Safe continual learning in MPC with prescribed bounds on the tracking error

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Abstract

Recently, robust funnel Model Predictive Control (MPC) was introduced, which consists of model-based funnel MPC and model-free funnel control for its robustification w.r.t. model-plant mismatches, bounded disturbances, and uncertainties. It achieves output-reference tracking within prescribed bounds on the tracking error for a class of unknown nonlinear systems. We extend robust funnel MPC by a learning component to adapt the underlying model to the system data and hence to improve the contribution of MPC. Since robust funnel MPC is inherently robust and the evolution of the tracking error in the prescribed performance funnel is guaranteed, the additional learning component is able to perform the learning task online – even without an initial model or offline training.

Keywords: Data-based control, funnel control, model predictive control, robust control, safe learning

1. Introduction

Model Predictive Control (MPC) is a well-established control technique for constrained linear as well as for nonlinear multi-input multi-output systems. Using a model of the system to be controlled, the idea is to iteratively solve Optimal Control Problems (OCPs) based on predictions about the system behavior on a finite-time horizon, see e.g. the textbook [1], and the references therein. MPC is nowadays widely used in various applications, see e.g. [2] and the references therein.

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Since MPC heavily relies on the availability and accuracy of the underlying model of the actual system, some research effort has been made to compensate for model-plant mismatches and external disturbances. One research branch focuses on the MPC algorithm itself and its robustification, see e.g. [3, 4, 5, 6, 7], and the references therein, respectively. For instance, in [5] a robust MPC framework was developed for nonlinear discrete-time systems. The uncertainties and disturbances are compensated by introducing tightening tubes around the input and output constraints to ensure robust satisfaction of the constraints. A similar tube-based technique was used in [7]. In [3] it was shown that combining funnel MPC, which achieves reference tracking with guaranteed error performance, cf. [8], with an additional robust feedback controller leads to a control scheme, which achieves a tracking objective even in case of a severe model-plant mismatch.

A different research branch focuses on the idea to achieve robust constraint satisfaction of the actual system via adaption of the underlying model, see e.g. [9, 10, 11] and the references therein. For instance, in [10] the notion of persistently exciting data, cf. [12, 13], is exploited to update the model and show initial and recursive feasibility of the resulting MPC scheme. In [14, 15] combining MPC with Gaussian process-based learning schemes was proposed to achieve predictive control with stability guarantees. In [15] a NARX model was incorporated in the learning scheme. The proposed controller in [14] is formulated to address satisfaction of chance constraints, and its functioning is demonstrated with an autonomous racing vehicle. A similar approach was used in [16] to perform safe learning-based control in robotics. The issue of malfunctioning of the machine learning scheme is considered in [17], where the underlying neural network is forced to stay close to a predefined nominal model, and hence the MPC scheme safely achieves the control objective. Predictive safety filters are a MPC variant which is also closely related to tube-based approaches and allows the application of learning-based control techniques while guaranteeing compliance with constraints. The core idea is that the predictive safety filter verifies a control input signal proposed by a learning algorithm against a model. If the proposed control signal is deemed safe it is applied to the system, otherwise it is modified as little as necessary to guarantee constraint satisfaction, see e.g. [18, 19, 20]. A comprehensive overview of the application of various safe learning methods in MPC can be found in [21] and the references therein.

A particular control task, which is considered in the present article, is output-reference tracking with prescribed performance. To achieve this objective, funnel MPC was introduced in [22] and further developed in [8, 23, 24]. Funnel MPC incorporates a particular choice of the stage cost inspired by *funnel control*. The latter is a model-free adaptive high-gain feedback controller, see e.g. [25] and the references therein. By penalizing the distance of the tracking error to the prescribed error boundary via a "funnel-like" stage cost, the funnel MPC scheme achieves output tracking for systems with global strict relative degree one and bounded-input bounded-state stable internal dynamics, where the input and output are of the same dimension. Within this framework it was shown that the tracking objective is achieved and funnel MPC is initially and recursively feasible, without incorporating state constraints, terminal conditions, or the requirement of a sufficiently long prediction horizon. Although not proven yet mathematically, numerical results indicate that this control algorithm can successfully be applied to systems with higher relative degree, cf. [8, 22], as well as Section 5.2 in the present article. Extensions to arbitrary relative degree based on different stage cost functions are discussed in [23, 24] and initial and recursive feasibility is proved.

Many MPC schemes assume access to the full state of the system to be controlled, however this is not satisfied in general. Therefore, in [3] a distinction between the system to be controlled and the model to be used in the MPC algorithm is made, and only access to the model's state and to output measurements (but no state measurements) of the real system is used to initialize the model. In the present article we make use of that distinction, too, see Section 3. A different approach is pursed in [26, 27] with uncertain/disturbed linear discrete-time systems under consideration, where the state is estimated using a Luenberger observer. Combining the observer structure with a tube-based MPC scheme, robust constraint satisfaction and feasibility of the control algorithm were shown.

Along the lines of the second research branch discussed above, namely to update the model using measurement data from the system, we build on the results of [3], where a two component controller was used, consisting of funnel MPC in combination with a model-free funnel control feedback loop. We extend this approach by introducing a general online learning framework in order to continually improve the model utilizing the past system data, meaning system output, past model-based predictions, and applied control signals, both from the model-based and the model-free controller component. This not only allows to learn and fine-tune parameters of an already detailed model, but it is even possible to learn an unknown system without an initially given model. Continually improving the model and thereby the prediction capability required in MPC, the predictive funnel MPC control signal achieves the control objective for the unknown plant.

The present article is organized as follows. In Section 2 we provide the problem formulation. The control objective, as well as the controller components are introduced. In Section 3 we formally introduce the system class and the model class under study. Anticipating the later considerations, we highlight that the actual system to be controlled, and the model of that system can be quite different. E.g., it is possible to have a nonlinear system and a linear surrogate model on which the MPC algorithm operates. Section 4 contains the main result of the article. We introduce the combined three-component controller, establish the corresponding control Algorithm 1, and formulate the main result Theorem 4.2, which yields initial and recursive feasibility of the proposed control algorithm. Furthermore, we present a particular learning scheme, and rigorously prove its feasibility. In the last Section 5 we illustrate the control Algorithm 1 with two numerical simulations.

Nomenclature: $\mathcal{B}_\eta := \{x \in \mathbb{R}^n \mid \|x\| \leq \eta\}$ is the closed ball with radius

$\eta > 0$ around the origin in $\mathbb{R}^n$. $\mathrm{Gl}_n(\mathbb{R})$ is the group of invertible $\mathbb{R}^{n\times n}$ matrices. For an interval $I \subseteq \mathbb{R}$, $L^\infty(I,\mathbb{R}^n)$ is the Lebesgue space of measurable, essentially bounded functions $f\colon I \to \mathbb{R}^n$ with norm $\|f\|_\infty = \operatorname{ess\,sup}_{t\in I}\|f(t)\|$, and $W^{k,\infty}(I,\mathbb{R}^n)$ is the Sobolev space of all functions $f: I \to \mathbb{R}^n$ with k-th order weak derivative $f^{(k)}$ and $f, f^{(1)}, \ldots, f^{(k)} \in L^\infty(I,\mathbb{R}^n)$; and $f|_J$ denotes the restriction of a function $f : I \to \mathbb{R}^n$ to the interval $J \subseteq I$.

2. Problem formulation

The overall task is output-reference tracking within prescribed bounds on the tracking error. To this end, we apply a model predictive controller to achieve a superior controller performance while maintaining input and output constraints. However, since model-plant mismatches are unavoidable, we safeguard the MPC controller by an additional component, i.e., the model-free funnel controller, to guarantee satisfaction of the output-tracking criterion. The third component, besides MPC and funnel control, is a learning one, which repeatedly updates the model to reduce the model-plant mismatch and, thus, improves the overall controller performance. A major challenge is to ensure proper functioning of the interplay of these three components, which requires a consistency condition (see Section 3) for the model updates – the key novelty of our approach in comparison to the recently proposed robust funnel MPC (which combines the first two components) in [3].

