Partial detectability and generalized functional observer design for linear descriptor systems

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Abstract

This paper studies linear time-invariant descriptor systems, which are not necessarily regular. We introduce the notion of partial detectability and characterize this concept by means of a simple rank criterion involving the system coefficient matrices. Some particular cases of this characterization are discussed in detail. Furthermore, we show that partial detectability is equivalent to the existence of a generalized functional estimator. Furthermore, it is necessary for the existence of a generalized functional observer, but not sufficient. We identify a condition which, together with partial detectability, gives sufficiency.

Keywords: Descriptor systems, Partial detectability, Wong sequences, Observer design, Generalized functional observer

1. Introduction

We consider linear time-invariant (LTI) descriptor systems of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{1a}$$
$$y(t) = Cx(t) + Du(t), \tag{1b}$$
$$z(t) = Kx(t), \tag{1c}$$

where $E,\ A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{m\times k},\ C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times k}$, and $K \in \mathbb{R}^{r\times n}$ are known matrices. Systems of type (1) are also called singular systems or systems described by differential-algebraic equations (DAEs). The first order matrix polynomial $(\lambda E - A)$, in the indeterminate λ, is called matrix pencil for (1). System (1) is called regular if $m = n$ and $\det(\lambda E - A)$ is not the zero polynomial in λ. In the present paper, we do not assume that the system is regular; in fact, no assumptions on the matrix pencil $(\lambda E - A)$ are made, and the system may be under- and/or over-determined. Descriptor systems occur naturally when dynamical systems are subject to algebraic constraints; for further motivation, we refer the readers to [1–6] and the references therein. Another situation that necessitates modeling with rectangular descriptor systems is encountered in observer design for standard state space systems with unknown inputs [7]. The augmentation of state variables with unknown inputs results in systems of the form (1).

We call $x : \mathbb{R} \to \mathbb{R}^n$ the semistate of the system (1), because unlike state space systems, $x(t)$ does not satisfy the semigroup property and cannot be arbitrarily initialized [8]. However, $x(t)$ contains the full information about all intrinsic properties of the system at time t. The functions $u : \mathbb{R} \to \mathbb{R}^k$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called the input and the output of system (1), respectively, and they are obtained by measurements, e.g., via sensors. The functional vector $z(t) \in \mathbb{R}^r$ contains those variables which cannot be measured, and estimators or observers are required to estimate them, cf. Definitions 3 and 4. If K is not the identity

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matrix, such an observer (estimator) is called a functional (or partial state) observer (estimator); otherwise, we call it a full state observer (estimator). The tuple $(x, u, y, z) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^p \times \mathbb{R}^r$ is said to be a solution of (1), if it belongs to the set

$$\mathscr{B} := \{(x, u, y, z) \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n+k+p+r}) \mid Ex \in \mathcal{AC}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^m) \text{ and } (x, u, y, z) \text{ satisfies (1) for almost all } t \in \mathbb{R}\},$$

where $\mathscr{L}^1_{\mathrm{loc}}$ is the set of measurable and locally Lebesgue integrable functions and $\mathcal{AC}_{\mathrm{loc}}$ represents the set of locally absolutely continuous functions. Descriptor systems based on the *behavior* $\mathscr{B}$ have been studied in detail e.g. in [9]. Exploiting the behavior $\mathscr{B}$, various observability and detectability concepts for descriptor systems (1) are studied in [10]. Moreover, existence conditions for full state and functional observers of descriptor systems have been investigated in [11–19], see also the references therein. Throughout the article, we assume that the behavior $\mathscr{B}$ is nonempty, which amounts to the existence of an admissible pair for (1), consisting of an admissible initial condition and input function, see also [11].

The major contribution of the current study is twofold. First, we prove that the system (1) is partially detectable if, and only if, a simple rank condition involving the system coefficient matrices holds. This result first requires to establish a precise definition of partial detectability for (1) based on the behavior $\mathscr{B}$. Roughly speaking, detectability means that the inputs and outputs determine the state variables asymptotically. From this point of view, the partial detectability of (1) is related to the asymptotic determination of $z(t)$ from the knowledge of $u(t)$ and $y(t)$. In a particular case of the characterization of partial detectability, we deduce that the existing algebraic characterization of partial detectability of state space systems in [20, Thm. 1] may give an erroneous result. The second major contribution is to show that partial detectability is equivalent to the existence of a generalized functional estimator. Moreover, we show that partial detectability is necessary for the existence of generalized functional observers, but not sufficient. Finally, we derive an additional condition, which, together with partial detectability, allows for the construction of a generalized functional observer. For this, we provide a step-by-step algorithm. Our approach is purely algebraic and based on simple matrix theory.

The paper is organized as follows. Section 2 collects some preliminary results used in the sequel of the article. In Section 3, the concept of partial detectability of system (1) is introduced along with algebraic test conditions and their equivalence is shown. Section 4 discusses some particular cases of the proposed results and emphasizes a significant modification to the existing theory of partial detectability for state space systems. In Section 5, we show that partial detectability is equivalent to the existence of a generalized functional estimator for system (1) and identify a condition, together with which it is also equivalent to the existence of a generalized functional observer. Section 6 considers a numerical example to illustrate the design algorithm. Finally, Section 7 concludes the article.

We use the following notations: The set of natural numbers is denoted by $\mathbb{N}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. 0 and I stand for appropriate dimensional zero and identity matrices, respectively. For more clarity, the identity matrix of size $n \times n$ is sometimes denoted by I_n. The set of complex numbers is denoted by $\mathbb{C}$, $\bar{\mathbb{C}}^+ := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) \geq 0\}$, and $\mathbb{C}^- := \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$. The symbols $A^\top$ and $\ker A$ denote the transpose and null space of a matrix $A \in \mathbb{R}^{m\times n}$, respectively. In a block partitioned matrix, all missing blocks are zero matrices of appropriate dimensions. The set $\sigma(M)$ denotes the spectrum of a matrix $M \in \mathbb{R}^{n\times n}$. The set $AM := \{\, Ax \,|\, x \in M \,\}$ is the image of a subspace $M \subseteq \mathbb{R}^n$ under $A \in \mathbb{R}^{m\times n}$ and $A^{-1}M := \{\, x \in \mathbb{R}^n \,|\, Ax \in M \,\}$ represents the pre-image of $M \subseteq \mathbb{R}^m$ under $A \in \mathbb{R}^{m\times n}$.

