Funnel control with input filter for nonlinear systems of relative degree two ⋆

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Abstract: We address the problem of output reference tracking for unknown nonlinear multi-input, multi-output systems with relative degree two and bounded-input bounded-state (BIBS) stable internal dynamics. We propose a novel model-free adaptive controller that ensures the evolution of the tracking error within prescribed performance funnel boundaries. By applying an output filter, the control objective is achieved without utilizing derivative information of system's output. The controller is illustrated by a numerical example.

Keywords: Funnel control, nonlinear systems, adaptive control, output feedback, output reference tracking, robust control, prescribed performance

1. INTRODUCTION

Introduced in Ilchmann et al. (2002), *funnel control* is a well-established adaptive high-gain control methodology for output-reference tracking of nonlinear multi-input, multi-output systems. The controller operates under minimal structural assumptions – namely stable internal dynamics and a known relative degree with a sign-definite high-frequency gain matrix – yet it guarantees that the tracking error remains within prescribed performance bounds. This framework offers robustness against disturbances and ensures transient performance without relying on an explicit system model. As a result, funnel control has been successfully applied to tracking problems in various domains, such as DC-link power flow control (Senfelds and Paugurs, 2014), control of industrial servo-systems (Hackl, 2017), and temperature control of chemical reactor models (Ilchmann and Trenn, 2004).

Recent research in funnel control has focused on overcoming practical implementation challenges. Key areas of investigation include handling measurement losses (Berger and Lanza, 2023) and actuator failures (Zhang et al., 2025b), multi-agent settings (Min et al., 2022; Zhang et al., 2025a), input constraints (Hu et al., 2022; Berger, 2024), and sampled-data (Lanza et al., 2024) or discrete-time systems (Cheng et al., 2023), as well as incorporating prior model knowledge to enhance performance (Berger et al., 2022). For a comprehensive literature overview, we refer to the survey by Berger et al. (2025). Despite these advancements, a long-standing and persistent difficulty for both funnel control and adaptive high-gain methods in general has been the handling of systems with a relative degree exceeding one, see Morse (1996).

The prevalent approach to achieve prescribed output tracking for higher-order systems relies on recursively defined error variables that incorporate output derivatives, cf. Berger et al. (2018, 2021). However, obtaining these derivatives in practice requires differentiating the output measurement, which is an ill-posed problem, especially in the presence of noise (Hackl, 2012, Sec. 1.4.4). This fundamental issue of derivative availability has been addressed in several ways. Early results on tracking with prescribed accuracy, while only using output information, for single-input, single-output linear time-invariant systems were achieved by Miller and Davison (1991). Ilchmann et al. (2006, 2007) employed a backstepping-based filter to obtain a funnel control mechanism which uses output feedback only, but its design is complex and impractical due to high powers of a typically large gain function. Alternatively, the *funnel pre-compensator* – a high-gain observer structure – was introduced by Berger and Reis (2018b) (with Lanza (2022) providing the feasibility proof for higher relative degrees) to supply necessary auxiliary derivative signals. While effective, this method also remains relatively complicated and hard to implement, especially for systems with relative degree higher than two. Lanza (2024) recently proposed a sampled-data funnel controller for control-affine systems of relative degree two that also avoids output derivatives. However, in order to derive a sufficiently fast sampling rate and sufficiently large control signals, knowledge about the system to be controlled is necessary; upper bounds of the system dynamics are assumed to be known.

Prescribed performance control – a methodology related to funnel control – also ensures that a reference signal is tracked within prescribed boundaries, see e.g. Bechlioulis and Rovithakis (2008, 2014). However, the controller is designed for systems in a certain normal form and requires full system state access. To overcome the need for full state information and rely solely on output measurements, Dimanidis et al. (2020) employed a high-gain observer. Similarly, Liu et al. (2021) and Chowdhury and Khalil (2019) combined a high-gain observer with the funnel control scheme. A drawback of the latter approaches is that the observer parameters must be chosen "sufficiently

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large" a priori. A specific lower bound, however, is not provided.

The present paper proposes a funnel controller that achieves prescribed output tracking for nonlinear multi-input, multi-output systems of relative degree two. By utilizing an output filter signal, the controller achieves its objective without incorporating output derivatives. The proposed design is both simple and has a lower complexity compared to previous methods. Furthermore, its reduced parameter set allows for more straightforward performance tuning. Although this paper focuses on systems with relative degree two, the approach lays a groundwork that we believe is generalizable to systems with higher relative degree.

Nomenclature: $\mathbb{N}$ and $\mathbb{R}$ denote natural and real numbers, respectively. $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mathbb{R}_{\geq 0} := [0, \infty)$. $\|x\| := \sqrt{\langle x, x \rangle}$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. $\mathcal{C}^p(V, \mathbb{R}^n)$ is the linear space of p-times continuously differentiable functions $f : V \to \mathbb{R}^n$, where $V \subset \mathbb{R}^m$, and $p \in \mathbb{N}_0 \cup \{\infty\}$. $\mathcal{C}(V, \mathbb{R}^n) := \mathcal{C}^0(V, \mathbb{R}^n)$. On an interval $I \subset \mathbb{R}$, $L^\infty(I, \mathbb{R}^n)$ denotes the space of measurable and essentially bounded functions $f : I \to \mathbb{R}^n$ with norm $\|f\|_\infty := \operatorname{ess\,sup}_{t \in I} \|f(t)\|$, $L^\infty_{\text{loc}}(I, \mathbb{R}^n)$ the set of measurable and locally essentially bounded functions. Furthermore, $W^{k,\infty}(I, \mathbb{R}^n)$ is the Sobolev space of all k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ such that $f, \dots, f^{(k)} \in L^\infty(I, \mathbb{R}^n)$.