The control objective is output reference tracking within prescribed transient error bounds. This means that, for an unknown control system of the form

$$\dot{z}(t) = F(z(t), d(t)) + G(z(t), d(t))u(t), \quad y(t) = H(z(t)), \tag{1}$$

where $z(t) \in \mathbb{R}^Z$, bounded disturbance $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^D)$, and $y(t) \in \mathbb{R}^m$ for $Z, D, m \in \mathbb{N}$, we seek an input $u(t) \in \mathbb{R}^m$ such that the output y tracks a given reference signal y_{ref}. Specific properties of the system parameters F, G, H are introduced in Section 3. Moreover, this tracking task is asked to be satisfied with a given precision, i.e. the tracking error $e(t) := y(t) - y_{\mathrm{ref}}(t)$ should evolve within (possibly time-varying) boundaries given by a so-called funnel function ψ, prescribed by the user. To be precise, the tracking error shall evolve within the funnel

$$\mathcal{F}_\psi := \{\, (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \|e\| < \psi(t) \,\},$$

which is determined by the choice of ψ belonging to the set

$$\mathcal{G} = \left\{ \psi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}) \;\middle|\; \inf_{t\geq 0} \psi(t) > 0 \right\}.$$

Note that by $\psi \in \mathcal{G}$ the tracking error is not forced to converge asymptotically to zero. For a function $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ we denote the set of all functions which evolve in the performance funnel defined by $\psi \in \mathcal{G}$ around y_{ref} by

$$Y(y_{\mathrm{ref}}, \psi) := \{ y \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m) \,|\, \forall\, t \geq 0 : \|y(t) - y_{\mathrm{ref}}(t)\| < \psi(t) \}.$$

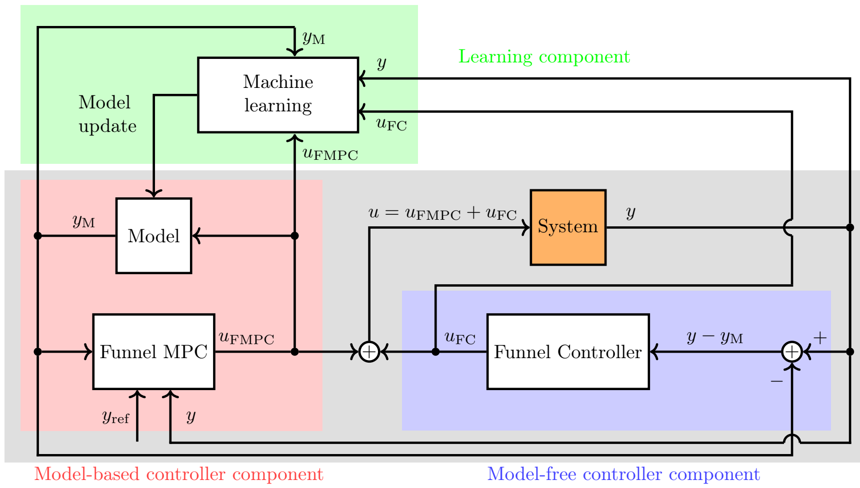

Figure 1: Design of the proposed three-component controller. The grey box (containing the red (funnel MPC) and the blue (funnel control) structures) represents the two-component controller *robust funnel MPC* proposed in [3]. The green box represents the learning component, which receives the four signals: system output y, model output y_{M}, funnel MPC control signal u_{FMPC}, and funnel control signal u_{FC}.

Robust funnel MPC. To solve the described problem for the unknown system (1), *robust funnel MPC* was proposed in [3]. This control scheme consists of two components, one model-based and one model-free, see Figure 1. The model-based funnel MPC component (red box in Figure 1) uses a model of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \quad y_{\mathrm{M}}(t) = h(x(t)) \tag{2}$$

where $x(t) \in \mathbb{R}^n$ and $y_{\mathrm{M}}(t), u(t) \in \mathbb{R}^m$, as an approximation of the system (1). Specific properties of the model parameters f, g, h are introduced in Section 3. At time instances $t_k \in \delta\mathbb{N}_0$ with $\delta > 0$, the current output $y_{\mathrm{M}}(t_k)$ of the model (2) is measured and predictions of the future model behavior are computed over the next time interval of length $T = N\delta > 0$, $N \in \mathbb{N}$. Using the time-varying *stage cost* $\ell : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ where $\ell(t, y_{\mathrm{M}}, u)$ is defined by

$$\ell(t, y_{\mathrm{M}}, u) = \begin{cases} \dfrac{\|y_{\mathrm{M}} - y_{\mathrm{ref}}(t)\|^2}{\psi(t)^2 - \|y_{\mathrm{M}} - y_{\mathrm{ref}}(t)\|^2} + \lambda_u \|u\|^2, & \|y_{\mathrm{M}} - y_{\mathrm{ref}}(t)\| \neq \psi(t) \\ \infty, & \text{else,} \end{cases} \tag{3}$$

with design parameter $\lambda_u \geq 0$, a control signal $u_{\mathrm{FMPC}} \in L^\infty([t_k, t_k + T], \mathbb{R}^m)$ is computed as a solution of an optimization problem. Additionally, the corresponding predicted output $y_{\mathrm{M}}(\cdot; t_k, x_k, u_{\mathrm{FMPC}}) = h(x(\cdot; t_k, x_k, u_{\mathrm{FMPC}})) \in L^\infty([t_k, t_k+T], \mathbb{R}^m)$ is calculated. Here $x(\cdot; t_k, x_k, u)$ denotes the unique solution of (2) with initial condition $x(t_k) = x_k$, which is well-defined for appropriate

f, g, h (the model is specified in Definition 3.2) on the interval $[t_k, t_k + T]$ for appropriate control input u.

The model-free funnel control component (blue box in Figure 1) computes an instantaneous control signal u_{FC} based on the deviation between the output y of system (1) and the funnel MPC-based predicted y_{M}. The sum $u(t) = u_{\mathrm{FMPC}}(t) + u_{\mathrm{FC}}(t)$ is then applied to the actual system (1) at time instant t. The signal u_{FC} from the funnel controller is used to compensate for occurring disturbances, uncertainties in the model (2) and unmodelled dynamics. It is solely determined by the instantaneous values of the system output y, the funnel function ψ, and the prediction y_{M}. Therefore, the model-free component cannot *plan ahead*. This may result in large control values and a rapidly changing control signal if the actual output significantly deviates from its predicted counterpart, where the term *significant* is to be understood in comparison to the current funnel size. Numerical simulations in [8, 22] show that funnel MPC exhibits a considerably better controller performance than pure funnel control. In order to reduce the control effort of the funnel controller, a continuous *activation function* β is incorporated and has the following effect: whenever the deviation between predicted y_{M} and plant output y is acceptable (designer's choice), only u_{FMPC} is applied. A reasonable and simple choice for β is a ReLU-like map, which is zero below a given threshold and linear above.

Learning the model is the third component of the overall task addressed in the present article. Since funnel MPC exhibits better controller performance, and robust funnel MPC is able to compensate for model-plant mismatches, it is desirable to improve the model so that, preferably, the control u_{FMPC} is sufficient to achieve the tracking task with prescribed performance for the unknown system while satisfying the input constraints – in other words, it is desirable that the funnel controller is inactive most of the time. In Definition 3.2 we identify and establish properties of the learning component such that learning and updating the model preserves the structure necessary for robust funnel MPC [3]. The particular robustness w.r.t. model-plant mismatches of robust funnel MPC even allows to start with "no model", e.g., only an integrator chain, and then learn the remaining drift-dynamics. This is considered in a numerical simulation in Section 5.

3. System class and model class

In this section we formally introduce the class of systems to be controlled as well as the class of models to be used in the MPC part. We consider nonlinear multi-input multi-output systems. Since in our later analysis we mainly refer to the results of [3, 8], where the system is considered in so-called Byrnes-Isidori normal form, we assume that the system (1) can be transformed (by a transformation which does not need to be known) into input-output normal form

$$\dot{y}(t) = P(y(t), \zeta(t), d(t)) + \Gamma(y(t), \zeta(t), d(t))u(t), \qquad y(0) = y^0, \tag{4a}$$

$$\dot{\zeta}(t) = Q(y(t), \zeta(t), d(t)), \qquad \zeta(0) = \zeta^0, \tag{4b}$$

with control input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, a bounded internal or external disturbance $d : \mathbb{R}_{\geq 0} \to \mathbb{R}^a$, and output $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$. The system dynamics are governed by *unknown* nonlinear functions $P \in \mathcal{C}(\mathbb{R}^m \times \mathbb{R}^\kappa \times \mathbb{R}^a, \mathbb{R}^m)$ and $\Gamma \in \mathcal{C}(\mathbb{R}^m \times \mathbb{R}^\kappa \times \mathbb{R}^a, \mathbb{R}^{m\times m})$, where the latter is the so-called high-gain matrix. The last equation in (4) with $Q \in \mathcal{C}^1(\mathbb{R}^m \times \mathbb{R}^\kappa \times \mathbb{R}^a, \mathbb{R}^q)$ describes the internal dynamics, i.e. the dynamics within the system, which cannot be directly measured at the output, with dimension $\kappa := Z - m \in \mathbb{N}$ (Z is the state dimension in (1)). Referring to system (4) above, we formally introduce the class of systems under consideration in the present article.