2. Preliminaries

In this section, we first recall some preliminary results from matrix theory and the theory of descriptor systems. These are fundamental to the development of the main results in this paper.

Lemma 1. *[21, Quasi-Kronecker Form (QKF)] For every matrix pencil $(\lambda E - A)$ with $E,\ A \in \mathbb{R}^{m\times n}$ there exist nonsingular matrices $P \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ such that*

$$P(\lambda E - A)Q = \begin{bmatrix} \lambda E_\epsilon - A_\epsilon & & & \\ & \lambda I_f - J_f & & \\ & & \lambda J_\sigma - I_\sigma & \\ & & & \lambda E_\eta - A_\eta \end{bmatrix} \tag{2}$$

where

1. $E_\epsilon,\ A_\epsilon \in \mathbb{R}^{m_\epsilon \times n_\epsilon}$, $m_\epsilon < n_\epsilon$, *and* $\operatorname{rank} E_\epsilon = \operatorname{rank}(\lambda E_\epsilon - A_\epsilon) = m_\epsilon$ *for all* $\lambda \in \mathbb{C}$.
2. $J_f \in \mathbb{R}^{n_f \times n_f}$.
3. $J_\sigma \in \mathbb{R}^{n_\sigma \times n_\sigma}$ *is nilpotent.*
4. $E_\eta,\ A_\eta \in \mathbb{R}^{m_\eta \times n_\eta}$, $m_\eta > n_\eta$, *and* $\operatorname{rank} E_\eta = \operatorname{rank}(\lambda E_\eta - A_\eta) = n_\eta$ *for all* $\lambda \in \mathbb{C}$.

The proof of the following lemma is evident and hence omitted.

Lemma 2. *For any matrices X and Y of compatible dimensions,* $\operatorname{rank}\begin{bmatrix} X \\ Y \end{bmatrix} = \operatorname{rank} X$ *if, and only if,* $\ker X \subseteq \ker Y$.

Lemma 3. *[16] Let $X \in \mathbb{R}^{m_1 \times r_1}$, $S \in \mathbb{R}^{m_1 \times r_2}$, and $Y \in \mathbb{R}^{m_2 \times r_2}$. If X has full row rank and/or Y has full column rank, then*

$$\operatorname{rank}\begin{bmatrix} X & S \\ 0 & Y \end{bmatrix} = \operatorname{rank} X + \operatorname{rank} Y.$$

We conclude this section by recalling the concept of a complex Wong sequence corresponding to (1a) from [14].

Definition 1. *For a given system* (1a) *with $E,\ A \in \mathbb{R}^{m \times n}$ and $\lambda \in \mathbb{C}$ the Wong sequence $\left\{\mathcal{W}^i_{[E,A],\lambda}\right\}_{i=0}^{\infty}$ is a sequence of complex subspaces, defined by*

$$\mathcal{W}^0_{[E,A],\lambda} := \{0\}, \quad \mathcal{W}^{i+1}_{[E,A],\lambda} := (A - \lambda E)^{-1}(E\mathcal{W}^i_{[E,A],\lambda}) \subseteq \mathbb{C}^n, \quad \mathcal{W}^*_{[E,A],\lambda} := \bigcup_{i \in \mathbb{N}} \mathcal{W}^i_{[E,A],\lambda}.$$

3. Partial detectability

The main aim of this section is to derive a simple rank condition for partial detectability of (1) in terms of the system coefficient matrices. First, we define the concept of partial detectability of (1) in terms of the behavior $\mathscr{B}$, which is a natural extension of the detectability of (1a)–(1b). Throughout the paper, the notation "$x(t) \to 0$ as $t \to \infty$" means "$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} ||x(t)|| = 0$".

Definition 2. *The descriptor system* (1) *is said to be partially detectable, if for all $(x_1, u, y, z_1), (x_2, u, y, z_2) \in \mathscr{B}$ we have that $z_1(t) - z_2(t) \to 0$ as $t \to \infty$.*

By linearity of the behavior $\mathscr{B}$ it is clear that (1) is partially detectable if, and only if, for all $(x, 0, 0, z) \in \mathscr{B}$ we have that $z(t) \to 0$ as $t \to \infty$. Therefore, partial detectability is independent of the matrices B and D.

Remark 1. *In* (1)*, if $K = I_n$, the above definition reduces to the detectability of* (1a)–(1b)*, see [10–12, 16]. Note that detectability is called "behavioral detectability" in [10].*

The aim of the remainder of this section is to derive a rank criterion for partial detectability of (1). Additionally to the new rank criterion, we include a characterization of partial detectability in terms of the Wong sequences, which was already implicitly contained in [14].

Let $l \in \mathbb{N}$, $\lambda \in \mathbb{C}$ and introduce the following notations:

$$\mathcal{E} = \begin{bmatrix} E \\ 0 \end{bmatrix}, \ \mathcal{A} = \begin{bmatrix} A \\ C \end{bmatrix}, \ \mathcal{G}_{l,[\mathcal{E},\mathcal{A}],\lambda} = \overbrace{\begin{bmatrix} \lambda\mathcal{E}-\mathcal{A} & & & & & \\ \mathcal{E} & \lambda\mathcal{E}-\mathcal{A} & & & & \\ & \ddots & \ddots & & & \\ & & \mathcal{E} & \lambda\mathcal{E}-\mathcal{A} & & \\ & & & \mathcal{E} & \lambda\mathcal{E}-\mathcal{A} & \\ & & & & \mathcal{E} & \lambda\mathcal{E}-\mathcal{A} \end{bmatrix}}^{l \text{ block columns}},$$

$$\text{and} \ \ \mathcal{G}_{l,[\mathcal{E},\mathcal{A},K],\lambda} = \overbrace{\begin{bmatrix} \lambda\mathcal{E}-\mathcal{A} & & & & & \\ \mathcal{E} & \lambda\mathcal{E}-\mathcal{A} & & & & \\ & \ddots & \ddots & & & \\ & & \mathcal{E} & \lambda\mathcal{E}-\mathcal{A} & & \\ & & & \mathcal{E} & \lambda\mathcal{E}-\mathcal{A} & \\ & & & & K & \end{bmatrix}}^{l \text{ block columns}}.$$

We are now ready to state the first main result of this paper.