2. PROBLEM FORMULATION

2.1 System class

We consider second order nonlinear multi-input, multi-output systems of the form

$$\begin{aligned} \ddot{y}(t) &= R_1 y(t) + R_2 \dot{y}(t) + f\big(\mathbf{T}(y, \dot{y})(t)\big) + \Gamma u(t) \\ y|_{[0,t_0]} &= y^0 \in \mathcal{C}^1([0, t_0], \mathbb{R}^m) \end{aligned} \tag{1}$$

with $t_0 \geq 0$, initial trajectory y^0, control input $u \in L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^m)$, and output $y(t) \in \mathbb{R}^m$ at time $t \geq 0$. In the case of $t_0 = 0$, we identify $\mathcal{C}^1([0, t_0], \mathbb{R}^m)$ with the vector space $\mathbb{R}^{2m}$. Then, the initial value condition in (1) is replaced by $(y(t_0), \dot{y}(t_0)) = (y_1^0, y_2^0) \in \mathbb{R}^{2m}$. Note that the control input u and the system's output y have the same dimension $m \in \mathbb{N}$. The system consists of the *unknown* matrices $R_1, R_2 \in \mathbb{R}^{m \times m}$, *unknown* nonlinear function $f \in \mathcal{C}(\mathbb{R}^q, \mathbb{R}^m)$, an *unknown* positive definite matrix $\Gamma \in \mathrm{GL}_m(\mathbb{R})$ and *unknown* nonlinear operator $\mathbf{T} : \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \to L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^q)$. The operator $\mathbf{T}$ is causal, locally Lipschitz and satisfies a bounded-input bounded-output property. It is characterized in detail in the following definition.

Definition 1. For $n, q \in \mathbb{N}$, and $t_0 \geq 0$, the set $\mathcal{T}^{2n,q}_{t_0}$ denotes the class of operators $\mathbf{T} : \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^n) \to L^\infty_{\text{loc}}([t_0, \infty), \mathbb{R}^q)$ for which the following properties hold:

(i) *Causality*: $\forall\, y_1, y_2 \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)^2\ \forall\, t \geq t_0$:
$$y_1|_{[0,t]} = y_2|_{[0,t]} \implies \mathbf{T}(y_1)|_{[t_0,t]} = \mathbf{T}(y_2)|_{[t_0,t]}.$$

(ii) *Local Lipschitz*: $\forall\, t \geq t_0\ \forall\, y \in \mathcal{C}([0,t], \mathbb{R}^n)^2\ \exists\, \Delta, \delta, c > 0\ \forall\, y_1, y_2 \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)^2$ with $y_1|_{[0,t]} = y_2|_{[0,t]} = y$ and $\|y_1(s) - y(t)\| < \delta$, $\|y_2(s) - y(t)\| < \delta$ for all $s \in [t, t+\Delta]$:
$$\operatorname*{ess\,sup}_{s \in [t, t+\Delta]} \|\mathbf{T}(y_1)(s) - \mathbf{T}(y_2)(s)\| \leq c \sup_{s \in [t, t+\Delta]} \|y_1(s) - y_2(s)\|.$$

(iii) *Bounded-input bounded-output (BIBO)*: $\forall\, c_0 > 0\ \exists\, c_1 > 0\ \forall\, y_1, y_2 \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^n)$:
$$\sup_{t \in \mathbb{R}_{\geq 0}} \|y_1(t)\| \leq c_0 \implies \operatorname*{ess\,sup}_{t \in [t_0, \infty)} \|\mathbf{T}(y_1, y_2)(t)\| \leq c_1.$$

Remark 2. The BIBO property (iii) of operator $\mathbf{T}$ allows us to conclude from the available information (the system output y) that the internal dynamics of the system (modeled by $\mathbf{T}$) stay bounded. This was also assumed in previous works on pure output feedback funnel control, see, e.g., (Berger and Reis, 2018b; Ilchmann et al., 2007; Lanza, 2022), but is a stronger property than the corresponding property

(iii') $\forall\, c_0 > 0\ \exists\, c_1 > 0\ \forall\, z \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^{2n})$:
$$\sup_{t \in \mathbb{R}_{\geq 0}} \|z(t)\| \leq c_0 \implies \operatorname*{ess\,sup}_{t \in [t_0, \infty)} \|\mathbf{T}(z)(t)\| \leq c_1$$

used in classical funnel control, where the output derivative information is assumed to be available, see, e.g., Berger et al. (2018, 2021). The system (1) explicitly contains linear terms as they would be otherwise excluded by property (iii), contrary to property (iii').

