asymptotic shape" $\psi_{\mathrm{des}}(t) = ae^{-bt}+c$ under inactive saturation can be prescribed by choice of the parameters a,b,c.

D. Organization of the present paper

The paper is structured as follows. In Section II, we introduce an improved version of the funnel controller from [1] for systems (1) under input constraints (2). Feasibility of the control is proved in the main result in Section III: existence of a global solution for systems of class $\mathcal{N}^{m,r}$ is shown in Theorem 3.1. The performance of the improved controller is compared to that from [1] by an illustrative example in Section IV. The paper concludes with Section V.

II. Funnel control structure

We introduce the following improved input-constrained funnel controller for systems (1), (2).

$$\boxed{\begin{aligned} e_1(t) &= e(t) = y(t) - y_{\mathrm{ref}}(t),\\ e_{i+1}(t) &= \dot e_i(t) + k_i e_i(t), \quad i=1,\dots,r-1,\\ \dot\psi(t) &= -\alpha\psi(t) + \beta + \psi(t)\frac{\kappa(v(t))}{\|e_r(t)\|}, \quad \psi(0)=\psi^0,\\ \kappa(v(t)) &= \|v(t) - \mathrm{sat}(v(t))\|,\\ k(t) &= \left(1 - \frac{\|e_r(t)\|^2}{\psi(t)^2}\right)^{-1},\\ v(t) &= N\big(k(t)\big)e_r(t)\end{aligned}} \tag{3}$$

with the controller design parameters

$$\boxed{\begin{aligned}&\alpha,\beta > 0,\ k_1,\dots,k_{r-1} > \alpha,\ \psi^0 > \tfrac{\beta}{\alpha},\\ &N\in C(\mathbb{R}_{\ge 0},\mathbb{R})\ \text{ a surjection.}\end{aligned}} \tag{4}$$

The controller (3) is similar to that presented in [1] with some significant differences: a) the gains $k_1,\dots,k_{r-1}$ are not dynamic, but static, b) instead of a chain of r dynamically generated funnel functions, only one function is used, and c) much fewer controller design parameters are involved. Clearly, all three improvements a)–c) decrease the complexity of the controller design. On the other hand, it is not directly clear that the simplifications preserve its feasibility and effectiveness – this requires a proof which we provide in Theorem 3.1.

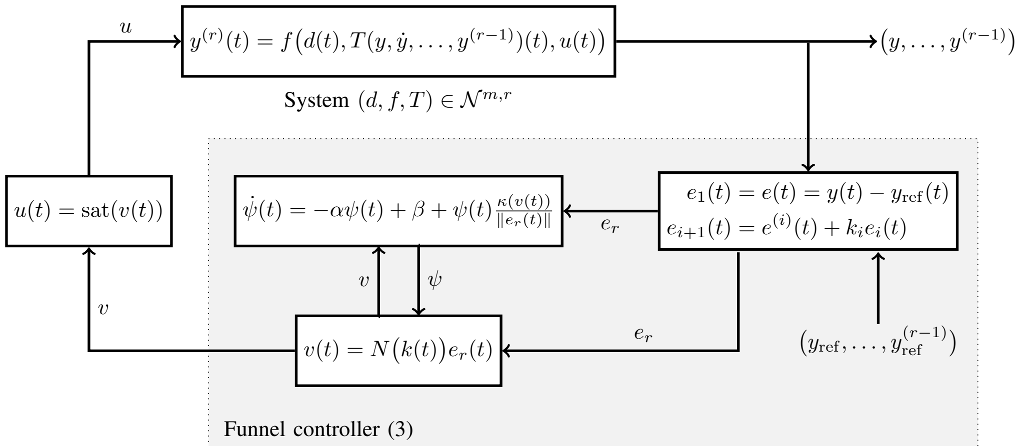

Fig. 2: Construction of the funnel controller (3) and its internal feedback loop.

Note that the controller (3) exhibits an internal feedback loop: the funnel function ψ is widened based on the term $\kappa(v)$, which becomes positive when the saturation is active. The gain function k is then calculated on the basis of ψ and determines the value of the control signal v. Therefore, a feedback structure arises (depicted in Fig. 2), for which we seek to prove existence of global solutions.

The surjective function N in (4) accommodates for possibly unknown control directions. A typical choice for N would be $N(s) = s\sin s$. For more details see also [5, Rem. 1.8].

Like its precursor presented in [1], the novel control design (3) is feasible under arbitrary input constraints (2). If the saturation is inactive (i.e., $\kappa(v(t)) = 0$), then $\psi(t) = \left(\psi^0 - \frac{\beta}{\alpha}\right)e^{-\alpha t} + \frac{\beta}{\alpha}$; if the saturation is active (i.e., $\kappa(v(t)) \neq 0$), then $\kappa(v(t))$ provides a positive contribution to $\dot{\psi}$ and hence widens the funnel. After a period of active saturation, ψ reverts to its prescribed shape exponentially fast. However, one difference to [1] is that for (3) it is not directly clear in which performance funnel the tracking error e evolves in, if any. While $\|e_r(t)\| < \psi(t)$ is enforced by the control design, a boundary for $\|e(t)\|$ is not obvious. This is clarified by the following lemma.