Theorem 1. *For a given system* (1) *the following statements are equivalent:*

(a) The system (1) *is partially detectable.*

(b) The following condition holds:

$$\forall\, \lambda \in \bar{\mathbb{C}}^+ : \ \operatorname{rank} \mathcal{G}_{n,[\mathcal{E},\mathcal{A}],\lambda} = \operatorname{rank} \mathcal{G}_{n,[\mathcal{E},\mathcal{A},K],\lambda}. \tag{3}$$

(c) $\forall\, \lambda \in \bar{\mathbb{C}}^+ : \ \mathcal{W}^*_{[\mathcal{E},\mathcal{A}],\lambda} \subseteq \ker K$.

Proof. (a) $\Leftrightarrow$ (c): This follows from [14, Lem. A.4].

(b) $\Rightarrow$ (c): Fix $\lambda \in \bar{\mathbb{C}}^+$. In view of Lemma 2, condition (b) is equivalent to

$$\ker \mathcal{G}_{n,[\mathcal{E},\mathcal{A}],\lambda} \subseteq \ker \overbrace{\begin{bmatrix} 0 & 0 & \dots & 0 & K \end{bmatrix}}^{n \text{ block columns}}. \tag{4}$$

Now let $x \in \mathcal{W}^*_{[\mathcal{E},\mathcal{A}],\lambda}$ and observe that, since the Wong sequence $\left\{\mathcal{W}^i_{[\mathcal{E},\mathcal{A}],\lambda}\right\}_{i=0}^{\infty}$ terminates after finitely many steps, and in each step before termination the dimension of the associated space increases by at least one, it is clear that $\mathcal{W}^n_{[\mathcal{E},\mathcal{A}],\lambda} = \mathcal{W}^*_{[\mathcal{E},\mathcal{A}],\lambda}$. Thus $x \in \mathcal{W}^n_{[\mathcal{E},\mathcal{A}],\lambda}$ and there exist $x_1, \dots, x_n \in \mathbb{C}^n$ such that $x = x_n$ and

$$(\lambda\mathcal{E}-\mathcal{A})x_1 = 0, \ \mathcal{E}x_1 + (\lambda\mathcal{E}-\mathcal{A})x_2 = 0, \ \dots, \ \mathcal{E}x_{n-1} + (\lambda\mathcal{E}-\mathcal{A})x_n = 0. \tag{5}$$

Therefore, $(x_1^\top, \dots, x_n^\top)^\top \in \ker \mathcal{G}_{n,[\mathcal{E},\mathcal{A}],\lambda}$ and by (4) this gives $x = x_n \in \ker K$.

(c) $\Rightarrow$ (b): Fix $\lambda \in \bar{\mathbb{C}}^+$ and let $x \in \ker \mathcal{G}_{n,[\mathcal{E},\mathcal{A}],\lambda}$. We show that $x \in \ker[0, \dots, 0, K]$, which proves (4) and hence also (b). Write $x = (x_1^\top, \dots, x_n^\top)^\top$ with $x_1, \dots, x_n \in \mathbb{C}^n$, then (5) holds.

Therefore, $x_i \in \mathcal{W}^i_{[\mathcal{E},\mathcal{A}],\lambda}$ for $i = 1, \dots, n$ and since $\mathcal{W}^*_{[\mathcal{E},\mathcal{A}],\lambda} = \mathcal{W}^n_{[\mathcal{E},\mathcal{A}],\lambda}$ it follows from (c) that $x_n \in \ker K$. This completes the proof. □

In the following lemma, we derive a characterization for partial detectability in terms of the QKF of the matrix pencil $(\lambda\mathcal{E}-\mathcal{A})$. Later, this will be used in Section 5 to design a generalized functional estimator.

Lemma 4. *Consider a system* (1) *and let the matrix pencil* $(\lambda\mathcal{E}-\mathcal{A})$ *have QKF* (2) *such that* K *is partitioned accordingly, i.e.,*

$$KQ = \begin{bmatrix} K_\epsilon & K_f & K_\sigma & K_\eta \end{bmatrix}.$$

Then (1) *is partially detectable if, and only if,*

(a) $K_\epsilon = 0$ *and*
(b) $\forall\, \lambda \in \bar{\mathbb{C}}^+ :\ \ker(\lambda I_f - J_f)^n \subseteq \ker K_f$.

Moreover, if $J_f = \begin{bmatrix} J_{f_1} & \\ & J_{f_2} \end{bmatrix}$ *and* $K_f = \begin{bmatrix} K_{f_1} & K_{f_2} \end{bmatrix}$ *such that* $\sigma(J_{f_1}) \subseteq \bar{\mathbb{C}}^+$ *and* $\sigma(J_{f_2}) \subseteq \mathbb{C}^-$*, then (b) is equivalent to* $K_{f_1} = 0$.

Proof. Without loss of generality, assume that $(\lambda\mathcal{E} - \mathcal{A})$ is in QKF (2) and

$$K = \begin{bmatrix} K_\epsilon & K_f & K_\sigma & K_\eta \end{bmatrix}.$$

Then, as shown in [14, Eq. (A.4)], we have that

$$\mathcal{W}^*_{[\mathcal{E},\mathcal{A}],\lambda} = \mathbb{C}^{n_\epsilon} \times \ker(\lambda I_f - J_f)^n \times \{0\}^{n_\sigma} \times \{0\}^{n_\eta}.$$

By condition (c) in Theorem 1 it follows that partial detectability is equivalent to

$$\mathbb{C}^{n_\epsilon} \subseteq \ker K_\epsilon \quad \text{and} \quad \ker(\lambda I_f - J_f)^n \subseteq \ker K_f$$

for all $\lambda \in \bar{\mathbb{C}}^+$, which is equivalent to (a) and (b).