Definition 3.1. *For a given funnel function $\psi \in \mathcal{G}$ and reference $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, we say that the system (4) belongs to the system class $\mathcal{N}$, written $(P, \Gamma, Q, d) \in \mathcal{N}$, if $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^a)$, the symmetric part of the high gain matrix Γ is positive definite, i.e. $\Gamma(\cdot) + \Gamma(\cdot)^\top > 0$, and the internal dynamics satisfy the following bounded-input bounded-state property*

$$\begin{aligned}
&\forall\, c_0 > 0\ \exists\, c_1 > 0\ \forall\, \zeta^0 \in \mathbb{R}^\kappa\ \forall\, d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^a)\\
&\forall\, y \in Y(y_{\mathrm{ref}}, \psi):\\
&\|\zeta^0\| + \|y\|_\infty + \|d\|_\infty \leq c_0 \Rightarrow \|\zeta(\cdot; 0, \zeta^0, y, d)\|_\infty \leq c_1,
\end{aligned}$$

where $\zeta(\cdot; 0, \zeta^0, y, d) : \mathbb{R}_{\geq 0} \to \mathbb{R}^\kappa$ denotes the unique global solution of (4b).

We remark that the globality of the solution ζ is ensured by the imposed conditions. Note that it is also possible to allow for a negative definite matrix $\Gamma + \Gamma^\top$. Then we may simply change the sign in the control (9) defined below. The last condition, namely bounded-input bounded-state stability of the internal dynamics, is a common condition for control systems, cf. [28, 29, 30]. In our particular case this ensures that the internal states, which cannot be measured directly, are bounded if the output of the system evolves within a funnel around the reference signal and the disturbance is bounded. For linear systems the function $Q(\cdot)$ consists of three matrices $Q_\zeta \in \mathbb{R}^{\kappa\times\kappa}$, $Q_y \in \mathbb{R}^{\kappa\times m}$ and $Q_d \in \mathbb{R}^{\kappa\times a}$ ($\dot{\zeta} = Q_\zeta \zeta + Q_y y + Q_d d$), and the system satisfies the bounded-input bounded-state property, if the matrix Q_ζ is Hurwitz (all eigenvalues have negative real part). Such systems are called *minimum phase*, cf. [31, 32].

For the unknown system (4) we consider a surrogate model

$$\dot{y}_{\mathrm{M}}(t) = p(y_{\mathrm{M}}(t), \eta(t)) + \gamma(y_{\mathrm{M}}(t), \eta(t))u(t), \qquad y_{\mathrm{M}}(0) = y_{\mathrm{M}}^0, \tag{5a}$$

$$\dot{\eta}(t) = q(y_{\mathrm{M}}(t), \eta(t)), \qquad \eta(0) = \eta^0, \tag{5b}$$

where properties of the functions p, γ and q are specified in Definition 3.2. Equation (5b) describes the internal dynamics. Referring to (5), we introduce the class of feasible models. This class is parameterized by the value $\bar{u}$, which ensures that for each member of the class the tracking task can be performed successfully by robust funnel MPC for a given reference signal and funnel boundary with the same upper bound $\bar{u}$ for the maximal control value.

Definition 3.2. *Let $\bar{u} > 0$. For given $\psi \in \mathcal{G}$ and $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ a model (5) is said to belong to the model class $\mathcal{M}_{\bar{u}}$, written $(p, \gamma, q, \eta^0) \in \mathcal{M}_{\bar{u}}$, if there exist $\nu \in \mathbb{N}_0$ and $\bar{\eta} \geq 0$ such that the vector $\eta^0 \in \mathbb{R}^\nu$ and the functions $p \in \mathcal{C}^1(\mathbb{R}^m \times \mathbb{R}^\nu, \mathbb{R}^m)$, $\gamma \in \mathcal{C}^1(\mathbb{R}^m \times \mathbb{R}^\nu, \mathbb{R}^{m \times m})$, and $q \in \mathcal{C}^1(\mathbb{R}^m \times \mathbb{R}^\nu, \mathbb{R}^\nu)$ satisfy the following conditions:*

(M.1) *$\gamma(\rho, \eta) \in \mathrm{Gl}_m(\mathbb{R})$ for all $(\rho, \eta) \in \mathbb{R}^m \times \mathbb{R}^\nu$, that is (5) has global strict relative degree one,*

(M.2) *the solutions of the internal dynamics (5b) are bounded by $\bar{\eta}$, that is*

$$\forall\, \eta^0 \in \mathbb{R}^\nu \ \forall\, y_{\mathrm{M}} \in Y(y_{\mathrm{ref}}, \psi): \ \left\| \eta(\cdot; 0, \eta^0, y_{\mathrm{M}}) \right\|_\infty \leq \bar{\eta},$$

where $\eta(\cdot; 0, \eta^0, y_{\mathrm{M}}) : \mathbb{R}_{\geq 0} \to \mathbb{R}^\nu$ denotes the global solution of (5b),

(M.3) *for the a-priori fixed maximal control input value $\bar{u}$ the model parameters, reference signal and funnel function satisfy the estimate*

$$G_{\max} \left(P_{\max} + \|\dot{\psi}\|_\infty + \|\dot{y}_{\mathrm{ref}}\|_\infty \right) \leq \bar{u},$$

where

$$\begin{aligned} G_{\max} &= \max_{(\rho,\eta) \in \Psi \times \mathcal{B}_{\bar{\eta}}} \left\| \gamma(\rho, \eta)^{-1} \right\|, \\ P_{\max} &= \max_{(\rho,\eta) \in \Psi \times \mathcal{B}_{\bar{\eta}}} \| p(\rho, \eta) \|, \\ \Psi &= \bigcup_{t \in \mathbb{R}_{\geq 0}} \{ \rho \in \mathbb{R}^m \,|\, \|\rho - y_{\mathrm{ref}}(t)\| \leq \psi(t) \}. \end{aligned}$$

Note that, although present in (5), the initial value y_{M}^0 is not part of the tuple $(p, \gamma, q, \eta^0) \in \mathcal{M}_{\bar{u}}$. The reason is that initializing the model output $y_{\mathrm{M}}(t_k)$ with the measured system output $y(t_k)$ at time instances t_k is part of Algorithm 1.

Condition (M.2) seems quite technical at first glace. It's intention is to ensure the following. If the internal state is bounded at every beginning of the MPC cycle, and the model output evolves within the funnel boundaries around the reference signal, then the internal dynamics evolve within an a-priori-fixed compact set. This condition is used in the feasibility proof of the funnel MPC algorithm, cf. [8, Lem. 4.8, Prop. 4.9], and also for the robust funnel MPC algorithm, cf. [3, Prop. 5.1, Thm. 3.13]. Condition (M.3) ensures that the a-priori prescribed maximal control effort $\bar{u}$ is sufficient to keep the tracking error within the funnel boundaries. Since $\bar{u}$ is fixed in advance, the models determined by the learning component must be amenable to the achievement of the tracking objective under control constraints $\|u\|_\infty \leq \bar{u}$.

There is some difference between the two classes $\mathcal{N}$ and $\mathcal{M}_{\bar{u}}$. Under certain technical assumptions, cf. [33, Cor. 5.7], system (4) as well as model (5) may be obtained via a transformation from a state-space representation (1) or (2), respectively. To avoid these technicalities, we restrict ourselves to the input-output representations (4) and (5). The system class $\mathcal{N}$ encompasses systems with bounded external or internal disturbances $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^a)$, while no such disturbances are allowed in the model class $\mathcal{M}_{\bar{u}}$. The high-gain matrix Γ of the system is required to have a positive definite symmetric part; in contrast, the high-gain matrix γ in the model is only required to be invertible.