Lemma 2.1: Let $e \in C^{r-1}([0,\omega), \mathbb{R}^m)$, $\omega \in (0,\infty]$, and consider the signals $e_1(t) = e(t)$ and $e_{i+1}(t) = \dot{e}_i(t) + k_i e_i(t)$, $k_i > 0$, for $i = 1, \ldots, r-1$. Further let $\psi \in C^1([0,\omega), \mathbb{R})$ be such that $\psi(t) > 0$ and $\dot{\psi}(t) \geq -\alpha\psi(t)$ for all $t \in [0,\omega)$, where $0 \leq \alpha < \min_{i=1,\ldots,r-1} k_i$. If $\|e_r(t)\| < \psi(t)$ for all $t \in [0,\omega)$, then we have for $i = 1, \ldots, r-1$ and all $t \in [0,\omega)$:

$$\|e_i(t)\| < \max\left\{ \left(\prod_{j=i}^{r-1} \frac{1}{k_j - \alpha} \right), \max_{j=i,\ldots,r-1} \left(\prod_{p=r-j+i}^{r-1} \frac{1}{k_p - \alpha} \right) \frac{\|e_j(0)\|}{\psi(0)} \right\} \psi(t). \tag{5}$$

Proof: Fix $i = 1, \ldots, r-1$ and set $\ell_i(t) := \max\left\{ \frac{\|e_i(0)\|}{\psi(0)}, \frac{\|e_{i+1}(t)\|}{(k_i-\alpha)\psi(t)} \right\}$ for $t \in [0,\omega)$, which is well defined since $k_i > \alpha$. We first show that $\|e_i(t)\| \leq \ell_i(t)\psi(t)$ for all $t \in [0,\omega)$. We may calculate that

$$\begin{aligned} \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{\|e_i(t)\|^2}{\psi(t)^2} &= -\tfrac{\dot{\psi}(t)}{\psi(t)} \tfrac{\|e_i(t)\|^2}{\psi(t)^2} + \tfrac{1}{\psi(t)^2} e_i(t)^\top \big(e_{i+1}(t) - k_i e_i(t)\big) \\ &\leq (\alpha - k_i)\tfrac{\|e_i(t)\|^2}{\psi(t)^2} + \tfrac{\|e_{i+1}(t)\|}{\psi(t)} \tfrac{\|e_i(t)\|}{\psi(t)} \\ &= \left((\alpha - k_i)\tfrac{\|e_i(t)\|}{\psi(t)} + \tfrac{\|e_{i+1}(t)\|}{\psi(t)} \right) \tfrac{\|e_i(t)\|}{\psi(t)}. \end{aligned}$$

for all $t \in [0,\omega)$. Seeking a contradiction, assume that there exists $t_1 \in [0,\omega)$ with $\|e_i(t_1)\|/\psi(t_1) > \ell_i(t_1)$. Set $t_0 := \sup\{\, t \in [0,t_1) \mid \|e_i(t)\|/\psi(t) = \ell_i(t) \,\}$, which is well defined by $\frac{\|e_i(0)\|}{\psi(0)} \leq \ell_i(0)$. Then the above estimate implies $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\frac{\|e_i(t)\|^2}{\psi(t)^2} \leq 0$ for all $t \in [t_0, t_1]$, whence

$$\ell_i(t_0) = \|e_i(t_0)\|/\psi(t_0) \geq \|e_i(t_1)\|/\psi(t_1) > \ell_i(t_1),$$

a contradiction. Now observe that

$$\ell_i(t) \leq \max\left\{ \frac{\|e_i(0)\|}{\psi(0)}, \frac{\ell_{i+1}(t)}{(k_i - \alpha)} \right\}$$

for all $t \in [0,\omega)$ and $i = 1, \ldots, r-1$, where $\ell_r(t) := 1$ and we used $\|e_r(t)\| < \psi(t)$. Then the assertion follows from a straightforward induction over $i = r-1, \ldots, 1$. ∎

III. Funnel control – main result

In this section we show that the application of the funnel controller (3) to a system (1) under input constraints (2) leads to a closed-loop initial-value problem which has a global solution. By a solution of (1), (2), (3) on $[-h,\omega)$ we mean a pair of functions $(y,\psi) \in C^{r-1}([-h,\omega), \mathbb{R}^m) \times C([-h,\omega), \mathbb{R})$ with $\omega \in (0,\infty]$, which satisfies $y|_{[-h,0]} = y^0$, $\psi(0) = \psi^0$ and $(y^{(r-1)}, \psi)|_{[0,\omega)}$ is locally absolutely continuous and satisfies the differential equations in (1) and (3) with u defined by (2), (3) for almost all $t \in [0,\omega)$; (y,ψ) is called maximal, if it has no right extension that is also a solution.