If $J_f = \begin{bmatrix} J_{f_1} & \\ & J_{f_2} \end{bmatrix}$ and $K_f = \begin{bmatrix} K_{f_1} & K_{f_2} \end{bmatrix}$, where $\sigma(J_{f_1}) \subseteq \bar{\mathbb{C}}^+$ and $\sigma(J_{f_2}) \subseteq \mathbb{C}^-$, then (b) is equivalent to

$$\forall\, \lambda \in \bar{\mathbb{C}}^+ :\ \ker \begin{bmatrix} (\lambda I - J_{f_1})^n & \\ & (\lambda I - J_{f_2})^n \end{bmatrix} \subseteq \ker \begin{bmatrix} K_{f_1} & K_{f_2} \end{bmatrix}.$$

Since $\sigma(J_{f_2}) \subseteq \mathbb{C}^-$ we have $\ker(\lambda I - J_{f_2})^n = \{0\}$, so the above condition is equivalent to

$$\forall\, \lambda \in \bar{\mathbb{C}}^+ : \qquad \ker(\lambda I - J_{f_1})^n \subseteq \ker K_{f_1}.$$

Since $\sigma(J_{f_1}) \subseteq \bar{\mathbb{C}}^+$, this condition is in turn equivalent to

$$\mathbb{C}^{n_{f_1}} = \bigcup_{\lambda \in \bar{\mathbb{C}}^+} \ker(\lambda I - J_{f_1})^n \subseteq \ker K_{f_1},$$

where n_{f_1} is the dimension of the square matrix J_{f_1}, thus $K_{f_1} = 0$. □

The following remarks are warranted on Theorem 1 and Lemma 4.

Remark 2. *It is apparent from the proof of Theorem 1 that, if* $\mathcal{W}^*_{[\mathcal{E},\mathcal{A}],\lambda} = \mathcal{W}^s_{[\mathcal{E},\mathcal{A}],\lambda}$ *for some* $s \in \mathbb{N}$*, then the number* n *in statement (b) of Theorem 1 can be replaced by* s*. Here, we use* n *because* s *is not known in advance and our main aim is to provide a condition directly in terms of the known data,* i.e.*, the system coefficient matrices. Moreover, to verify the partial detectability of* (1)*, it is sufficient to check condition* (3) *in Theorem 1 only for those finite eigenvalues of the matrix pencil* $(\lambda\mathcal{E} - \mathcal{A})$ *which belong to* $\bar{\mathbb{C}}^+$*. For the computation of finite eigenvalues, it is recommended to use the QKF* (2)*.*

Remark 3. *Any solution* $x(t)$ *of* (1a) *is uniquely determined if, and only if, the* ϵ*-blocks in the QKF* (2) *of* $(\lambda E - A)$ *are not present, see [22, Cor.* 2.4*]. So,* $z(t)$ *in* (1) *is uniquely determined if, and only if,* $K_\epsilon = 0$*. Thus, Lemma 4 reveals that partial detectability of system* (1) *implies that* $z(t)$ *in* (1) *is always uniquely determined, even if* $x(t)$ *is not unique.*

4. Particular cases

In this section, we discuss some particular cases of Theorem 1.

4.1. Detectability of descriptor system (1a)–(1b)

This case corresponds to $K = I_n$ in (1). By substituting $K = I_n$ in statement (c) in Theorem 1, it reduces to $\mathcal{W}^*_{[\mathcal{E},\mathcal{A}],\lambda} = \{0\}$, which means that, for each $i \in \mathbb{N}$ and $\lambda \in \bar{\mathbb{C}}^+$ we have $\mathcal{W}^i_{[\mathcal{E},\mathcal{A}],\lambda} = \{0\}$. For $i = 1$ we obtain that, for all $\lambda \in \bar{\mathbb{C}}^+$, $\ker(\lambda\mathcal{E} - \mathcal{A}) = \{0\}$, *i.e.*,

$$\forall\, \lambda \in \bar{\mathbb{C}}^+ : \ \operatorname{rank} \begin{bmatrix} \lambda E - A \\ C \end{bmatrix} = n, \tag{6}$$

which is the standard characterization of detectability of (1a)–(1b). In addition, when $E = I_n$, this becomes the classical characterization of detectability for standard state space systems.

4.2. Partial detectability of state space systems

If $E = I_n$ in (1), then Definition 2 provides the notion of partial detectability of standard state space systems. Moreover, the algebraic criterion (3) reduces to the condition

$$\forall\, \lambda \in \bar{\mathbb{C}}^+ : \ \operatorname{rank} \begin{bmatrix} (\lambda I - A)^n \\ \mathcal{O}^n_{[\lambda I - A, C]} \\ K \end{bmatrix} = \operatorname{rank} \begin{bmatrix} (\lambda I - A)^n \\ \mathcal{O}^n_{[\lambda I - A, C]} \end{bmatrix} \tag{7}$$

where $\mathcal{O}^n_{[\lambda I - A, C]} := \begin{bmatrix} C(\lambda I - A)^{n-1} \\ \vdots \\ C(\lambda I - A) \\ C \end{bmatrix}$. To prove (7), first consider $\mathcal{G}_{2,[\mathcal{E},\mathcal{A}],\lambda}$ and substitute $\mathcal{E} = \begin{bmatrix} I \\ 0 \end{bmatrix}$ and $\mathcal{A} = \begin{bmatrix} A \\ C \end{bmatrix}$, thus obtaining

$$\mathcal{G}_{2,[\mathcal{E},\mathcal{A}],\lambda} = \begin{bmatrix} \lambda I - A & 0 \\ -C & 0 \\ I & \lambda I - A \\ 0 & -C \end{bmatrix}. \tag{8}$$

Then, by multiplying the matrix on the right hand side of (8) with $U = \begin{bmatrix} I_n & & -(\lambda I - A) & \\ & I & C & \\ & & I_n & \\ & & & I \end{bmatrix}$ from the left, and then applying Lemma 3 with $X = I_n$, we obtain

$$\operatorname{rank} \mathcal{G}_{2,[\mathcal{E},\mathcal{A}],\lambda} = n + \operatorname{rank} \begin{bmatrix} (\lambda I - A)^2 \\ C(\lambda I - A) \\ C \end{bmatrix}.$$

By a similar calculation, it follows that

$$\operatorname{rank} \mathcal{G}_{2,[\mathcal{E},\mathcal{A},K],\lambda} = n + \operatorname{rank} \begin{bmatrix} (\lambda I - A)^2 \\ C(\lambda I - A) \\ C \\ K \end{bmatrix}.$$

With this argument applied to $\mathcal{G}_{n,[\mathcal{E},\mathcal{A}],\lambda}$ and $\mathcal{G}_{n,[\mathcal{E},\mathcal{A},K],\lambda}$ for $n > 2$, it can be shown that (3) reduces to (7).