Remark 3.3. *The dimension κ of the internal dynamics of system (4) is unknown but fixed. In contrast, the dimension $\nu \in \mathbb{N}_0$ of the model's internal dynamics can be considered as a parameter in the learning step. This means, in order to improve the model such that it "explains" the system measurements, the dimension of the internal state can be varied. Note that $\nu = 0$ (no internal dynamics) is explicitly allowed for the model.*

Remark 3.4. *One way to verify satisfaction of (M.2) is to apply [34, Thm. 4.3], which states the following. If there exists $V \in \mathcal{C}^1(\mathbb{R}^\nu, \mathbb{R}_{\geq 0})$ such that $V(\eta) \to \infty$ as $\|\eta\| \to \infty$, and for $q \in \mathcal{C}(\mathbb{R}^\nu \times \mathbb{R}^m, \mathbb{R}^\nu)$ we have $V'(\eta) \cdot q(\eta, \rho) \leq 0$ for all $\rho \in \mathbb{R}^m$ with $\|\rho\| \leq \|\psi\|_\infty + \|y_{\mathrm{ref}}\|_\infty$ and all $\eta \in \mathbb{R}^\nu$ with $\|\eta\| > \alpha$ for a predefined $\alpha > 0$. Then $\|\eta(t; 0, \eta^0, y_{\mathrm{M}})\|_\infty \leq \max\{\|\eta^0\|, \alpha\}$ for all $t \geq 0$, $\eta^0 \in \mathbb{R}^\nu$ and all $y_{\mathrm{M}} \in Y(y_{\mathrm{ref}}, \psi)$. Hence, fixing $V(\cdot)$ and $\alpha > 0$ in advance can be used to restrict choices of $q(\cdot)$ satisfying (M.2). We will make use of this fact later in Sections 4.2 and 4.3, where we discuss a particular learning scheme.*

4. Learning-based robust funnel MPC

In this section we develop the control algorithm and present our main result. First, we establish the control methodology to achieve the control objective introduced in Section 2. For the sake of completeness and readability, we recall the robust funnel MPC algorithm proposed in [3, Alg. 3.7], adapted to our framework. Before, we define an abstract learning scheme $\mathcal{L}$, which is incorporated in the algorithm. The idea of the learning component is to use measurement data from the model output y_{M}, the system output y, the funnel MPC signal u_{FMPC} and the funnel control signal u_{FC} to improve the model used for computation of u_{FMPC} in every MPC sampling interval (cf. Figure 1), under the condition that the updated model is still a member of the model class $\mathcal{M}_{\bar{u}}$ given in Definition 3.2.

For example, let $(y_{\mathrm{M}}, y, u_{\mathrm{FMPC}}, u_{\mathrm{FC}})|_{I_0^k}$ be given data collected up to $t = t_k$ over the interval $I_0^k := [0, t_k]$. This data is then used to choose $(p, \gamma, q, \eta^0) \in \mathcal{M}_{\bar{u}}$, i.e.,

$$\mathcal{L}\big((y_{\mathrm{M}}, y, u_{\mathrm{FMPC}}, u_{\mathrm{FC}})|_{I_0^k}\big) = (p, \gamma, q, \eta^0) \in \mathcal{M}_{\bar{u}}.$$

The function $\mathcal{L}$ receives the data, which are available up to the time instant $t \in \mathbb{R}_{\geq 0}$, and maps it to suitable model functions $(p, \gamma, q, \eta^0) \in \mathcal{M}_{\bar{u}}$. Note that $\mathcal{L}$ does not necessarily use all available data, see the discussion in Section 4.3.

Since all four signal types, the outputs y and y_{M} and the control signals u_{FMPC} and u_{FC}, are bounded, the space of data is L^∞; moreover, all of these four signals have the same dimension $m \in \mathbb{N}$. This motivates the following definition.

Definition 4.1. *For $\bar{u} > 0$, we call a function $\mathcal{L} : \bigcup_{t\geq 0} L^\infty([0,t],\mathbb{R}^m)^4 \to \mathcal{M}_{\bar{u}}$ a feasible learning scheme for robust funnel MPC with respect to the model class $\mathcal{M}_{\bar{u}}$, in short notation $\mathcal{M}_{\bar{u}}$-feasible.*

4.1. Control algorithm and main result

Now, we summarize the reasoning so far in the following algorithm, which achieves the control objective formulated in Section 2. It is a modification of [3, Alg. 3.7]. Here, the proper re-initialization of the model at every time step done in [3] is substituted by the learning component $\mathcal{L}$. Concerning notation, for an input $u \in L^\infty([t_k, t_k+T],\mathbb{R}^m)$, we denote by $y_{\mathrm{M}}(\cdot; t_k, y_k^0, \eta_k^0, u)$ the first component of the unique maximal solution of (5) under the initial condition $(y_{\mathrm{M}}(t_k), \eta(t_k)) = (y_k^0, \eta_k^0) \in \mathbb{R}^{m+\nu}$.

Algorithm 1 (Learning-based robust funnel MPC)**.**
Input*: instantaneous measurements of the output $y(t)$ of system (4), reference signal $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0},\mathbb{R}^m)$, funnel function $\psi \in \mathcal{G}$, an initial model $(p_0, \gamma_0, q_0, \eta_0^0) \in \mathcal{M}_{\bar{u}}$, and an $\mathcal{M}_{\bar{u}}$-feasible learning scheme $\mathcal{L}$.*
Initialisation*: Set time shift $\delta > 0$, prediction horizon $T \geq \delta$, and index $k := 0$. Define the time sequence $(t_k)_{k\in\mathbb{N}_0}$ by $t_k := k\delta$.*
Steps:

(a) Initialize the model (5) given by $(p_k, \gamma_k, q_k, \eta_k^0) \in \mathcal{M}_{\bar{u}}$ with the data $(y_{\mathrm{M}}(t_k), \eta(t_k)) = (y(t_k), \eta_k^0)$.

(b) **Funnel MPC**
For ℓ as in (3) compute a solution u_{FMPC} of the optimal control problem

$$\underset{\substack{u\in L^\infty(I^k,\mathbb{R}^m),\\ \|u\|_\infty \leq \bar{u}}}{\text{minimize}} \int_{I^k} \ell(t, y_{\mathrm{M}}(t; t_k, y(t_k), \eta_k^0, u), u(t))\,\mathrm{d}t, \tag{6}$$

over the interval $I^k := [t_k, t_k+T]$. Predict the output $y_{\mathrm{M}}(t) = y_{\mathrm{M}}(t; t_k, y(t_k), \eta_k^0, u_{\mathrm{FMPC}})$ of the model (5) on the interval $[t_k, t_{k+1}]$ and define the adaptive funnel $\varphi : [t_k, t_{k+1}] \to \mathbb{R}_{>0}$ by

$$\varphi(t) := \psi(t) - \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\|. \tag{7}$$

(c) **Funnel control**
Define the funnel control law with $y_{\mathrm{M}}|_{[t_k,t_{k+1}]}$ and funnel φ as in (7) by

$$u_{\mathrm{FC}}(t, y(t)) := -\beta\left(\tfrac{\|y(t)-y_{\mathrm{M}}(t)\|}{\varphi(t)}\right) \tfrac{\varphi(t)(y(t)-y_{\mathrm{M}}(t))}{\varphi(t)^2 - \|y(t)-y_{\mathrm{M}}(t)\|^2}, \tag{8}$$

for $t \in [t_k, t_{k+1})$, where the activation function $\beta \in \mathcal{C}([0,1],[0,\beta^+])$ with $\beta^+ > 0$ is such that $\beta(1) = \beta^+$. Apply the feedback control $\mu : [t_k, t_{k+1}) \times \mathbb{R}^m \to \mathbb{R}^m$ to system (4), given by

$$\mu(t, y(t)) = u_{\mathrm{FMPC}}(t) + u_{\mathrm{FC}}(t, y(t)). \tag{9}$$

(d) **Continual learning**
Increment k by 1, find a feasible model for the next sampling interval

$$\mathcal{L}\big((y_{\mathrm{M}}, y, u_{\mathrm{FMPC}}, u_{\mathrm{FC}})|_{I_0^k}\big) = (p_k, \gamma_k, q_k, \eta_k^0)$$

with $I_0^k := [0, t_k]$. Then go to Step (a).

Now we are in the position to formulate the main result of the present article, which extends [3, Thm. 3.13] by the learning component.