Next we present the main result of the present paper.

Theorem 3.1: Consider a system (1) with $(d,f,T) \in \mathcal{N}^{m,r}$ for $m, r \in \mathbb{N}$, under input constraints (2) with saturation function sat that satisfies (P5). Let $y^0 \in C^{r-1}([-h,0], \mathbb{R}^m)$ be the initial trajectory, $y_{\mathrm{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ the reference signal and choose funnel control design parameters as in (4). Set $e = y - y_{\mathrm{ref}}$ and assume that the instantaneous values

$e(t), \dot e(t), \ldots, e^{(r-1)}(t)$ are available for feedback and satisfy $\|e_r(0)\| < \psi^0$ for e_r defined in (3). Then the funnel controller (3) applied to (1), (2) yields an initial-value problem which has a solution, every solution can be maximally extended and every maximal solution $(y,\psi) : [-h,\omega) \to \mathbb{R}^{m+1}$, $\omega \in (0,\infty]$, has the following properties:

(i) global existence: $\omega = \infty$;

(ii) the functions $e_1, \ldots, e_{r-1}$ satisfy (5) and e_r satisfies

$$\exists\, \varepsilon \in (0,1)\ \forall\, t \geq 0 :\ \|e_r(t)\| \leq \varepsilon \psi(t),$$

in particular, k and v are bounded;

(iii) if the saturation is not active on some interval $[t_0, t_1) \subseteq \mathbb{R}_{\geq 0}$ with $t_1 \in (t_0, \infty]$, i.e., $v(t) = \mathrm{sat}(v(t))$ for all $t \in [t_0, t_1)$, then the performance funnel ψ reverts to its prescribed shape exponentially fast:

$$\forall\, t \in [t_0, t_1) :\ \psi(t) \leq \frac{\beta}{\alpha} + \mu(t_0) e^{-\alpha(t-t_0)},$$

where $\mu(t_0) := \psi(t_0) - \frac{\beta}{\alpha}$.

Proof: The proof consists of several steps.

Step 1: We recast the closed-loop system in the form of an initial-value problem to which a well-known existence theory applies. First define the polynomial $p(s) = (s+k_{r-1}) \cdots (s+k_1)$ and observe that $p(s) = s^{r-1} + \sum_{i=1}^{r-1} \mu_i s^{i-1}$ for some $\mu_1, \ldots, \mu_{r-1} > 0$; set $\mu_r := 1$. Define the non-empty and relatively open set

$$\mathcal{D} := \left\{ (t, \xi_1, \ldots, \xi_r, \zeta) \in \mathbb{R}_{\geq 0} \times (\mathbb{R}^m)^r \times \mathbb{R} \;\middle|\; \left\| \sum\nolimits_{i=1}^{r} \mu_i \xi_i - p(\tfrac{\mathrm{d}}{\mathrm{d}t}) y_{\mathrm{ref}}(t) \right\| < \zeta,\ \zeta > \tfrac{\beta}{\alpha} \right\},$$

and the functions

$$E : \mathcal{D} \to \mathbb{R}^m,\ (t, \xi_1, \ldots, \xi_r, \zeta) \mapsto \sum_{i=1}^{r} \mu_i \xi_i - p(\tfrac{\mathrm{d}}{\mathrm{d}t}) y_{\mathrm{ref}}(t),$$

$$V : \mathcal{D} \to \mathbb{R}^m,\ (t, \xi_1, \ldots, \xi_r, \zeta) \mapsto N\left(\frac{1}{1 - \frac{\|E(t,\xi_1,\ldots,\xi_r,\zeta)\|^2}{\zeta^2}} \right) E(t, \xi_1, \ldots, \xi_r, \zeta).$$

Further define

$$F \colon \mathcal{D} \times \mathbb{R}^q \to \mathbb{R}^{rm+1},\ (t, \xi_1, \ldots, \xi_r, \zeta, \eta) \mapsto \begin{pmatrix} \xi_2 \\ \vdots \\ \xi_r \\ f\big(d(t), \eta, \mathrm{sat}(V(t, \xi_1, \ldots, \xi_r, \zeta))\big) \\ -\alpha\zeta + \beta + \zeta \frac{\|V(t,\xi_1,\ldots,\xi_r,\zeta) - \mathrm{sat}(V(t,\xi_1,\ldots,\xi_r,\zeta))\|}{\|E(t,\xi_1,\ldots,\xi_r,\zeta)\|} \end{pmatrix}.$$