In the articles [20, 23, 24], it has been reported (see e.g. [20, Thm. 1]) that a state space system is partially detectable (note that the notion is called "functional detectability" in these works) if, and only if,

$$\forall\, \lambda \in \bar{\mathbb{C}}^+ : \ \operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \\ K \end{bmatrix} = \operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix}. \tag{9}$$

However, this condition is obviously not equivalent to (7) and is incorrect in general. In the aforementioned works, it has been implicitly used that all eigenvalues of the matrix A with nonnegative real part are semisimple (i.e., their algebraic and geometric multiplicities coincide), but this assumption was not stated explicitly. In fact, if all eigenvalues of A having nonnegative real part are semisimple, then it is not hard to show that (7) reduces to (9). As an explicit counterexample for condition (9), consider the state space system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$, $z(t) = Kx(t)$, with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \text{ and } K = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}.$$

It is easy to verify that (9) is satisfied for $\lambda = 1$ and hence for all $\lambda \in \bar{\mathbb{C}}^+$. On the other hand, for $\lambda = 1$ condition (7) is not satisfied. And indeed, the system is not partially detectable: for initial data $x(0) = \begin{bmatrix} x_1^0 & x_2^0 & x_3^0 \end{bmatrix}^\top$ the solution is given by

$$\begin{aligned} x(t) &= \exp(At)x(0) = \begin{bmatrix} \exp(t) & 0 & 0 \\ 0 & \exp(t) & t\exp(t) \\ 0 & 0 & \exp(t) \end{bmatrix} \begin{bmatrix} x_1^0 \\ x_2^0 \\ x_3^0 \end{bmatrix}, \\ y(t) &= x_1(t) = x_1^0 \exp(t), \\ z(t) &= x_1(t) + x_3(t) = (x_1^0 + x_3^0)\exp(t). \end{aligned}$$

Thus, for $x_1^0 = 0$ and $x_3^0 \neq 0$, we have $y = 0$, but $z(t) \nrightarrow 0$ as $t \to \infty$.

4.3. Partial observability of state space systems

Again consider $E = I_n$ in (1) and assume that condition (7) holds true for all $\lambda \in \mathbb{C}$, *i.e.*,

$$\bigcup_{\lambda \in \mathbb{C}} \ker \begin{bmatrix} (\lambda I - A)^n \\ \mathcal{O}^n_{[\lambda I - A, C]} \end{bmatrix} \subseteq \ker K. \tag{10}$$

We show that this condition is equivalent to partial observability of state space systems as considered in [25]. In view of $\bigcup_{\lambda \in \mathbb{C}} \ker(\lambda I - A)^n = \mathbb{C}^n$, (10) is equivalent to

$$\bigcup_{\lambda \in \mathbb{C}} \ker \mathcal{O}^n_{[\lambda I - A, C]} \subseteq \ker K.$$

We show that the left hand side of the inclusion is equal to $\ker \mathcal{O}^n_{[A,C]}$. If $v \in \ker \mathcal{O}^n_{[A,C]}$, then

$$Cv = 0, \; CAv = 0, \; \ldots, \; CA^{n-1}v = 0.$$

Then, for arbitrary $\lambda \in \mathbb{C}$,

$$Cv = 0, \; C(\lambda I - A)v = 0, \; \ldots, \; C(\lambda I - A)^{n-1}v = 0, \quad i.e., \quad \ker \mathcal{O}^n_{[A,C]} \subseteq \bigcup_{\lambda \in \mathbb{C}} \ker \mathcal{O}^n_{[\lambda I - A, C]}.$$

On the other hand, $v \in \bigcup_{\lambda \in \mathbb{C}} \ker \mathcal{O}^n_{[\lambda I - A, C]}$ implies, for some $\lambda \in \mathbb{C}$,

$$Cv = 0, \; C(\lambda I - A)v = 0, \; \ldots, \; C(\lambda I - A)^{n-1}v = 0, \quad i.e., \quad Cv = 0, \; CAv = 0, \; \ldots, \; CA^{n-1}v = 0.$$

Thus,

$$\bigcup_{\lambda \in \mathbb{C}} \ker \mathcal{O}^n_{[\lambda I - A, C]} = \ker \mathcal{O}^n_{[A,C]}.$$

Hence, condition (10) is equivalent to

$$\ker \mathcal{O}^n_{[A,C]} \subseteq \ker K, \; i.e., \; \operatorname{rank} \begin{bmatrix} \mathcal{O}^n_{[A,C]} \\ K \end{bmatrix} = \operatorname{rank} \mathcal{O}^n_{[A,C]},$$

which coincides with the algebraic characterization of partial observability of state space systems given in [25, Eq. (8)].

5. Observer design

In this section, we propose an observer (estimator) of the following form to estimate the functional vector $z(t)$ in (1):

$$\dot{w}(t) = Nw(t) + H\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \tag{11a}$$

$$\hat{z}(t) = Rw(t) + \sum_{i=0}^{h-1} M_i \begin{bmatrix} u^{(i)}(t) \\ y^{(i)}(t) \end{bmatrix}, \tag{11b}$$

where $l,\ h \in \mathbb{N}_0$, $N \in \mathbb{R}^{l\times l}$, $H \in \mathbb{R}^{l\times(k+p)}$, $R \in \mathbb{R}^{r\times l}$, $M_i \in \mathbb{R}^{r\times(k+p)}$, $i = 0, \ldots, h-1$. A system of the form (11) is called a generalized functional observer (estimator), because it includes derivatives of the input and output variables. The integers l and h denote the order and index of the observer (estimator), respectively. To the best of our knowledge, generalized observers were first proposed by Hou and Müller [11] to estimate the full state of linear descriptor systems.