Theorem 4.2. *Consider a system (4) with $(P, \Gamma, Q, d) \in \mathcal{N}$ and initial values $y^0 \in \mathbb{R}^m$ and $\zeta^0 \in \mathbb{R}^\kappa$. For given $\bar{u} > 0$, choose an initial model (5) with $(p_0, \gamma_0, q_0, \eta_0^0) \in \mathcal{M}_{\bar{u}}$. Let a reference signal $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and a funnel function $\psi \in \mathcal{G}$ be given such that $\|y^0 - y_{\mathrm{ref}}(0)\| < \psi(0)$. Further, let $\mathcal{L}$ be an $\mathcal{M}_{\bar{u}}$-feasible learning scheme. Then, the learning-based robust funnel MPC Algorithm 1 with $\delta > 0$ and $T \geq \delta$ is initially and recursively feasible, i.e., at every time instance $t_k := k\delta$ for $k \in \mathbb{N}_0$ the OCP (6) has a solution $u_k^* \in L^\infty([t_k, t_k + T], \mathbb{R}^m)$, and the closed-loop system consisting of system (4) and the feedback (9) has a global solution $(y, \zeta) : [0, \infty) \to \mathbb{R}^m \times \mathbb{R}^\kappa$. In particular, each global solution (y, ζ) satisfies:*

(i) all signals are bounded, in particular, we have $u_{\mathrm{FMPC}}, u_{\mathrm{FC}}, y \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$,

(ii) the tracking error $y - y_{\mathrm{ref}}$ evolves within prescribed boundaries, i.e.,

$$\forall\, t \geq 0 : \ \|y(t) - y_{\mathrm{ref}}(t)\| < \psi(t).$$

Proof. To begin with, we show initial feasibility of Algorithm 1. By construction of $\mathcal{M}_{\bar{u}}$, and since $\|y^0 - y_{\mathrm{ref}}(0)\| < \psi(0)$, invoking [8, Prop. 4.8] there exists a control $u \in L^\infty([0, T], \mathbb{R}^m)$ with $\|u\|_\infty \leq \bar{u}$ such that $\|y_{\mathrm{M}}(t; 0, y^0, \eta_0^0, u) - y_{\mathrm{ref}}(t)\| < \psi(t)$ for all $t \in [0, T]$. Then [8, Thm. 4.5] yields that the OCP (6) has a solution. Moreover, $\varphi(t) = \psi(t) - \|y_{\mathrm{M}}(t) - y_{\mathrm{ref}}(t)\| > 0$ for all $t \in [0, \delta] \subset [0, T]$. Therefore, the closed-loop system (4), (8) has a solution on $[0, \delta]$ according to [35, Thm. 7]. Then the result [3, Thm. 3.13] is applicable and yields the existence of a solution of the closed-loop system (4) and (9) on $[0, \delta]$, where the initialization strategy in [3] is substituted by the initial choice of the model in Algorithm 1. Recursive feasibility can then be obtained as follows. We aim to apply [3, Prop. 5.1], which yields that the OCP in (6) has a solution for all $k \in \mathbb{N}$, if y_{ref} and ψ are defined globally, and the bound $\bar{u} > 0$ is sufficiently large for the actual model. The first two conditions are satisfied since $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $\psi \in \mathcal{G}$, and the last since the learning scheme is chosen to be $\mathcal{M}_{\bar{u}}$-feasible.

Next we have to take care about the possible induced discontinuities in the function φ in (7). These discontinuities, however, occur in both cases either by re-initialization as in [3], or due to the learning step in Algorithm 1. Therefore, the existence of a global solution of the closed loop system (4) and (9) follows with the same reasoning as in [3, Thm. 3.13]. Namely, $\|y(t_k) - y_{\mathrm{ref}}(t_k)\| < \varphi(t_k)$ is satisfied for each $k \in \mathbb{N}$ by the previous argument. Then, [35, Thm. 7] yields the existence of a solution of the closed-loop system (4), (8) on $[t_k, t_k+\delta]$ for all $k \in \mathbb{N}$. This also yields assertion (i). Assertion (ii) follows by the estimation $\|y(t)-y_{\mathrm{ref}}(t)\| = \|y(t)-y_{\mathrm{M}}(t)+(y_{\mathrm{M}}(t)-y_{\mathrm{ref}}(t))\| < \psi(t) - \|y_{\mathrm{M}}(t)-y_{\mathrm{ref}}(t)\| + \|y_{\mathrm{M}}(t)-y_{\mathrm{ref}}(t)\| = \psi(t)$ for all $t \geq 0$. $\square$

4.2. Learning scheme

In this section we derive sufficient conditions on the parameters of models to be learned, which then guarantee that a respective model is contained in $\mathcal{M}_{\bar{u}}$ for a fixed $\bar{u}$. Later, we invoke these conditions to discuss the $\mathcal{M}_{\bar{u}}$-feasibility of various learning programs. Since in many applications a linear model may serve as a good prediction model, we derive sufficient conditions on the parameters of linear systems

$$\begin{aligned} \dot{y}_{\mathrm{M}}(t) &= Ry_{\mathrm{M}}(t) + S\eta(t) + D_1 + \gamma u(t), \quad & y_{\mathrm{M}}(0) &= y_{\mathrm{M}}^0, \\ \dot{\eta}(t) &= Q\eta(t) + Py_{\mathrm{M}}(t) + D_2, & \eta(0) &= \eta^0. \end{aligned} \tag{10}$$

In the following, we denote by $\lambda_Q^+ < 0$ the largest eigenvalue of a symmetric negative definite matrix $Q = Q^\top < 0$. For a given reference signal $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and funnel boundary $\psi \in \mathcal{G}$ let $\bar{\rho} := \max_{\rho \in \Psi} \|\rho\|$ for the set Ψ defined in (M.3). Furthermore, for given numbers $\bar{\eta}, \bar{u}, \bar{r}, \bar{s}, \bar{\gamma}, \bar{p} \geq 0$ we define the following set of matrices, where we do not indicate the dependence on the parameters. Let

$$\bar{\mathcal{K}} := \mathbb{R}^{m\times m} \times \mathbb{R}^{m\times\nu} \times \mathrm{Gl}_m(\mathbb{R}) \times \mathbb{R}^m \times \mathbb{R}^{\nu\times\nu} \times \mathbb{R}^{\nu\times m} \times \mathbb{R}^\nu \times \mathbb{R}^\nu,$$

and define

$$\mathcal{K} := \left\{ \ (R, S, \gamma, D_1, Q, P, D_2, \eta^0) \in \bar{\mathcal{K}} \ \middle| \ (12) \ \right\}, \tag{11}$$

where

$$\|R\| \leq \bar{r}, \tag{12a}$$
$$\|S\| \leq \bar{s}, \tag{12b}$$
$$\|\gamma^{-1}\| \leq \bar{\gamma}, \tag{12c}$$
$$\bar{r}\bar{\rho} + s\bar{\eta} + \|D_1\| \leq \frac{\bar{u}}{\bar{\gamma}} - \|\dot{\psi}\|_\infty - \|\dot{y}_{\mathrm{ref}}\|_\infty, \tag{12d}$$
$$\lambda_Q^+ \leq -\frac{\bar{p}\bar{\rho} + \|D_2\|}{\bar{\eta}}, \tag{12e}$$
$$\|P\| \leq \bar{p}, \tag{12f}$$
$$\|\eta^0\| \leq \bar{\eta}. \tag{12g}$$

For $(R, S, \gamma, D_1, Q, P, D_2, \eta^0) \in \bar{\mathcal{K}}$ we define the functions

$$\begin{aligned} p_{R,S,D_1} &: \mathbb{R}^m \times \mathbb{R}^\nu \to \mathbb{R}^m, \quad & (y, \eta) &\mapsto Ry + S\eta + D_1, \\ q_{Q,P,D_2} &: \mathbb{R}^m \times \mathbb{R}^\nu \to \mathbb{R}^\nu, & (y, \eta) &\mapsto Py + Q\eta + D_2. \end{aligned} \tag{13}$$

Then, we may derive the following statement.

Proposition 4.3. *Let $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $\psi \in \mathcal{G}$ be given and $\bar{\rho} := \max_{\rho \in \Psi} \|\rho\|$. Moreover, let parameters $\bar{\eta}, \bar{u}, \bar{r}, \bar{s}, \bar{\gamma}, \bar{p} \geq 0$ be given. Then the set $\mathcal{K}$ defined in (11) satisfies the implication*

$$(R, S, \gamma, D_1, Q, P, D_2, \eta^0) \ \in \ \mathcal{K} \quad \Longrightarrow \quad (p_{R,S,D_1}, \gamma, q_{Q,P,D_2}, \eta^0) \ \in \ \mathcal{M}_{\bar{u}},$$

where p_{R,S,D_1} and q_{Q,P,D_2} are given in (13).