Note that the function F, and in particular its last component, is well-defined on $\mathcal{D} \times \mathbb{R}^q$: Since N is continuous and $\zeta > \frac{\beta}{\alpha}$, there exists $\delta > 0$ such that for all $(t, \xi_1, \ldots, \xi_r, \zeta) \in \mathcal{D}$ with $\|E(t, \xi_1, \ldots, \xi_r, \zeta)\| < \delta$ we have that $\|V(t, \xi_1, \ldots, \xi_r, \zeta)\| < \theta$ for θ as in (P5), and hence $\|V(t, \xi_1, \ldots, \xi_r, \zeta) - \mathrm{sat}(V(t, \xi_1, \ldots, \xi_r, \zeta))\| = 0$.

Writing

$$x(t) = \big(y(t)^\top, \ldots, y^{(r-1)}(t)^\top, \psi(t)\big)$$

we see that the closed-loop initial-value problem (1), (2), (3) may now be formulated as

$$\begin{aligned} \dot x(t) &= F\big(t, x(t), T(x)(t)\big), \\ x|_{[-h,0]} &= x^0 \in C([-h,0], \mathbb{R}^{rm+1}), \end{aligned} \tag{6}$$

where, for $t \in [-h, 0]$,

$$x^0(t) := \big(y^0(t)^\top, \ldots, (y^0)^{(r-1)}(t)^\top, \psi^0\big)^\top.$$

The function F is measurable in t, continuous in $(\xi_1, \ldots, \xi_r, \zeta, \eta)$ and locally essentially bounded. By the assumptions $\|e_r(0)\| < \psi^0$ and $\psi^0 > \frac{\beta}{\alpha}$ we see that $(0, x^0(0)) \in \mathcal{D}$. Therefore, an application of a variant of [8, Thm. B.1][1] yields the existence of a solution of (6) and every solution can be extended to a maximal solution. Furthermore, any maximal solution $x : [-h, \omega) \to \mathbb{R}^n$, $\omega \in (0, \infty]$, of (6) has the property that its graph

$$\mathcal{G} := \{\, (t, x(t)) \mid t \in [0, \omega) \,\} \subset \mathcal{D}$$

has a closure which is not a compact subset of $\mathcal{D}$.

Step 2: In this step we record some observations for later use. For $t \in [0, \omega)$, define $e_1(t) := e(t) = y(t) - y_{\mathrm{ref}}(t)$ for $y(t) = (x_1(t), \ldots, x_m(t))^\top$, and $e_{i+1}(t) := \dot e_i(t) + k_i e_i(t)$ for $i = 1, \ldots, r-1$ as well as $\psi(t) := x_{rm+1}(t)$. Then $e_r(t) = E(t, x(t))$ and we have $\|e_r(t)\| < \psi(t)$, by which $k(t) := \left(1 - \frac{\|e_r(t)\|^2}{\psi(t)^2}\right)^{-1}$ is well defined. Thus we arrive at the quantities in the control law (3); in particular $V(t, x(t)) = N(k(t)) e_r(t) = v(t)$. Furthermore, invoking $p(s)$ from Step 1, it follows that for almost all $t \in [0, \omega)$ we have

$$\dot e_r(t) = e^{(r)}(t) + \sum_{i=1}^{r-1} \mu_i e^{(i)}(t). \tag{7}$$

Furthermore, since $\psi(t) > 0$ it follows $\dot\psi(t) \geq -\alpha\psi(t) + \beta$ and hence $\psi(t) \geq \mu(0) e^{-\alpha t} + \frac{\beta}{\alpha}$ for all $t \in [0, \omega)$, where $\mu(\cdot)$ is defined in statement (iii).

Step 3: Fix $i \in \{0, \ldots, r-1\}$. We show that there exists $\kappa_i > 0$ such that $\|y^{(i)}(t)\| \leq \kappa_i \psi(t)$ for all $t \in [0, \omega)$. Observe that a straightforward induction utilizing $e_{i+1}(t) = \dot e_i(t) + k_i e_i(t)$ and $e_1(t) = e(t)$ gives that

$$e^{(i)}(t) = e_{i+1}(t) - \sum_{j=1}^{i} k_j e_j^{(i-j)}(t) = e_{i+1}(t) + \sum_{j=1}^{i} c_{i,j} e_j(t)$$

for some $c_{i,j} \in \mathbb{R}$, $i = 1, \ldots, r-1$, $j = 1, \ldots, i$. Invoking $\|e_r(t)\| < \psi(t)$ and Lemma 2.1 (notice that ψ satisfies the assumptions of the lemma by the observations in Step 2), it follows that (5) holds, i.e., there exist $\sigma_1, \ldots, \sigma_{r-1}$ such that, with $\sigma_r := 1$, we have $\|e_i(t)\| \leq \sigma_i \psi(t)$ for all $t \in [0, \omega)$ and $i = 1, \ldots, r$. Then

$$\begin{aligned} \|y^{(i)}(t)\| &\leq \|e_{i+1}(t)\| + \sum_{j=1}^{i} |c_{i,j}| \|e_j(t)\| + \|y_{\mathrm{ref}}^{(i)}(t)\| \\ &\leq \left(\sigma_{i+1} + \sum_{j=1}^{i} |c_{i,j}| \sigma_j + \frac{\alpha}{\beta} \|y_{\mathrm{ref}}^{(i)}\|_\infty \right) \psi(t) \end{aligned}$$