We exploit the behavior $\mathscr{B}$ to give a precise definition of generalized functional observers (estimators) for (1), similar to [14, Def. 3.2]. We like to note that the notion of functional observers for state space systems goes back to the seminal work [26] by Luenberger.

Definition 3. *System* (11) *is said to be a generalized functional estimator for* (1)*, if for every* $(x, u, y, z) \in \mathscr{B}$ *there exist* $w \in \mathcal{AC}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l)$ *and* $\hat{z} \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^r)$ *such that* $(w, u, y, \hat{z})$ *satisfy* (11) *for almost all* $t \in \mathbb{R}$*, and for all* w*,* $\hat{z}$ *with this property we have*

$$\hat{z}(t) - z(t) \to 0 \text{ for } t \to \infty.$$

Definition 4. *System* (11) *is said to be a generalized functional observer for* (1)*, if*

(a) (11) *is a generalized functional estimator for* (1)*, and*
(b) for any $(x, u, y, z) \in \mathscr{B}$*, there exist* $w \in \mathcal{AC}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l)$ *and* $\hat{z} \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^r)$ *such that* $(w, u, y, \hat{z})$ *satisfy* (11) *with* $\hat{z}(0) = z(0)$*, we have* $\hat{z}(t) = z(t)$ *for almost all* $t > 0$*.*

Notably, any system (11) can be written as a regular descriptor system in the form

$$E_o\dot{x}_o(t) = A_o x_o(t) + B_o\begin{bmatrix} u(t) \\ y(t) \end{bmatrix}, \tag{12a}$$

$$\hat{z}(t) = C_o x_o(t), \tag{12b}$$

where $x_o := \begin{bmatrix} w \\ w_1 \end{bmatrix}$, $E_o := \begin{bmatrix} I & \\ & E_1 \end{bmatrix}$, E_1 is a nilpotent matrix with nilpotency index h, $A_o := \begin{bmatrix} N & \\ & I \end{bmatrix}$, $B_o := \begin{bmatrix} H \\ H_1 \end{bmatrix}$, and $C_o := \begin{bmatrix} R & R_1 \end{bmatrix}$ such that $R_1 E_1^i H_1 = -M_i$ for $0 \le i \le h-1$. Clearly, upon substituting $w_1(t) = -\sum_{i=0}^{h-1} E_1^i H_1 \begin{bmatrix} u^{(i)}(t) \\ y^{(i)}(t) \end{bmatrix}$ from (12a) in (12b), (12) reduces to (11). Since the matrix pencil $(\lambda E_o - A_o)$ is regular, this shows that the class of observers (estimators) (11) is a special subclass of the observer (estimator) systems considered in [14], where the notion "partial state observer (estimator)" is used and the observers (estimator) are constructed as descriptor systems using a system copy. Thus, remarkably, system (11) constitutes a regular descriptor system where the algebraic constraints have been resolved, and results from fixing the dimension of the innovations in the observer class in [14].

Now, we establish a relation between partial detectability and the existence of a generalized functional estimator for (1).

Theorem 2. *There exists a generalized functional estimator of the form* (11) *for a given system* (1) *if, and only if,* (1) *is partially detectable.*

Proof. ($\Rightarrow$): Let $(x, 0, 0, Kx) \in \mathscr{B}$ be arbitrary. Then $w = 0$ and $\hat{z} = 0$ satisfy (11) with $u = 0$ and $y = 0$, thus we have (since (11) is a generalized functional estimator)

$$z(t) - \hat{z}(t) = Kx(t) \to 0 \text{ for } t \to \infty. \tag{13}$$

Therefore, since in particular $\mathcal{E}\dot{x}(t) = \mathcal{A}x(t)$, it follows from (13) and [14, Lem. A.4] that condition (c) in Theorem 1 is satisfied and hence (1) is partially detectable.

($\Leftarrow$): Assume that the system (1) is partially detectable. First, utilizing Lemma 1, we compute nonsingular matrices P_1 and Q_1 such that the matrix pencil $P_1(\lambda\mathcal{E} - \mathcal{A})Q_1$ is in QKF (2). Then, we compute a non-singular matrix U_1 such that

$$U_1^{-1} J_f U_1 = \begin{bmatrix} J_{f_1} & \\ & J_{f_2} \end{bmatrix}, \tag{14}$$

where $\sigma(J_{f_1}) \subseteq \bar{\mathbb{C}}^+$ and $\sigma(J_{f_2}) \subseteq \mathbb{C}^-$. The existence of such U_1 is guaranteed by the Jordan canonical form of J_f. Further, we find (e.g. using the SVD or QR factorization) a non-singular matrix U_2 such that

$$U_2(\lambda E_\eta - A_\eta) = \begin{bmatrix} \lambda I_{n_\eta} - A_{\eta_1} \\ -A_{\eta_2} \end{bmatrix}.$$

Finally, we define

$$P := \begin{bmatrix} I & & & \\ & U_1^{-1} & & \\ & & I & \\ & & & U_2 \end{bmatrix} P_1,\ Q := Q_1 \begin{bmatrix} I & & & \\ & U_1 & & \\ & & I & \\ & & & I \end{bmatrix},\ P \begin{bmatrix} B & 0 \\ D & -I_p \end{bmatrix} = \begin{bmatrix} B_\epsilon \\ B_{f_1} \\ B_{f_2} \\ B_\sigma \\ B_{\eta_1} \\ B_{\eta_2} \end{bmatrix},\ x = Q \begin{bmatrix} x_\epsilon \\ x_{f_1} \\ x_{f_2} \\ x_\sigma \\ x_\eta \end{bmatrix},$$

$$\text{and } KQ = \begin{bmatrix} K_\epsilon & K_{f_1} & K_{f_2} & K_\sigma & K_\eta \end{bmatrix}.$$