Proof. The proof consists of three parts, with one part devoted to each condition (M.1) – (M.3) in Definition 3.2, respectively.
Step one (M.1). The choice $\gamma \in \mathrm{Gl}_m(\mathbb{R})$ trivially satisfies condition (M.1).
Step two (M.2). We show for the given parameter $\bar{\eta} \geq 0$, that for any tuple of parameters contained in $\mathcal{K}$, the solution of (10) satisfies $\|\eta(t; 0, \eta^0, y_{\mathrm{M}})\| \leq \bar{\eta}$ for all $t \geq 0$ and all $y_{\mathrm{M}} \in Y(y_{\mathrm{ref}}, \psi)$. Note that any such y_{M} satisfies $\|y_{\mathrm{M}}\|_\infty \leq \bar{\rho}$. In virtue of Remark 3.4 we calculate for $t \geq 0$ that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{1}{2} \|\eta(t; 0, \eta^0, y_{\mathrm{M}})\|^2 &= \eta(t; 0, \eta^0, y_{\mathrm{M}})^\top \Big(Q\eta(t; 0, \eta^0, y_{\mathrm{M}}) + P y_{\mathrm{M}}(t) + D_2 \Big) \\ &\leq \|\eta(t; 0, \eta^0, y_{\mathrm{M}})\| \Big(\lambda_Q^+ \|\eta(t; 0, \eta^0, y_{\mathrm{M}})\| + \bar{p}\bar{\rho} + \|D_2\| \Big),\end{aligned}$$

which by $\lambda_Q^+ < 0$ is non-positive for $\|\eta(t; \eta^0, y_{\mathrm{M}})\| \geq (\bar{p}\bar{\rho} + \|D_2\|)/|\lambda_Q^+|$. Therefore, [34, Thm. 4.3] yields

$$\forall\, t \geq 0 : \ \|\eta(t; 0, \eta^0, y_{\mathrm{M}})\| \leq \max\{(\bar{p}\bar{\rho} + \|D_2\|)/|\lambda_Q^+|, \|\eta^0\|\}.$$

By assumption (12) we have $\|\eta^0\| \leq \bar{\eta}$ and $|\lambda_Q^+| \geq (\bar{p}\bar{\rho} + \|D_2\|)/\bar{\eta}$, hence it follows $\|\eta(t; 0, \eta^0, y_{\mathrm{M}})\| \leq \bar{\eta}$ for all $t \geq 0$.
Step three (M.3). We may estimate

$$\begin{aligned} G_{\max} &= \left\|\gamma^{-1}\right\| \leq \bar{\gamma}, \\ P_{\max} &= \max_{(\rho,\eta) \in \Psi \times \mathcal{B}_{\bar{\eta}}} \|p_{R,S,D_1}(\rho, \eta)\| \leq \bar{r}\bar{\rho} + \bar{s}\bar{\eta} + \|D_1\|,\end{aligned}$$

and hence

$$\begin{aligned} &G_{\max} \Big(P_{\max} + \|\dot{\psi}\|_\infty + \|\dot{y}_{\mathrm{ref}}\|_\infty \Big) \\ &\leq \bar{\gamma} \Big(\bar{r}\bar{\rho} + \bar{s}\bar{\eta} + \|D_1\| + \|\dot{\psi}\|_\infty + \|\dot{y}_{\mathrm{ref}}\|_\infty \Big) \overset{(12d)}{\leq} \bar{u}.\end{aligned}$$

This completes the proof. ☐

4.3. A particular learning scheme

With the set of parameters $\mathcal{K}$, the functions p_{R,S,D_1}, q_{Q,P,D_2} defined in (13), and Proposition 4.3, we may define a learning scheme $\mathcal{L}$ mapping from $\bigcup_{t \geq 0} L^\infty([0,t], \mathbb{R}^m)^4$ to the subset

$$\left\{ \, (p_{R,S,D_1}, \gamma, q_{Q,P,D_2}, \eta^0) \, \middle| \, (R, S, \gamma, D_1, Q, P, D_2, \eta^0) \in \mathcal{K} \, \right\}$$

of $\mathcal{M}_{\bar{u}}$, defined by

$$\mathcal{L} : ((y_{\mathrm{M}}, y, u_{\mathrm{FMPC}}, u_{\mathrm{FC}})|_{[0,t]}) \mapsto (p_{R,S,D_1}, \gamma, q_{Q,P,D_2}, \eta^0)$$

for some $t \geq 0$, where $(p_{R,S,D_1}, \gamma, q_{Q,P,D_2}, \eta^0)$ is determined by the solution of an optimization problem involving discrete measurements of the system data y and the applied control signals u_{FMPC} and u_{FC} at time instances $i\tau$ with $\tau > 0$, $i \in \mathbb{N}$ and $i\tau \leq t$ of the form

$$\begin{aligned} &\underset{(R,S,\gamma,D_1,Q,P,D_2,\eta^0)\in\mathcal{K}}{\text{minimize}} \quad J((y,z)|_{[0,t]}) \\ &\quad \text{s.t.} \;\; z(0) = z^0 \text{and for all } i \leq t/\tau : \\ &\qquad\quad z(i\tau) = \chi(\tau; z((i-1)\tau), (u_{\mathrm{FMPC}} + u_{\mathrm{FC}})((i-1)\tau)), \end{aligned} \tag{14}$$

where $J(\cdot)$ is a suitable cost function, $z = (\tilde{y}_{\mathrm{M}}, \eta)$ denotes the states of the linear model (10), and the expression $\chi(\cdot; z((i-1)\tau), (u_{\mathrm{FMPC}} + u_{\mathrm{FC}})((i-1)\tau))$ denotes its solution under the initial condition $\chi(0) = z((i-1)\tau)$ and with constant control $u(\cdot) \equiv (u_{\mathrm{FMPC}} + u_{\mathrm{FC}})((i-1)\tau)$. In the following we discuss some possible choices for the cost function $J(\cdot)$.

(i) $J((y,z)|_{[0,t]}) := \sum_{i=0}^{\lfloor t/\tau \rfloor} \xi_i \|\tilde{y}_{\mathrm{M}}(i\tau) - y(i\tau)\|^2$ with weights $\xi_i \geq 0$. The idea is to find a model in the set $\mathcal{K}$ which minimizes the weighted squared measured output errors. The weights ξ_i reflect the relative importance of the measurements $y(i\tau)$. In certain cases it might be beneficial to weight data points that are far in the past lower than current data points. By choosing $\xi_i > 0$ for all $i > 0$ all measured past data is taken into account. With increasing runtime of the algorithm, this results in a growing complexity of the optimization problem, computation time, and required memory space for the measurements. Therefore, this is not suitable in practice. Thus, it is beneficial to use a moving horizon estimation approach and only take the last N measurements into account and set $\xi_i = 0$ for $i < \lfloor t/\tau \rfloor - N$. In application, one has to find a good balance between considering many data points (large N), thus having a probably more accurate model, and low computation time and memory requirements (small N).

(ii) If the computation of the solution of the optimization problem has to be done very quickly, it is also possible to only consider the last measurement $y(\lfloor t/\tau \rfloor \tau)$. Thus, one might choose the cost function $J((y,z)|_{[0,t]}) := \|\tilde{y}_{\mathrm{M}}(\lfloor t/\tau \rfloor \tau) - y(\lfloor t/\tau \rfloor \tau)\|^2$. The idea is to find a model, which best explains the last MPC period in terms of output error, i.e., a model on the prediction interval $[t_k, t_{k+1}]$, so that with $\tau = \delta$ the error $\|\tilde{y}_{\mathrm{M}}(t_{k+1}) - y(t_{k+1})\|$ at the end of the interval is minimal.

(iii) In addition, it is worth considering to include regularization terms for the model parameters in the cost function. For the parameters $\mathcal{K}_i = (R_i, S_i, \gamma_i, D_{1,i}, Q_i, P_i, D_{2,i}\eta_i^0) \in \mathcal{K}$ one could either penalize the weighted distance of $\mathcal{K}_i$ to an a priori known tuple of parameters $\mathcal{K}^\star =$

$(R^\star, S^\star, \gamma^\star, D^\star, Q^\star, P^\star, D_2^\star, \eta^{0\star})$ and thus allow only small adaptions of the a priori known model or penalize the change of parameters $\mathcal{K}_i$ such that the model does only change slightly between two learning steps. This results in a cost function of the form $J((y,z)|_{[0,t]}) := \sum_{i=0}^{\lfloor t/\tau \rfloor} \big(\xi_i \|\tilde{y}_{\mathrm{M}}(i\tau) - y(i\tau)\|^2 + \sum_{j=1}^{8} k_i^j \|(\mathcal{K}_i^j - \tilde{\mathcal{K}}^j)\|\big)$, where $\tilde{\mathcal{K}} = \mathcal{K}^\star$ or $\tilde{\mathcal{K}} = \mathcal{K}_{i-1}$ and with weights $\xi_i, k_i^j \geq 0$. Here the expressions $\mathcal{K}_i^j, \tilde{\mathcal{K}}_i^j$ with $j = 1, \dots, 8$ refer to the j^{th} entry of the tuple $\mathcal{K}_i$, $\tilde{\mathcal{K}}_i$, respectively; for instance, $\mathcal{K}_i^2 = S_i$.