Definition 3. We say that the system (1) belongs to the system class $\mathcal{N}^{m,2}$, written $(f, \Gamma, \mathbf{T}, R_1, R_2) \in \mathcal{N}^{m,2}$, if, for $t_0 \geq 0$ and some $q \in \mathbb{N}$, the following holds: $f \in \mathcal{C}(\mathbb{R}^q, \mathbb{R}^m)$, $R_1, R_2 \in \mathbb{R}^{m \times m}$, $\Gamma \in \mathrm{GL}_m(\mathbb{R})$ is a positive definite matrix, and $\mathbf{T} \in \mathcal{T}^{2m,q}_{t_0}$.

2.2 Control objective

The objective is to design an output feedback control law, which achieves that, for any reference signal $y_{\text{ref}} \in W^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, the output tracking error $e(t) = y(t) - y_{\text{ref}}(t)$ evolves within a prescribed performance funnel

$$\mathcal{F}_\varphi := \{(t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \,|\varphi(t)\|e\| < 1\}.$$

This funnel is determined by the choice of the function φ belonging to

$$\mathcal{G} := \left\{ \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}) \,\middle|\, \inf_{t \geq 0} \varphi(t) > 0 \right\},$$

see also Figure 1. Note that the tracking error e evolving

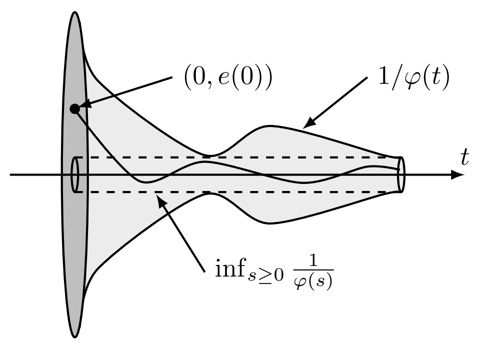

Fig. 1. Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $1/\varphi(t)$. The figure is based on (Berger et al., 2018, Fig. 1), edited for present purpose.

in $\mathcal{F}_\varphi$ is not constrained to converge to zero asymptotically. The choice of the performance function φ is guided by the application, which dictates the requisite constraints on the tracking error.

3. CONTROLLER DESIGN

We propose the following funnel controller to achieve the control objective described in Section 2.2 for systems of class $\mathcal{N}^{m,2}$.

$$\begin{aligned} e(t) &= y(t) - y_{\text{ref}}(t), \\ \dot{\xi}(t) &= -\xi(t) + u(t), \qquad \xi(t_0) = \xi^0, \\ \theta(t) &= \xi(t) + \frac{e(t)}{1 - \varphi(t)^2 \|e(t)\|^2}, \\ u(t) &= \frac{-\theta(t)}{\hat{\theta}^2 - \|\theta(t)\|^2}, \qquad \hat{\theta} > 0. \end{aligned} \tag{2}$$

Note that the controller *does neither* use the derivative $\dot{y}$ of the system's output nor of the reference trajectory. Instead, we introduce a filter variable ξ that is designed to qualitatively reproduce $\dot{y}$. The control law u is then selected to ensure that ξ tracks the virtual input signal

$$\xi^\star(t) = -\frac{e(t)}{1 - \varphi^2(t) \|e(t)\|},$$

which is determined by the funnel controller for systems of relative degree one, with the tracking error $\theta(t) = \xi(t) - \xi^\star(t)$ constrained to remain in a funnel of constant radius $\hat{\theta} > 0$. In this way, the filter variable acts as a surrogate for the unavailable derivative $\dot{y}$, enabling the controller to achieve the desired control objective. The controller has two parameters $\hat{\theta}$ and ξ^0, which determine the tracking performance. However, note that no system knowledge is required to select the parameters, except for the initial system output $y^0(t_0)$, see Theorem 4.

A function $(x, \xi) : [0, \omega) \to \mathbb{R}^{2m} \times \mathbb{R}^m$, $\omega \in (t_0, \infty]$, is called solution of the closed-loop system (1), (2) in the sense of *Carathéodory*, if it satisfies the initial conditions $(x, \xi)|_{[0,t_0]} = (y^0, \dot{y}^0, \xi^0)$, and $(x, \xi)|_{[0,\omega)}$ is absolutely continuous and satisfies

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= R_1 x_1(t) + R_2 x_2(t) + f\big(\mathbf{T}(x)(t)\big) + \Gamma u(t) \end{aligned}$$

(which corresponds to (1) with $y = x_1$) and

$$\dot{\xi}(t) = -\xi(t) + u(t)$$

with u as in (2) for almost all $t \in [t_0, \omega)$. We call a solution x *maximal*, if it does not have a right extension, which is also a solution.

We are now in the position to present the main result of the paper.

Theorem 4. For $(f, \Gamma, \mathbf{T}, R_1, R_2) \in \mathcal{N}^{m,2}$, consider system (1). Let $\varphi \in \mathcal{G}, y_{\text{ref}} \in W^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $\hat{\theta} > 0$. Further, let the initial trajectory $y^0 \in \mathcal{C}^1([0, t_0], \mathbb{R}^m)$ be given with $\varphi(t_0)^2 \|y^0(t_0) - y_{\text{ref}}(t_0)\|^2 < 1$ and choose $\xi^0 \in \mathbb{R}^m$ such that

$$\left\| \xi^0 + \frac{y^0(t_0) - y_{\text{ref}}(t_0)}{1 - \varphi(t_0)^2 \|y^0(t_0) - y_{\text{ref}}(t_0)\|^2} \right\| < \hat{\theta}.$$