Since system (1) is partially detectable, it follows from Lemma 4 that $K_\epsilon = 0$ and $K_{f_1} = 0$. Thus, in the new coordinates, the problem of generalized functional estimator design for system (1) reduces to the problem of generalized functional estimator design for

$$\begin{aligned} J_\sigma \dot{x}_\sigma(t) &= x_\sigma(t) + B_\sigma \bar{u}(t), && (15a) \\ \begin{bmatrix} \dot{x}_{f_2}(t) \\ \dot{x}_\eta(t) \end{bmatrix} &= \begin{bmatrix} J_{f_2} & \\ & A_{\eta_1} \end{bmatrix} \begin{bmatrix} x_{f_2}(t) \\ x_\eta(t) \end{bmatrix} + \begin{bmatrix} B_{f_2} \\ B_{\eta_1} \end{bmatrix} \bar{u}(t), && (15b) \\ 0 &= A_{\eta_2} x_\eta(t) + B_{\eta_2} \bar{u}(t), && (15c) \\ z(t) &= K_\sigma x_\sigma(t) + \begin{bmatrix} K_{f_2} & K_\eta \end{bmatrix} \begin{bmatrix} x_{f_2}(t) \\ x_\eta(t) \end{bmatrix}, && (15d) \end{aligned}$$

where $\bar{u} = \begin{bmatrix} u \\ y \end{bmatrix}$. Since $\operatorname{rank} \begin{bmatrix} \lambda I_{n_\eta} - A_{\eta_1} \\ -A_{\eta_2} \end{bmatrix} = n_\eta$ for all $\lambda \in \mathbb{C}$ by Lemma 1, there exists $L \in \mathbb{R}^{n_\eta \times (m_\eta - n_\eta)}$ such that $\sigma(A_{\eta_1} - LA_{\eta_2}) \subseteq \mathbb{C}^-$. Define the following system:

$$\begin{aligned} \dot{w}(t) &= Nw(t) + \begin{bmatrix} B_{f_2} \\ B_{\eta_1} - LB_{\eta_2} \end{bmatrix} \bar{u}(t), && (16a) \\ \hat{z}(t) &= Rw(t) - \sum_{i=0}^{h-1} K_\sigma J_\sigma^i B_\sigma \bar{u}^{(i)}(t), && (16b) \end{aligned}$$

where $N = \begin{bmatrix} J_{f_2} & 0 \\ 0 & A_{\eta_1} - LA_{\eta_2} \end{bmatrix}$ and $R = \begin{bmatrix} K_{f_2} & K_\eta \end{bmatrix}$. For any $(x, u, y, z) \in \mathscr{B}$ we have that $(x_\sigma, x_{f_2}, x_\eta, \bar{u}, z)$ is a solution of (15). Clearly, $\bar{u}$ must be sufficiently smooth by (15a) and hence for all $(x, u, y, z) \in \mathscr{B}$

there exists a solution $(w, \bar{u}, \hat{z})$ of (16). For any such solution, we define $e(t) = z(t) - \hat{z}(t)$ and $e_1(t) = \begin{bmatrix} x_{f_2}(t) \\ x_\eta(t) \end{bmatrix} - w(t)$, then using (15c) we obtain

$$\dot{e}_1(t) = Ne_1(t) + \begin{bmatrix} 0 \\ L\underbrace{\big(A_{\eta_2} x_\eta(t) + B_{\eta_2}\bar{u}(t)\big)}_{=0} \end{bmatrix} = Ne_1(t), \tag{17a}$$

$$e(t) = Re_1(t). \tag{17b}$$

Since $\sigma(N) \subseteq \mathbb{C}^-$ by construction, $e(t) \to 0$ for $t \to \infty$. Therefore, $\hat{z}(t) \to z(t)$ for $t \to \infty$. □

Remark 4. *The estimator design procedure can actually be turned into a numerically stable algorithm by*

- *using the staircase form from [27] instead of the QKF,*
- *using the GUPTRI form from [28] for the decomposition of the stable and unstable eigenvalues of the matrix J_f in* (14)*,*

and some other small but straightforward modifications. However, the presentation of this algorithm is quite technical and we leave the details to the reader.

Now we turn to the question of existence of generalized functional observers. We like to note that a necessary and sufficient condition for the existence of generalized functional observers in a more general form (which are descriptor systems again) was derived in [14, Thm. 3.5]. Since we consider a smaller class of admissible observers here, the conditions from [14] are not sufficient for the existence of an observer anymore. Likewise, although partial detectability is necessary, it is not sufficient for the existence of generalized functional observers. In the following result, which is a direct consequence of the error dynamics (17) and [14, Lem. A.1], we derive a condition which together with partial detectability also yields the existence of a generalized functional observer.

Corollary 1. *Consider a system* (1) *which is partially detectable and consider an observer candidate of the form* (16) *with gain matrix $L \in \mathbb{R}^{n_\eta \times (m_\eta - n_\eta)}$ such that $\sigma(A_{\eta_1} - LA_{\eta_2}) \subseteq \mathbb{C}^-$. Then*

$$\operatorname{rank} R = \operatorname{rank} \mathcal{O}^l_{[N,R]} \tag{18}$$

if, and only if, (16) *is a generalized functional observer for* (1)*, where $N = \begin{bmatrix} J_{f_2} & 0 \\ 0 & A_{\eta_1} - LA_{\eta_2} \end{bmatrix}$, $R = \begin{bmatrix} K_{f_2} & K_\eta \end{bmatrix}$, and $l = n_{f_2} + n_\eta$.*

Remark 5. *Denote by Σ_1 the class of descriptor systems* (1) *which are partially detectable, by Σ_2 the class for which there exists a generalized functional observer of the form* (11)*, and by Σ_3 the class for which there exists a gain matrix L with $\sigma(N) \subseteq \mathbb{C}^-$ and which satisfies* (18)*. By Corollary 1 the set Σ_3 consists of those descriptor systems for which* (16) *is a generalized functional observer. Furthermore, Theorem 2 and Corollary 1 imply the following inclusion:*

$$\Sigma_1 \supseteq \Sigma_2 \supseteq \Sigma_3.$$

We show that each inclusion is strict. To see that $\Sigma_1 \neq \Sigma_2$, consider a system (1) *with the following coefficient matrices:*