5. Numerical simulation

In this section we provide two numerical simulations to illustrate Algorithm 1. In the first example, we simulate the tracking task for a system of relative degree one, which is contained in the system class $\mathcal{N}$, and the corresponding surrogate model belongs to $\mathcal{M}_{\bar{u}}$, for a given $\bar{u}$. The second example goes beyond the system class $\mathcal{N}$, it is a system of relative degree two. We show that, although no theoretical results on its functioning are available yet, Algorithm 1 is successful in this case.

5.1. *Relative degree one.*

To illustrate the application of Algorithm 1, we consider a model of an exothermic chemical reaction which was also used in [8] to study funnel MPC and in [3] to study robust funnel MPC. The dynamics for one reactant ζ_1, a product ζ_2, and reactor temperature y is described by a system (4) where $m = 1$, $\kappa = 2$, $a = 0$, $\Gamma = 1$, $P(y,\zeta) = b_1\alpha(y,\zeta) - b_2 y$ and $Q(y,\zeta) = \left[\begin{smallmatrix} c_1 & 0 \\ 0 & c_2 \end{smallmatrix}\right] \alpha(y,\zeta) + d(\zeta^{\text{in}} - \zeta)$, with the Arrhenius law $\alpha : \mathbb{R}_{>0} \times \mathbb{R}^2_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by $\alpha(y,\zeta) := k_0 \mathrm{e}^{-k_1/y} \zeta_1$, and parameters k_0, k_1, b_1, d, $b_2 > 0$, $c_1 < 0$, $c_2 \in \mathbb{R}$, $\zeta^{\text{in}} \in \mathbb{R}^2_{\geq 0}$. The control objective is to steer the reactor's temperature y to a desired constant value $y_{\text{ref}} \equiv \bar{y}$ within prescribed funnel boundaries $\psi \in \mathcal{G}$, i.e., $\|y(t) - \bar{y}\| < \psi(t)$ for all $t \geq 0$. For the funnel MPC component of Algorithm 1 we consider linear models of the form (10) with $R, D_1 \in \mathbb{R}$, $S, D_2^\top, P^\top \in \mathbb{R}^{1\times 2}$, and $Q \in \mathbb{R}^{2\times 2}$. We assume $\gamma = 1$ and as initial model we choose $R = D_1 = 0 \in \mathbb{R}$, $S = D_2^\top = P^\top = 0 \in \mathbb{R}^{1\times 2}$, $Q = 0 \in \mathbb{R}^{2\times 2}$, and $\eta^0 = (0.02, 0.9)$. To improve this model over time, we adapt the matrices over a compact set $\mathcal{K}$ as in (11) at every fifth time step t_k by minimizing the plant-model mismatch based on the data of the last system output $y(t_{k-1})$, i.e., we solve the optimization problem

$$
\begin{aligned}
&\underset{(R,S,1,D_1,Q,P,D_2,\zeta(0))\in\mathcal{K}}{\text{minimize}} \quad \|y_{\mathrm{M}}(t_k) - y(t_k)\|^2 \\
&\text{s.t.} \ \tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} y_{\mathrm{M}}(t) \\ \eta(t) \end{pmatrix} = \begin{bmatrix} R & S \\ P & Q \end{bmatrix}\begin{pmatrix} y_{\mathrm{M}}(t) \\ \eta(t) \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \\
&\qquad \begin{pmatrix} y_{\mathrm{M}}(t_{k-1}) \\ \eta(t_{k-1}) \end{pmatrix} = \begin{pmatrix} y(t_{k-1}) \\ \zeta(0) \end{pmatrix},
\end{aligned}
$$

where $u(t) = u_{\mathrm{FMPC}}(t_{k-1}) + u_{\mathrm{FC}}(t_{k-1})$ which was applied to the model at the last time step t_{k-1} and $\zeta(0) = (\zeta_1(0), \zeta_2(0))$ is the vector of initial concentrations of the substances ζ_1 and ζ_2.

For the simulation we choose the funnel function $\psi(t) = 100\mathrm{e}^{-2t} + 1.5$. As in [3, 8], the initial data is $(y^0, \zeta_1(0), \zeta_2(0)) := (270, 0.02, 0.9)$, the reference signal is $y_{\mathrm{ref}} \equiv 337.1$, and the parameters are $c_1 = -1$, $c_2 = 1$, $k_0 = \mathrm{e}^{25}$, $k_1 = 8700$, $d = 1.1$, $b_1 = 209.2$, $b_2 = 1.25$, $\zeta_1^{\mathrm{in}} = 1$, $\zeta_2^{\mathrm{in}} = 0$. In this example we restrict the funnel MPC control to $\|u_{\mathrm{FMPC}}\|_\infty \le \bar{u} := 735$, choose $\lambda_u = 10^{-4}$, prediction horizon $T = 1$, and time shift $\delta = 0.1$. In accordance with Proposition 4.3, we choose for the set $\mathcal{K}$ as in (11) the parameters as $\bar{r} = 1.3$, $\bar{s} = 1.4$, $\bar{\eta} = 0.91$, $\bar{\rho} = 408.6$, $\bar{\gamma} = 1$, $\bar{p} = 1/400$, $\|D_1\| \le \frac{\bar{u}}{\bar{\gamma}} - \|\dot{\psi}\|_\infty - \|\dot{y}_{\mathrm{ref}}\|_\infty - \bar{r}\bar{\rho} - \bar{s}\bar{\eta} = 2.546$, and $\|D_2\| \le 3$ so that (12) is satisfied. Thus, the learning scheme is $\mathcal{M}_{\bar{u}}$–feasible. Due to discretisation, only step functions with a constant step length of 0.1 were considered to solve the OCP (6). The activation function of the funnel controller is constant $\beta \equiv 10$. Figure 2 shows the control signals and the

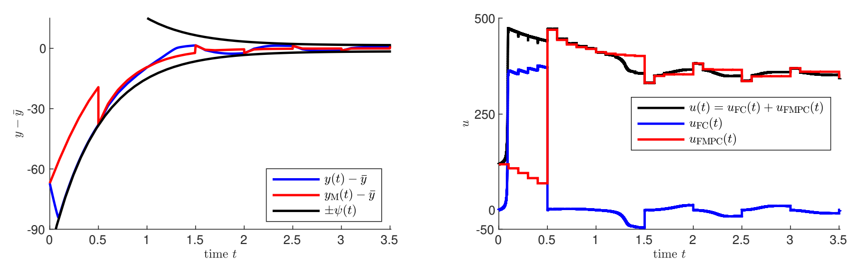

Figure 2: Application of learning based robust funnel MPC to the exothermic chemical reactor system

.

system and model output errors, respectively. It is evident that both $y_{\mathrm{M}} - y_{\mathrm{ref}}$ and $y - y_{\mathrm{ref}}$ remain within the predefined funnel boundaries ψ. Before the first learning step ($t \in [0, 0.5)$) the tracking error $y - y_{\mathrm{ref}}$ and the predicted error $y_{\mathrm{M}} - y_{\mathrm{ref}}$ diverge due to the poor quality of the initial model. However, since the tracking error is not close to the funnel boundary, the funnel controller remains inactive in the beginning and only reacts when the tracking error is close to the boundary. After the first learning step, the general direction of the predicted tracking error is consistent with the actual tracking error. The funnel controller still has to compensate for the model inaccuracies in order to guarantee that the tracking error remains within the boundaries, but with a significantly smaller contribution to the control signal. After each learning step, the model output jumps to the system output due to the newly updated model. The control signal u_{FC} is zero after each learning step since the system and model output coincide, and it becomes larger afterwards to compensate for the model inaccuracy. After $t = 1.5$ the system output is close to the desired constant reference signal. Thenceforth, the linear model is adequate to predict the system behavior and the control signal computed by funnel MPC is sufficient to achieve the tracking objective. The funnel controller only has to slightly compensate for model errors. We note that this example merely serves to illustrate that robust funnel MPC can be combined with $\mathcal{M}_{\bar{u}}$–feasible learning techniques. We do not claim that the learning algorithm used is superior to other methods.

5.2. Second numerical example: higher relative degree.

In this section, we present some promising preliminary results on extending the learning-based robust funnel MPC Algorithm 1 to a larger system class. Numerical simulations in [8] suggest that the funnel MPC algorithm is also applicable to systems of higher relative degree, meaning systems of the form

$$\begin{aligned} y^{(r)}(t) &= P(y(t), \dot{y}(t), \dots, y^{(r-1)}(t), \zeta(t)) + \Gamma(y(t), \dot{y}(t), \dots, y^{(r-1)}(t), \zeta(t))u(t), \\ \dot{\zeta}(t) &= Q(y(t), \dot{y}(t), \dots, y^{(r-1)}(t), \zeta(t)), \end{aligned}$$

with $r > 1$. Restricting the class of admissible funnel functions, utilizing an adapted cost function, and incorporating so-called feasibility constraints in the optimization problem, feasibility of funnel MPC for this system class was proved in [23]. It is an open problem whether it is sufficient to utilize the far simpler optimization (6) with cost function (3) and without the mentioned constraints. Accordingly, only the case of a relative degree one system was considered for robust funnel MPC in [3]. A generalisation to systems with higher relative degree has yet to be found.