[1] Although the property (P3) of the operator T is weaker than required in [8], this “local” property suffices for the proof.

for all $t \in [0,\omega)$, where we used $1 \leq \frac{\alpha}{\beta}\psi(t)$ by Step 2. This proves the assertion for $\kappa_i := \left(\sigma_{i+1} + \sum_{j=1}^{i} |c_{i,j}|\sigma_j + \frac{\alpha}{\beta}\|y_{\text{ref}}^{(i)}\|_\infty\right)$.

Step 4: We show that there exists $C > 0$ such that for all $t \in [0,\omega)$ we have

$$\|f\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t), \text{sat}(v(t))\big)\| \leq C\psi(t).$$

By the sector bound property (P4) and Step 3 we have that

$$\begin{aligned}
&\|f\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t), \text{sat}(v(t))\big)\| \\
&\leq M_1\big(t, d(t), \text{sat}(v(t))\big) \\
&\quad + \sum_{i=1}^{r} M_{i+1}\big(t, d(t), \text{sat}(v(t))\big)\|y^{(i-1)}|_{[-h,t]}\|_\infty \\
&\leq M_1\big(t, d(t), \text{sat}(v(t))\big)\frac{\alpha}{\beta}\psi(t) \\
&\quad + \sum_{i=1}^{r} M_{i+1}\big(t, d(t), \text{sat}(v(t))\big)\big(\|(y^0)^{(i-1)}\|_\infty + \|y^{(i-1)}\|_\infty\big).
\end{aligned}$$

for all $t \in [0,\omega)$. Since $M_1,\ldots,M_{r+1}$ are continuous and bounded in t and d, $\text{sat}(v)$ are bounded, there exist $\overline{M}_1,\ldots,\overline{M}_{r+1} > 0$ such that $\|M_i\big(t, d(t), \text{sat}(v(t))\big)\| \leq \overline{M}_i$ for all $t \in [0,\omega)$ and all $i = 1,\ldots,r+1$. Then, by Step 3, the assertion holds for

$$C := \overline{M}_1\frac{\alpha}{\beta} + \sum_{i=1}^{r}\overline{M}_{i+1}\left(\frac{\alpha}{\beta}\|(y^0)^{(i-1)}\|_\infty + \kappa_{i-1}\right).$$

Step 5: We show that $k \in L^\infty([0,\omega),\mathbb{R})$. Note that, invoking (3) we may estimate

$$\forall\, t \in [0,\omega):\ \kappa(v(t)) \geq |N(k(t))| \cdot \|e_r(t)\| - M, \qquad (8)$$

where $M > 0$ is some upper bound of sat, i.e., $\|\text{sat}(v)\| \leq M$ for all $v \in \mathbb{R}^m$. Now set

$$\hat{C} := C + \sum_{i=1}^{r}\mu_i\kappa_i + \frac{\alpha}{\beta}\left(\|y_{\text{ref}}^{(r)}\|_\infty + \sum_{i=1}^{r-1}\mu_i\|y_{\text{ref}}^{(i)}\|_\infty\right)$$

and choose

$$\delta > \alpha + \hat{C} + \frac{\alpha}{\beta}M \qquad (9)$$

and $\varepsilon \in (0,1)$ so that, invoking $\|e_r(0)\| < \psi^0$,

$$\varepsilon > \frac{\|e_r(0)\|}{\psi^0} \quad \text{and} \quad \varepsilon\left|N\left(\frac{1}{1-\varepsilon^2}\right)\right| \geq 2\delta,$$

where the latter is possible because of the properties of N in (4). We show that $\|e_r(t)\| \leq \varepsilon\psi(t)$ for all $t \in [0,\omega)$, which is equivalent to $k \in L^\infty([0,\omega),\mathbb{R})$. Seeking a contradiction, assume there exists $t_1 \in [0,\omega)$ such that $\|e_r(t_1)\| > \varepsilon\psi(t_1)$ and define

$$t_0 := \max\{\, t \in [0,t_1) \mid \|e_r(t)\| = \varepsilon\psi(t)\,\}.$$

Then, for all $t \in [t_0,t_1]$, we have

$$\|e_r(t)\| \geq \varepsilon\psi(t) \quad \text{and} \quad k(t) \geq \frac{1}{1-\varepsilon^2}. \qquad (10)$$