Then, the application of the controller (2) to the system (1) yields an initial-value problem which has a solution, and every maximal solution $(x, \xi) : [0, \omega) \to \mathbb{R}^{2m} \times \mathbb{R}^m$, $\omega \in (t_0, \infty]$, has the following properties:

(i) The solution is global, i.e. $\omega = \infty$.
(ii) The input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ is bounded.
(iii) The tracking error $e = x_1 - y_{\text{ref}}$ evolves uniformly within the performance funnel given by φ, i.e.

$$\exists\, \varepsilon \in (0,1)\ \forall\, t \geq t_0 : \quad \varphi(t) \|e(t)\| < \varepsilon.$$

Proof. *Step 1*: We show that there exists a maximal solution $(x, \xi) : [0, \omega) \to \mathbb{R}^{2m} \times \mathbb{R}^m$, $\omega \in (t_0, \infty]$, of the closed-loop system (1), (2). Define the non-empty open set

$$\mathcal{E} := \left\{ (t, z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{3m} \left| \begin{array}{l} \varphi(t) \|z_1 - y_{\text{ref}}(t)\| < 1, \\ \left\| z_3 + \dfrac{z_1 - y_{\text{ref}}(t)}{1 - \varphi(t)^2 \|z_1 - y_{\text{ref}}(t)\|^2} \right\| < \hat{\theta} \end{array} \right. \right\}.$$

Using the notation $\tilde{\theta}(t, z_1, z_3) := z_3 + \frac{z_1 - y_{\text{ref}}(t)}{1 - \varphi(t)^2 \|z_1 - y_{\text{ref}}(t)\|^2}$, define the function $F : \mathcal{E} \times \mathbb{R}^m \to \mathbb{R}^m$ by

$$F(t, z_1, z_2, z_3, \eta) := \begin{bmatrix} z_2 \\ R_1 z_1 + R_2 z_2 + f(\eta) - \Gamma \dfrac{\tilde{\theta}(t, z_1, z_3)}{\hat{\theta}^2 - \left\|\tilde{\theta}(t, z_1, z_3)\right\|^2} \\ -z_3 - \dfrac{\tilde{\theta}(t, z_1, z_3)}{\hat{\theta}^2 - \left\|\tilde{\theta}(t, z_1, z_3)\right\|^2} \end{bmatrix}.$$

Then, the initial value problem (1), (2) takes the form

$$\begin{pmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{pmatrix} = F\left(t, \begin{pmatrix} x(t) \\ \xi(t) \end{pmatrix}, \mathbf{T}(x)(t)\right), \quad \begin{pmatrix} x|_{[0,t_0]} \\ \xi|_{[0,t_0]} \end{pmatrix} = \begin{pmatrix} y^0 \\ \dot{y}^0 \\ \xi^0 \end{pmatrix}.$$

We have $(t_0, x(t_0), \xi^0) \in \mathcal{E}$ by assumption. Application of a variant of (Ilchmann and Ryan, 2009, Thm. B.1) yields the existence of a maximal solution $(x, \xi) : [t_0, \omega) \to \mathbb{R}^{2m} \times \mathbb{R}^m$, $\omega \in (t_0, \infty]$, with $\text{graph}((x, \xi)|_{[t_0,\omega)}) \subset \mathcal{E}$. Moreover, the closure of graph $\big((x, \xi)|_{[t_0,\omega)}\big)$ is not a compact subset of $\mathcal{E}$.

Step 2: We define some constants for later use. Let $y(t) := x_1(t)$ for $t \in [t_0, \omega)$. By definition of the set $\mathcal{E}$, we have $\varphi(t) \|y(t)\| < 1$ for all $t \in [t_0, \omega)$. On the entire interval $[t_0, \infty)$, let y^e be a continuous extension of y with $\varphi(t) \|y^e(t)\| < 1$ and $\tilde{y}^e$ be a continuous extension of $\dot{y}$ (set $y^e := y$ and $\tilde{y}^e := \dot{y}$ in the case $\omega = \infty$). By the BIBO property of operator $\mathbf{T}$, there exists a constant $c_1 > 0$ such that $\operatorname{ess\,sup}_{t \in [t_0,\infty)} \|\mathbf{T}(y^e, \tilde{y}^e)(t)\| \leq c_1$. Therefore, $(R_1 - R_2 - I) y^e(t) + f(\mathbf{T}(y^e, \tilde{y}^e)(t))$ is bounded on the entire interval $[t_0, \infty)$ due to the continuity of the involved functions. Consider the linear differential equation

$$\dot{\tilde{x}}(t) = -\tilde{x}(t) + (R_1 - R_2 - I) y^e(t) + f(\mathbf{T}(y^e, \tilde{y}^e)(t)) \tag{3}$$

with initial value $\tilde{x}(t_0) = \dot{y}(t_0) - (R_2 + I) y(t_0) - \Gamma \xi^0$. Then, (3) has a unique global solution $\tilde{x} : [t_0, \infty) \to \mathbb{R}^m$ that is bounded. Set $c_3 := \sup_{t \geq t_0} \|\tilde{x}(t)\|$ and define the constants

$$\tilde{\lambda} := \|R_2 + I\| \left(\|y_{\text{ref}}\|_\infty + \left\|\tfrac{1}{\varphi}\right\|_\infty \right) + c_3 + \|\dot{y}_{\text{ref}}\|_\infty$$