To illustrate that, nevertheless, the learning-based robust funnel MPC Algorithm 1 shows promising results for this lager system class with fixed relative degree $r > 1$, we consider the example of a mass-on-car system from [36] which was was also used in [8]. The mass m_2 is mounted on a car with mass m_1 via a spring and damper system with spring constant $k > 0$ and damper constant $d > 0$ and moves on a ramp which is inclined by the angle $\vartheta \in [0, \frac{\pi}{2})$. The car can be controlled via the force u acting on it. The situation is depicted in Figure 3. The system can be described by the equations

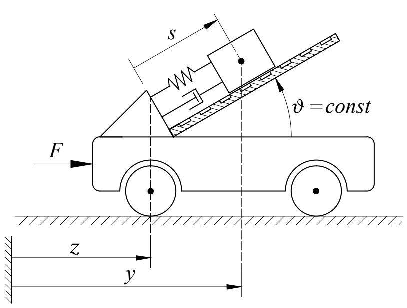

Figure 3: Mass-on-car system. The figure is based on the respective figures in [25], and [36].

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos(\vartheta) \\ m_2 \cos(\vartheta) & m_2 \end{bmatrix} \begin{pmatrix} \ddot{z}(t) \\ \ddot{s}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t) + d\dot{s}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix}. \tag{15}$$

The horizontal position of the car is $z(t)$ and the relative position of the mass on the ramp at time t is $s(t)$. The output y of the system is

$$y(t) = z(t) + s(t)\cos(\vartheta),$$

the horizontal position of the mass on the ramp. For the system we chose the same parameters $m_1 = 4$, $m_2 = 1$, $k = 2$, $d = 1$, $\vartheta = \pi/4$, and initial values $z(0) = s(0) = \dot{z}(0) = \dot{s}(0) = 0$ as in [8]. Then, as shown there, the system has a relative degree of $r = 2$. The objective is tracking of the reference signal $y_{\mathrm{ref}}(t) = \cos(t)$ such that the tracking error $y(t) - y_{\mathrm{ref}}(t)$ evolves within the prescribed performance funnel given by the function $\psi \in \mathcal{G}$ with $\psi(t) = 5\mathrm{e}^{-2t} + 0.2$.

For the model-based component of the controller, we solve the optimal control problem (6) with stage cost (3). The prediction horizon and time shift are selected as $T = 0.6$ and $\delta = 0.06$, resp. We restrict the funnel MPC control to $\|u_{\mathrm{FMPC}}\|_\infty \leq 25$ and choose the parameter $\lambda_u = 10^{-2}$ for the stage cost ℓ. Due to discretisation, only step functions with a constant step length of 0.06 were considered to solve the OCP (6). For the model-free component of the controller, we use the funnel controller for systems with relative degree two from [25] instead of (8). This component takes the form

$$\begin{aligned} w(t) &= \tfrac{\dot{e}_{\mathrm{M}}(t)}{\varphi(t)} + \alpha\left(\tfrac{e_{\mathrm{M}}(t)^2}{\varphi(t)^2}\right)\tfrac{e_{\mathrm{M}}(t)}{\varphi(t)}, \\ u_{\mathrm{FC}}(t) &= -\alpha(w(t)^2)w(t), \end{aligned}$$

with $e_{\mathrm{M}}(t) = y_{\mathrm{M}}(t) - y(t)$ and $\alpha(s) = \frac{1}{1-s}$ for $s \in [0, 1)$. The controller is applied to the system with a step size $0.6 \cdot 10^{-5}$. Similar to [22], where this problem was studied in the context of model identification during runtime, for the learning component we assume some knowledge about the structure of the system, but only limited information about the parameters and the initial value. We assume to know $\vartheta \in (0, 2\pi]$, $m_1 \in [1, 6]$, $m_2 \in [0.5, 1.5]$, $k \in [1, 3]$, $d \in [0.5, 1.5]$ and $z^0 := (x(0), \dot{x}(0), s(0), \dot{s}(0)) \in [-2.5, 3.5] \times [-2, 2] \times [-2.75, 3.25] \times [-2, 2]$. As initial model all model parameters where chosen equal to 1 and the initial state $z^0 = (0, 1, 0, 1)$. To learn the system parameters, we take measurements of the input-output data $((u_{\mathrm{FMPC}} + u_{\mathrm{FC}})(i\tau), y(i\tau))$ for $\tau = 0.006$ and $i \in \mathbb{N}_0$ and at every time $jT = 100j\tau$ for $j \in \mathbb{N}$ we solve the optimization problem

$$\begin{aligned} &\underset{\vartheta, m_1, m_2, k, d, z^0}{\text{minimize}} \quad \sum_{i=0}^{100j} \|\tilde{y}_{\mathrm{M}}(i\tau) - y(i\tau)\|^2 \\ &\text{s.t. } z(0) = z^0 \text{ and for all } i = 1, \ldots, 100j : \\ &\qquad z(i\tau) = \chi(\tau; z((i-1)\tau), (u_{\mathrm{FMPC}} + u_{\mathrm{FC}})((i-1)\tau)), \\ &\qquad \tilde{y}_{\mathrm{M}}(i\tau) = [1, \cos(\vartheta), 0, 0]\, z(i\tau), \end{aligned}$$

where $z = (x, \dot{x}, s, \dot{s})$ denotes the state of the mass on car system (15) and $\chi(\cdot; z((i-1)\tau), (u_{\mathrm{FMPC}} + u_{\mathrm{FC}})((i-1)\tau))$ denotes its solution under the initial condition $\chi(0) = z((i-1)\tau)$ and with constant control $u(\cdot) \equiv (u_{\mathrm{FMPC}} + u_{\mathrm{FC}})((i-1)\tau)$. Since only interval $[0, 4]$ is considered for the simulation, the entire history

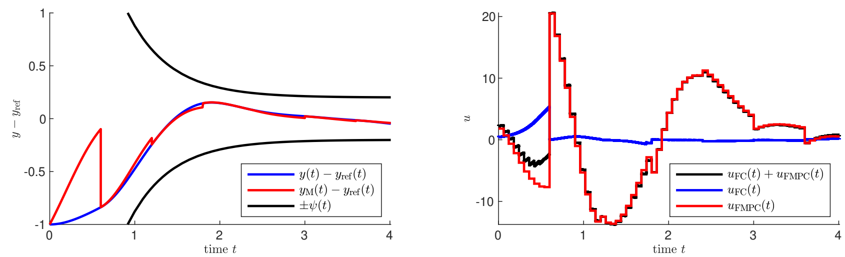

Figure 4: Simulation of system (15) under learning-based robust funnel MPC Algorithm 1.

of input-output data is considered in the optimization problem instead of a moving horizon approach as discussed in Section 4.3 (i).

All simulations are performed on the time interval $[0, 4]$ with MATLAB and the toolkit CASADI and are depicted in Figure 4. It is evident that the control scheme is feasible and achieves the control objective. Both errors $y_{\mathrm{M}} - y_{\mathrm{ref}}$ and $y - y_{\mathrm{ref}}$ evolve within the funnel boundaries given by ψ. Overall, a similar behavior as in the first simulation (initial divergence, small funnel control signals afterwards, jumps at each learning step, etc.) can be observed. We like to emphasize that already after the first learning step, the quality of the model is apparently quite good such that the funnel controller only has to slightly compensate for model errors and the control signal mainly consists of the control u_{FMPC} generated by the model-based controller component.

6. Conclusion

We extended the recently developed robust funnel MPC algorithm [3] by a learning component in order to improve the model-based control component by learning the model from system data. Based on the characterization of the model class and corresponding feasible learning structures, we showed that the interplay between pure feedback control, model predictive control and learning is consistent and successful; in particular, the control objective of output reference tracking with prescribed performance is achieved. Future research will focus on several particular aspects of the presented approach. Questions to be answered are, among other things, what it means to have a *good* learning scheme; which existing techniques (Willem's fundamental lemma [13], Koopman framework [37, 38], Neural Networks, etc.) can be used to exploit the data; how can knowledge about the system be incorporated into the learning scheme; to name but three aspects. Of special interest is the question of rigorously proving $\mathcal{M}_{\bar{u}}$-feasibility for more sophisticated learning algorithms.

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