Since $|N(k(t_0))| = |N(\frac{1}{1-\varepsilon^2})| \geq 2\delta/\varepsilon$, there exists $t_2 \in (t_0,t_1]$ such that

$$\forall\, t \in [t_0,t_2]:\ |N(k(t))| \geq \frac{\delta}{\varepsilon}.$$

Furthermore, by definition of t_0 we have that $\|e_r(t_2)\| > \varepsilon\psi(t_2)$. Then we obtain that

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_r(t)\|^2 &\overset{(7)}{=} e_r(t)^\top\left(e^{(r)}(t) + \sum_{i=1}^{r-1}\mu_i e^{(i)}(t)\right) \\
&\overset{(1)}{\leq} \Bigg(\|f\big(d(t), T(y,\dot{y},\ldots,y^{(r-1)})(t), \text{sat}(v(t))\big)\| \\
&\quad + \|y_{\text{ref}}^{(r)}(t)\| + \sum_{i=1}^{r-1}\mu_i\|e^{(i)}(t)\|\Bigg)\|e_r(t)\| \\
&\overset{\text{Steps 3,4}}{\leq} \Bigg(C\psi(t) + \|y_{\text{ref}}^{(r)}\|_\infty \\
&\quad + \sum_{i=1}^{r-1}\mu_i\big(\kappa_i\psi(t) + \|y_{\text{ref}}^{(i)}\|_\infty\big)\Bigg)\|e_r(t)\| \\
&\overset{\text{Step 2}}{\leq} \big(\varepsilon\dot{\psi}(t) + \hat{C}\psi(t) - \varepsilon\dot{\psi}(t)\big)\|e_r(t)\| \\
&\overset{(3)}{=} \left(\varepsilon\dot{\psi}(t) + \hat{C}\psi(t) + \varepsilon\alpha\psi(t) - \varepsilon\beta - \varepsilon\psi(t)\tfrac{\kappa(v(t))}{\|e_r(t)\|}\right)\|e_r(t)\| \\
&\overset{(8),(10)}{\leq} \Big(\varepsilon\dot{\psi}(t) - \varepsilon\beta + M - \big(\varepsilon|N(k(t))| - \varepsilon\alpha - \hat{C}\big)\psi(t)\Big)\|e_r(t)\| \\
&\leq \Big(\varepsilon\dot{\psi}(t) + M - \underbrace{\big(\delta - \alpha - \hat{C}\big)}_{>0 \text{ by } (9)}\psi(t)\Big)\|e_r(t)\| \\
&\overset{\text{Step 2}}{\leq} \Big(\varepsilon\dot{\psi}(t) + M - \big(\delta - \alpha - \hat{C}\big)\tfrac{\beta}{\alpha}\Big)\|e_r(t)\| \\
&\overset{(9)}{\leq} \varepsilon\dot{\psi}(t)\|e_r(t)\|
\end{aligned}$$

for almost all $t \in [t_0,t_2]$ and upon integration we get

$$\begin{aligned}
\|e_r(t_2)\| - \|e_r(t_0)\| &= \int_{t_0}^{t_2}\tfrac{1}{2}\|e_r(t)\|^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_r(t)\|^2\mathrm{d}t \\
&\leq \int_{t_0}^{t_2}\varepsilon\dot{\psi}(t)\mathrm{d}t = \varepsilon\psi(t_2) - \varepsilon\psi(t_0),
\end{aligned}$$

which yields the contradiction

$$0 = \varepsilon\psi(t_0) - \|e_r(t_0)\| \leq \varepsilon\psi(t_2) - \|e_r(t_2)\| < 0.$$

Step 6: We show that $\omega = \infty$, i.e., assertion (i) of the theorem. Suppose that $\omega < \infty$. From Step 5 it follows that there exists $\varepsilon \in (0,1)$ such that

$$\forall\, t \in [0,\omega):\ \|e_r(t)\| \leq \varepsilon\psi(t).$$

Furthermore, by Step 2 we have $\psi(t) \geq \mu(0)e^{-\alpha\omega} + \frac{\beta}{\alpha} > \frac{\beta}{\alpha}$ for all $t \in [0,\omega)$. Moreover, from boundedness of k it follows that $\kappa(v)$ is bounded and hence $\frac{\kappa(v)}{\|e_r\|}$ is bounded, since $\kappa(v)$ vanishes when $\|e_r\|$ is small enough, cf. Step 1. Therefore, it follows from (3) that there exist some $d_1, d_2 \geq 0$ such that $\psi(t) \leq d_1 e^{d_2 t} \leq d_1 e^{d_2\omega}$ for all $t \in [0,\omega)$. Define