and

$$\lambda := \left\|\tfrac{\dot{\varphi}}{\varphi}\right\|_\infty + \|\varphi\|_\infty \left(\tilde{\lambda} + \|\Gamma\| \,\hat{\theta} \right),$$

which are well defined by $\varphi \in \mathcal{G}$ and $y_{\text{ref}} \in W^{2,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$. Due to the unboundedness of the function $x \mapsto \frac{x}{1-x}$ on $[0, 1)$, it is possible to choose $\varepsilon_1 \in (0, 1)$ such that $\varepsilon_1 > \varphi(t_0) \|e(t_0)\|$ and

$$\frac{\varepsilon_1^2}{1 - \varepsilon_1^2} > \frac{\lambda}{\lambda_{\min}(\Gamma)},$$

where $\lambda_{\min}(\Gamma) > 0$ denotes the smallest eigenvalue of the positive definite matrix Γ.

Step 3: We show $\varphi(t)\|e(t)\| \le \varepsilon_1$ for all $t \in [t_0, \omega)$. Seeking a contradiction, assume there exists $t^\star \in [t_0, \omega)$ with $\varphi(t^\star)\|e(t^\star)\| > \varepsilon_1$. As $\varphi(t_0)\|e(t_0)\| < \varepsilon_1$, there exists

$$t_\star := \sup\{t \in [t_0, t^\star) \,|\, \varphi(t)\|e(t)\| = \varepsilon_1\} < t^\star,$$

because e is a continuous function on the entire interval $[t_0, t^\star]$. Using the shorthand notation

$$\xi^\star(t) := -\frac{e(t)}{1 - \varphi(t)^2 \|e(t)\|^2},$$

define the constants

$$\bar{\xi} := \sup_{s \in [t_0, t^\star]} \|\xi(s)\| \quad \text{and} \quad \bar{\xi}^\star := \sup_{s \in [t_0, t^\star]} \left\|\dot{\xi}^\star(s)\right\|$$

and choose $\varepsilon_2 \in (0, \hat{\theta})$ such that $\|\theta(t_0)\| < \varepsilon_2$ and $\frac{\varepsilon_2^2}{\hat{\theta}^2 - \varepsilon_2^2} > \hat{\theta}(\bar{\xi} + \bar{\xi}^\star)$.

Step 3.1: We show $\|\theta(s)\| \le \varepsilon_2$ for all $s \in [t_0, t^\star]$ for θ as in (2). Seeking a contradiction, assume there exists $s^\star \in [t_0, t^\star]$ with $\|\theta(s^\star)\| > \varepsilon_2$. Then, there exists

$$s_\star := \sup\{s \in [t_0, s^\star) \,|\, \|\theta(s)\| = \varepsilon_2\} < s^\star.$$

By construction of $s_\star$, we have $\|\theta(s)\| > \varepsilon_2$ and $\frac{\|\theta(s)\|^2}{\hat{\theta}^2 - \|\theta(s)\|^2} > \hat{\theta}(\bar{\xi} + \bar{\xi}^\star)$ for all $s \in (s_\star, s^\star]$. Omitting the dependency on s and invoking (2), we calculate

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}s}\|\theta\|^2 &= \left\langle \theta, \dot{\theta} \right\rangle = \left\langle \theta, \dot{\xi} - \dot{\xi}^\star \right\rangle \\
&= \left\langle \theta, -\dot{\xi}^\star \right\rangle + \langle \theta, -\xi \rangle + \langle \theta, u \rangle \\
&\le \hat{\theta}\left(\bar{\xi} + \bar{\xi}^\star\right) - \frac{\|\theta\|^2}{\hat{\theta}^2 - \|\theta\|^2} < 0
\end{aligned}$$

almost everywhere on $[s_\star, s^\star]$, where we used $\|\theta(s)\| \le \hat{\theta}$, which holds by $(s, x(s), \xi(s)) \in \mathcal{E}$. Integration yields

$$\varepsilon_2^2 < \|\theta(s^\star)\|^2 = \int_{s_\star}^{s^\star} \tfrac{\mathrm{d}}{\mathrm{d}\tau}\|\theta(\tau)\|^2 \,\mathrm{d}\tau + \|\theta(s_\star)\|^2 \le \|\theta(s_\star)\|^2 = \varepsilon_2^2,$$

a contradiction. Thus, $\|\theta(s)\| \le \varepsilon_2$ for all $s \in [t_0, t^\star]$.

Step 3.2: We show that the existence of $t^\star$ leads to a contradiction. By definition of $t_\star$ and choice of ε_1, we have $1 > \varphi(t)\|e(t)\| \ge \varepsilon_1$ and

$$\frac{\varphi^2(t)\|e(t)\|^2}{1 - \varphi(t)^2\|e(t)\|^2} \ge \frac{\lambda}{\lambda_{\min}(\Gamma)}$$

for all $t \in [t_\star, t^\star]$. Setting

$$\zeta(t) := \dot{y}(t) - (R_2 + I)y(t) - \Gamma\xi(t), \tag{4}$$

we have $\dot{y}(t) = (R_2 + I)y(t) + \zeta(t) + \Gamma\xi(t)$. Omitting the dependency on t, we calculate

$$\begin{aligned}
\dot{\zeta} &= \ddot{y} - (R_2 + I)\dot{y} - \Gamma\dot{\xi} \\
&= R_1 y + R_2\dot{y} + f\big(\mathbf{T}(y, \dot{y})\big) + \Gamma u - (R_2 + I)\dot{y} - \Gamma(-\xi + u) \\
&= R_1 y + f\big(\mathbf{T}(y, \dot{y})\big) - \dot{y} + \Gamma\xi \\
&= -\zeta + (R_1 - R_2 - I)y + f\big(\mathbf{T}(y, \dot{y})\big)
\end{aligned}$$

on the interval $[t_0, \omega)$. Hence, ζ fulfills the differential equation (3) since $(y(t), \dot{y}(t)) = (y^e(t), \tilde{y}^e(t))$ for all $t \in [t_0, \omega)$. Furthermore, the initial conditions coincide, $\zeta(t_0) = \tilde{x}(t_0)$, thus, by uniqueness of the solution of (3), $\zeta(t) = \tilde{x}(t)$ for all $t \in [t_0, \omega)$. Therefore, $\|\zeta(t)\| \le \|\tilde{x}(t)\| \le c_3$ for all $t \in [t_0, \omega)$. Omitting the dependency on t, we calculate

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi^2\|e\|^2 &= \varphi\dot{\varphi}\|e\|^2 + \varphi^2\langle e, \dot{e}\rangle = \tfrac{\dot{\varphi}}{\varphi}\underbrace{\varphi^2\|e\|^2}_{<1} + \varphi^2\langle e, \dot{e}\rangle \\
&\le \left\|\tfrac{\dot{\varphi}}{\varphi}\right\|_\infty + \varphi^2\langle e, \dot{y} - \dot{y}_{\mathrm{ref}}\rangle \\
&\overset{(4)}{=} \left\|\tfrac{\dot{\varphi}}{\varphi}\right\|_\infty + \varphi^2\langle e, (R_2 + I)y + \zeta + \Gamma\xi - \dot{y}_{\mathrm{ref}}\rangle \\
&\le \left\|\tfrac{\dot{\varphi}}{\varphi}\right\|_\infty + \|\varphi\|_\infty\Big(\|R_2 + I\|\Big(\|y_{\mathrm{ref}}\|_\infty + \left\|\tfrac{1}{\varphi}\right\|_\infty\Big) + c_3 \\
&\qquad + \|\dot{y}_{\mathrm{ref}}\|_\infty\Big) + \varphi^2\langle e, \Gamma\xi\rangle \\
&\le \left\|\tfrac{\dot{\varphi}}{\varphi}\right\|_\infty + \|\varphi\|_\infty\tilde{\lambda} + \varphi^2\langle e, \Gamma(\theta + \xi^\star)\rangle \\
&\le \left\|\tfrac{\dot{\varphi}}{\varphi}\right\|_\infty + \|\varphi\|_\infty\left(\tilde{\lambda} + \|\Gamma\|\,\hat{\theta}\right) + \varphi^2\left\langle e, -\frac{\Gamma e}{1 - \varphi^2\|e\|^2}\right\rangle \\
&\le \lambda - \lambda_{\min}(\Gamma)\frac{\varphi^2\|e\|^2}{1 - \varphi^2\|e\|^2} \le 0
\end{aligned}$$

on $[t_\star, t^\star]$. Therefore,

$$\varepsilon_1^2 < \varphi(t^\star)^2\|e(t^\star)\|^2 \le \varphi(t_\star)^2\|e(t_\star)\|^2 = \varepsilon_1^2,$$

a contradiction. Thus, we have $\varphi(t)\|e(t)\| \le \varepsilon_1$ for all $t \in [t_0, \omega)$.

Step 4: We show that ξ is bounded on $[t_0, \omega)$. As a consequence of Step 3, we have $\varphi(t)\|e(t)\| \le \varepsilon_1$ for all $t \in [t_0, \omega)$. Hence, there exists $\hat{\xi}^\star > 0$ such that $\|\xi^\star(t)\| \le \hat{\xi}^\star$ for all $t \in [t_0, \omega)$. To show $\|\xi(t)\| \le \max\{\|\xi(t_0)\|, \hat{\xi}^\star\}$ for all $t \in [t_0, \omega)$, we assume that there exists $\hat{t} \in [t_0, \omega)$ such that $\|\xi(\hat{t})\| > \max\{\|\xi(t_0)\|, \hat{\xi}^\star\}$. Then,

$$t_1 := \sup\left\{ s \in [t_0, \hat{t}) \,\middle|\, \|\xi(s)\| = \max\{\|\xi(t_0)\|, \hat{\xi}^\star\} \right\}$$

is well-defined. Omitting the dependency on t, we calculate

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\xi\|^2 &= \langle \xi, -\xi\rangle + \langle \xi, u\rangle \\
&= -\|\xi\|^2 + \tfrac{1}{\hat{\theta}^2 - \|\theta\|^2}\langle \xi, -\theta\rangle \\
&= -\|\xi\|^2 + \tfrac{1}{\hat{\theta}^2 - \|\theta\|^2}\langle \xi, -\xi + \xi^\star\rangle \\
&\le -\|\xi\|^2 + \tfrac{1}{\hat{\theta}^2 - \|\theta\|^2}\left(\|\xi\|\,\|\xi^\star\| - \|\xi\|^2\right) \\
&\le -\|\xi\|^2 + \tfrac{1}{\hat{\theta}^2 - \|\theta\|^2}\|\xi\|\left(\hat{\xi}^\star - \|\xi\|\right) \le 0
\end{aligned}$$