$$\begin{aligned}
\hat{\mathcal{D}} := \Bigg\{(t,\xi_1,\ldots,\xi_r,\zeta) \in [0,\omega] \times (\mathbb{R}^m)^r \times \mathbb{R}\ \Bigg|\
&\Big\|\textstyle\sum_{i=1}^{r}\mu_i\xi_i - p(\frac{\mathrm{d}}{\mathrm{d}t})y_{\text{ref}}(t)\Big\| \leq \varepsilon\zeta, \\
&\tfrac{\beta}{\alpha} + \mu(0)e^{-\alpha\omega} \leq \zeta \leq d_1 e^{d_2\omega}\Bigg\},
\end{aligned}$$

which is evidently a compact subset of $\mathcal{D}$ since $y_{\text{ref}},\ldots,y_{\text{ref}}^{(r-1)}$ are bounded. Since $(t, x(t)) \in \hat{\mathcal{D}}$ for all $t \in [0,\omega)$, it follows that the closure of the set $\mathcal{G}$ from Step 1 is a compact subset of $\mathcal{D}$, a contradiction. Therefore, $\omega = \infty$.

Step 7: We complete the proof by establishing assertions (ii) and (iii) of the theorem. Assertion (ii) is a consequence of Steps 3 and 5. Let $[t_0, t_1) \subseteq \mathbb{R}_{\geq 0}$ with $t_1 \in (t_0, \infty]$ be an interval with $v(t) = \mathrm{sat}(v(t))$ for all $t \in [t_0, t_1)$, then statement (iii) is clear since $\dot\psi(t) = -\alpha\psi(t) + \beta$ for all $t \in [t_0, t_1)$. This completes the proof. ∎

We stress that although Theorem 3.1 provides the existence of a global solution of the closed-loop system, it cannot be concluded that the funnel function ψ is bounded in general. However, statement (iii) provides that *a posteriori* the funnel boundary reverts to its prescribed shape on any interval where the saturation is not active; in particular, if $t_1 = \infty$, then it is bounded.

Remark 3.2: We comment on the freedom of choice of the design parameters in (4). The parameters α, β and ψ^0 are chosen by the user to determine the desired shape of the funnel boundary in the form $\psi_{\mathrm{des}}(t) = \left(\psi^0 - \frac{\beta}{\alpha}\right) e^{-\alpha t} + \frac{\beta}{\alpha}$. Then, according to (5), a suitable choice for $k_1, \ldots, k_{r-1}$ could be $k_i = \alpha + 1$, resulting in the estimate $\|e_i(t)\| < \max_{j=i,\ldots,r-1} \left\{1, \frac{\|e_j(0)\|}{\psi^0}\right\} \psi(t)$ for all $t \geq 0$ and all $i = 1, \ldots, r-1$. Finally, a typical choice for the surjection N is $N(s) = s \sin s$.

Compared to the precursor of the improved controller (3) presented in [1], it is another advantage that it has much fewer design parameters. For the precursor it is generally hard to determine suitable parameters.

IV. Simulations

We compare the controller (3) to its precursor presented in [1] and consider the benchmark example of the mass-on-car system presented therein, which is originally taken from [9]. As shown in Fig. 3, the mass m_2 (in kg) moves on a ramp inclined by the angle $\vartheta \in [0, \frac{\pi}{2})$ (in rad) and is mounted on a car with mass m_1 (in kg). The control input is the force $u = F$ (in N) which acts on the car. The equations of motion for the system are given by

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\vartheta \\ m_2 \cos\vartheta & m_2 \end{bmatrix} \begin{pmatrix} \ddot z(t) \\ \ddot s(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t) + d\dot s(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix}, \tag{11}$$

where t is the current time (in s), z (in m) is the horizontal car position and s (in m) the relative position of the mass on the ramp. The coefficients of the spring and damper are given by $k > 0$ (in N/m) and $d > 0$ (in Ns/m), resp. The output y (in m) is the horizontal position of the mass on the ramp given by

$$y(t) = z(t) + s(t)\cos\vartheta.$$

It can be observed that for $u = 0$ the system admits the solution $t \mapsto z(t) := t$ and $t \mapsto s(t) := 0$, thus the system is not input-to-state stable and hence [4, Thm. 1] cannot be applied.