on $[t_1, \hat{t}]$. Therefore,

$$\begin{aligned}
\max\{\|\xi(t_0)\|, \hat{\xi}^\star\}^2 < \|\xi(\hat{t})\|^2 &\le \|\xi(t_1)\|^2 \\
&\le \max\{\|\xi(t_0)\|, \hat{\xi}^\star\}^2,
\end{aligned}$$

a contradiction. This shows that $\|\xi(t)\| \le \max\{\|\xi(t_0)\|, \hat{\xi}^\star\}$ for all $t \in [t_0, \omega)$. Hence ξ is bounded.

Step 5: We show that $\dot{y}$ is bounded on $[t_0, \omega)$. As a consequence of Step 3, we have $\varphi(t)\|e(t)\| \le \varepsilon_1$ for all $t \in [t_0, \omega)$. Since y_{ref} is bounded and $\inf_{t \ge t_0}\varphi(t) > 0$, the system output y is bounded, too. Invoking (4), we have

$$\dot{y}(t) = (R_2 + I)y(t) + \zeta(t) + \Gamma\xi(t)$$

for all $t \in [t_0, \omega)$. Since ζ fulfills the differential equation (3), we have $\|\zeta(t)\| \le c_3$ for all $t \in [t_0, \omega)$, as shown in Step 3.2. According to Step 4, the function ξ is bounded. Therefore, $\dot{y}$ is bounded as well on the entire interval $[t_0, \omega)$.

Step 6: We show that u is bounded. According to the Steps 3–5, the functions y, $\dot{y}$, and ξ are bounded functions and $\varphi\|e\|$ is uniformly bounded away from 1. Therefore, $\xi^\star$ is a bounded function as well. Since $y_{\mathrm{ref}} \in W^{2,\infty}(\mathbb{R}_{\ge 0}, \mathbb{R}^m)$, we find that $\dot{\xi}^\star$ is bounded and hence we may define the constants

$$\bar{\xi} := \sup_{s\in[t_0,\infty)} \|\xi(s)\| \quad \text{and} \quad \bar{\xi}^\star := \sup_{s\in[t_0,\infty)} \left\|\dot{\xi}^\star(s)\right\|.$$

By choosing $\varepsilon_3 \in (0,\hat{\theta})$ such that $\|\theta(t_0)\| < \varepsilon_3$ and $\frac{\varepsilon_3^2}{\hat{\theta}^2-\varepsilon_3^2} > \hat{\theta}(\bar{\xi}+\bar{\xi}^\star)$, we can adapt Step 3.1 to show $\|\theta(t)\| \le \varepsilon_3$ for all $t \in [t_0,\infty)$. As a consequence, u is bounded.

Step 6: We show $\omega = \infty$. According to Step 5, $\dot{y}$ is a bounded function. Thus, there exists $\varepsilon_4 > 0$ with $\|\dot{y}\|_\infty \le \varepsilon_4$. According to Steps 3–6, $\operatorname{graph}\left((x,\xi)|_{[t_0,\omega)}\right)$ is contained in the set

$$\left\{ (t,z) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^{3m} \,\middle|\, \begin{array}{l} \varphi(t)\,\|z_1 - y_{\mathrm{ref}}(t)\| \le \varepsilon_1, \\ \|z_2\| \le \varepsilon_4, \\ \left\| z_3 + \dfrac{z_1 - y_{\mathrm{ref}}(t)}{1-\varphi(t)^2\,\|z_1 - y_{\mathrm{ref}}(t)\|^2} \right\| \le \varepsilon_3 \end{array} \right\} \subset \mathcal{E}$$

Since the closure of $\operatorname{graph}\left(x|_{[t_0,\omega)}\right)$ is not a compact subset of $\mathcal{E}$ according to the observation from Step 1, this implies $\omega = \infty$ and thereby shows assertion (i). Further, $\varphi(t)\|e(t)\| \le \varepsilon_1 < 1$ for all $t \in [t_0,\infty)$ shows assertion (iii). Moreover, u is bounded on the whole interval $[t_0,\infty)$ according to Step 6 verifying assertion (ii). This completes the proof. □

4. SIMULATIONS

To illustrate the previous results, we simulate the controller (2) for different choices of the parameter $\hat{\theta}$ and compare it to the performance of the controller proposed by Berger and Reis (2018a). This controller introduces two auxiliary variables z_1, z_2, yielding the following structure:

$$\begin{aligned}
\dot{z}_1(t) &= z_2(t) + (q_1 + p_1 k_2(t))(y(t) - z_1(t)), \\
\dot{z}_2(t) &= (q_2 + p_2 k_2(t))(y(t) - z_1(t)) + \tilde{\Gamma} u(t), \\
e_0(t) &= z_1(t) - y_{\mathrm{ref}}(t), \\
e_1(t) &= \dot{e}_0(t) + k_0(t) e_0(t), \ \text{where} \\
k_0(t) &= \frac{1}{1 - \varphi_0(t)^2 \|e_0(t)\|^2}, \\
k_1(t) &= \frac{1}{1 - \varphi_1(t)^2 \|e_1(t)\|^2}, \\
k_2(t) &= \frac{1}{1 - \varphi_2(t)^2 \|y(t) - z_1(t)\|^2}, \\
u(t) &= -k_1(t) e_1(t),
\end{aligned}$$