For the simulation we consider the case of relative degree $r = 3$, that is $\vartheta = 0$. As shown in [1], system (11) with output y belongs to the class $\mathcal{N}^{1,3}$ in this case. We choose the

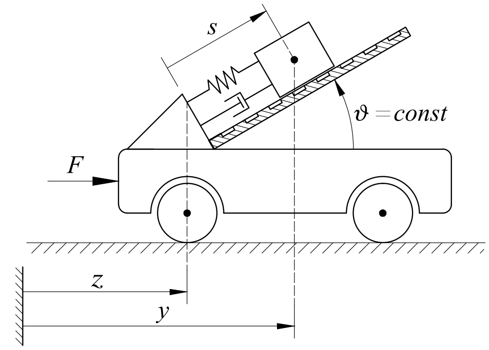

Fig. 3: Mass-on-car system.

parameters $m_1 = 4$, $m_2 = 1$, $k = 2$, $d = 1$, the initial values $z(0) = s(0) = 0$, $\dot z(0) = \dot s(0) = 0$ and the reference signal $y_{\mathrm{ref}}\colon t \mapsto \frac{1}{2}\cos t$. The saturation function in (2) is chosen as

$$v \mapsto \mathrm{sat}(v) = \begin{cases} v, & |v| \leq M, \\ \mathrm{sgn}(v) M, & |v| > M, \end{cases}$$

with $M = 8$. All simulations are MATLAB generated (solver: `ode45`, rel. tol.: 10^{-10}, abs. tol.: 10^{-8}) and over the time interval $[0, 20]$.

For both controllers (3) and its precursor from [1], we choose the common parameters $N(s) = s \sin s$ and $\alpha = \alpha_1 = 1.5$, $\beta = \beta_1 = 0.15$, $\psi^0 = \psi_1^0 = 3.1$, so that the desired funnel boundary is $\psi_{\mathrm{des}}(t) = 3e^{-1.5t} + 0.1$. Furthermore, for (3) we choose $k_1 = k_2 = \alpha + 1$, and for the controller from [1] we choose

$$\alpha_2 = 0.9\alpha_1,\ \alpha_3 = 0.9\alpha_2,\ \beta_2 = 0.5\alpha_2,\ \beta_3 = 0.5\alpha_3,$$
$$p_1 = p_2 = 1.1,\ \psi_2^0 = 2,\ \psi_3^0 = 1.$$

The application of the controller (3) and its precursor from [1] to (11) is depicted in Fig. 4. The corresponding tracking errors and funnel boundaries are shown in Fig. 4a, while Fig. 4b shows the respective input functions. It is evident that the performance of the improved controller (3) is superior to its precursor. While the latter constantly induces periods of active saturation with a tracking error frequently leaving the desired performance funnel, the improvement (3) only saturates over a short period at the beginning and quickly drives the tracking error to zero (never leaving the desired funnel), even under tight input constraints.

V. Conclusion

In the present paper we proposed an improvement of a recent funnel control design for a large class of nonlinear systems modeled by functional differential equations in the presence of input constraints presented in [1]. Compared to its precursor, the new controller comprises only one (and not r) dynamic equations for the funnel functions, it involves much less design parameters, it enables a proof for boundedness of the control signal v, and in simulations it exhibits a superior performance. Future research should focus on bridging the gap to the recent work of Trakas and Bechlioulis e.g. in [4] on the same topic, to find an approach which unifies the different considered system classes.

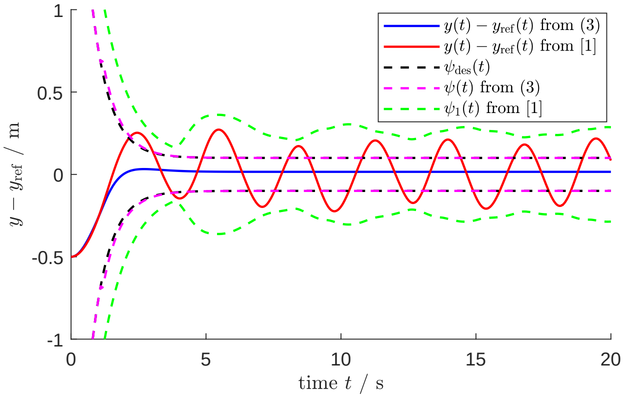

Fig. 4a: Performance funnels and tracking errors

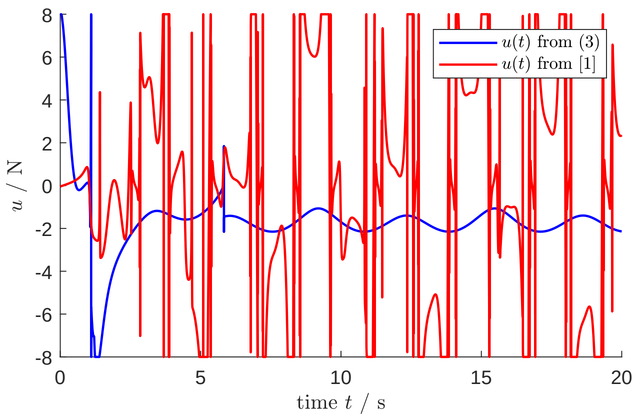

Fig. 4b: Input functions

Fig. 4: Simulation, under controllers (3) and its precursor from [1] of system (11) with $\vartheta = 0$.

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