for a suitable choice of constants $p_1, p_2, q_1, q_2 \in \mathbb{R}$, $\tilde{\Gamma} \in \mathbb{R}^{m\times m}$ and performance functions $\varphi_0, \varphi_1, \varphi_2$. Evidently, this controller involves much more design parameters, which need to be adjusted appropriately, than the controller proposed in (2). The latter also exhibits a lower complexity, as only one differential equation is involved. In order to compare the performance of both controllers, we consider the system also examined by Berger and Reis (2018a) given by

$$\ddot{y}(t) + a \sin y(t) = b u(t), \tag{5}$$

with the system parameters $a = 2, b = 2$. The simulations are performed in MATLAB (solver: `ode45`, rel.tol.: 10^{-8}, abs.tol.: 10^{-6}) over the time interval $[0, 20]$. The results for tracking the reference trajectory $y_{\mathrm{ref}}(t) = \frac{\cos(t)}{2}$ with the initial state $(y(0), \dot{y}(0)) = (0,0)$ for the choices

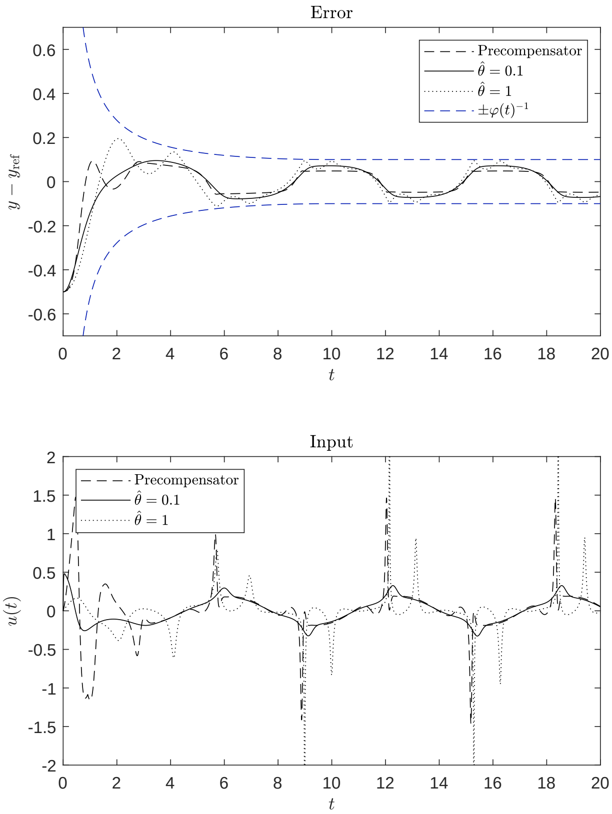

Fig. 2. Simulation of the behavior of the error $y(t) - y_{\mathrm{ref}}(t)$ and the input signal generated by the controller proposed by Berger and Reis (2018a) and the controller (2), for the system (5), using the parameters $a = 2, b = 2$.

$\xi^0 = 0.5$ for the initial value of the filter variable and $z_1(0) = z_2(0) = 0.5$ for the auxiliary variables for the controller from Berger and Reis (2018a) are depicted in Figure 2. The performance functions $\varphi_1(t) = \frac{1}{e^{-t}+1}$ and $\varphi_0(t) = \varphi_2(t) = 20 \begin{cases} 1 - (0.1t-1)^2, & t \le 10, \\ 1, & t > 10, \end{cases}$ and design parameters $\tilde{\Gamma} = 2$, $p_1 = 1$, $p_2 = \frac{5}{7}, q_1 = 1, q_2 = 5$ are obtained by sequential tuning of the parameters, while the funnel function $\varphi(t) = (\varphi_0(t)^{-1} + \varphi_2(t)^{-1})^{-1}$ is chosen such that it ensures the same transient performance of the error $y(t) - y_{\mathrm{ref}}(t)$.

We observe that the performance of the proposed controller (2) depends on the choice of parameter $\hat{\theta}$. For sufficiently small values of $\hat{\theta}$ (smaller than 0.1), further reductions do not noticeably alter the system response, which yields a straightforward tuning process. Since it is much more challenging to properly tune the controller proposed in Berger and Reis (2018a) due to the larger number of design parameters, finding a well-performing configuration is considerably more difficult. The input signal does not show any peaks, and has a smaller bandwidth compared to the controller proposed by Berger and Reis (2018a). Also, the output shows improved tracking behavior. However, for larger values of $\hat{\theta}$, the input signal shows larger bandwidth and substantial peaks, while the output tracking is inferior. This highlights that an appropriate selection of $\hat{\theta}$ is crucial for satisfactory controller performance.

5. CONCLUSION

In this work, we developed a funnel controller for systems of relative degree two that achieves trajectory tracking with prescribed performance and without requiring derivatives of the output signal by introducing a filter variable. The controller structure is simpler than in previous approaches, and the number of design parameters is much smaller, allowing for a more direct influence of the controller performance by appropriate selection, which is also exhibited in simulations. Future work will focus on extending this approach to systems of higher relative degree.

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