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 \end{bmatrix}, \; D = 0, \quad \text{and } K = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

Then, invoking Theorem 1, it is easy to verify that this system is partially detectable and hence belongs to Σ_1. To see that no generalized functional observer exists for this system, consider an arbitrary observer candidate of the form (11)*. Choose initial values $x_1(0) = 1$, $x_2(0) = -1$, $w(0) = 0$, and input function*

$u \equiv 0$. Invoking $y(t) = 0$ we find that $\dot{w}(t) = Nw(t)$ with $w(0) = 0$, thus $w(t) = 0$ for all $t \geq 0$. As a consequence $\hat{z}(t) = Rw(t) + \sum_{i=0}^{h-1} M_i \bar{u}^{(i)}(t) = 0$ for all $t \geq 0$, but $z(t) = x_1(t) + x_2(t) = \exp(-t) - \exp(-2t)$. Thus it holds $\hat{z}(0) = z(0)$, but $\hat{z}(t) \neq z(t)$ for all $t > 0$. This proves that condition (b) in Definition 4 is not satisfied and hence the system does not belong to Σ_2.

To see that $\Sigma_2 \neq \Sigma_3$, consider the following example for a system (1):

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad D = 1, \quad K = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Then with $P = I_3$ and $Q = I_2$ the system is already in the form (15) given by

$$\begin{aligned} \dot{x}_\eta(t) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_\eta(t) + \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \bar{u}(t), \\ 0 &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_\eta(t) + \begin{bmatrix} 1 & -1 \end{bmatrix} \bar{u}(t), \\ z(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_\eta(t). \end{aligned}$$

Resolving the equations we find that

$$z(t) = \dot{y}(t) - \dot{u}(t) - u(t),$$

and hence, clearly, there exists a generalized functional observer of the form (11) for the system. However, any observer of the form (16) with $L = \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right]$ reads

$$\begin{aligned} \dot{w}(t) &= \begin{bmatrix} 0 & -\alpha \\ 1 & -\beta \end{bmatrix} w(t) + \begin{bmatrix} 1-\alpha & \alpha \\ 1-\beta & \beta \end{bmatrix} \bar{u}(t), \\ \hat{z}(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} w(t) \end{aligned}$$

Therefore, $\sigma(N) \subseteq \mathbb{C}^-$ requires $\alpha \neq 0$, but then

$$\operatorname{rank} \begin{bmatrix} 1 & 0 \end{bmatrix} = 1 \neq 2 = \operatorname{rank} \begin{bmatrix} 1 & 0 \\ 0 & -\alpha \end{bmatrix},$$

so (18) is not satisfied.

Remark 6. *In a couple of works, e.g. [18, 20, 29, 30], estimators as in Definition 3 are considered, yet they are called "observer"; property (b) from Definition 4 is not considered. For more details on property (b) in Definition 4, see Example 1 in [31].*

6. Numerical illustration

In this section, we illustrate our theoretical findings and validate the proposed algorithm with a numerical example. Consider system (1) with the coefficient matrices

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \; D = 0, \text{ and } K = \begin{bmatrix} -1 & 1 & 1 & -1 & 1 \end{bmatrix}.$$

This system is partially detectable by Theorem 1. Following the estimator design procedure in the proof of Theorem 2, we obtain

$$P = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0.5 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Furthermore, the reduced system (15) is of the form

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x}_\sigma(t) &= x_\sigma(t) + \begin{bmatrix} 0 & 0 \\ 0.5 & -0.5 \end{bmatrix} \bar{u}(t), \\ \begin{bmatrix} \dot{x}_{f_2}(t) \\ \dot{x}_\eta(t) \end{bmatrix} &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_{f_2}(t) \\ x_\eta(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \bar{u}(t), \\ 0 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_\eta(t) + \begin{bmatrix} 0 & 0 \\ 0.5 & 0.5 \end{bmatrix} \bar{u}(t), \\ z(t) &= \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} x_{f_2}(t) \\ x_\eta(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \end{bmatrix} x_\sigma(t). \end{aligned}$$

Since the matrix $\begin{bmatrix} J_{f_2} & 0 \\ 0 & A_{\eta_1} \end{bmatrix}$ is stable, by choosing $L = 0$ we find that condition (18) is satisfied and we obtain a generalized functional observer of the form (11) given by

$$\begin{aligned} \dot{w}(t) &= \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} w(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \bar{u}(t), \\ \hat{z}(t) &= \begin{bmatrix} -1 & 1 \end{bmatrix} w(t) + \begin{bmatrix} 0.5 & -0.5 \end{bmatrix} \bar{u}(t) + \begin{bmatrix} 0.5 & -0.5 \end{bmatrix} \bar{u}^{(1)}(t). \end{aligned}$$

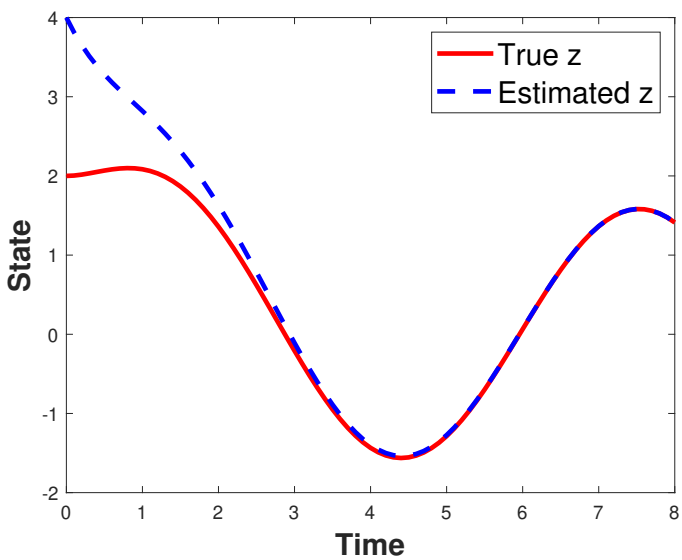

Figure 1: Time response of true and estimated $z(t)$.

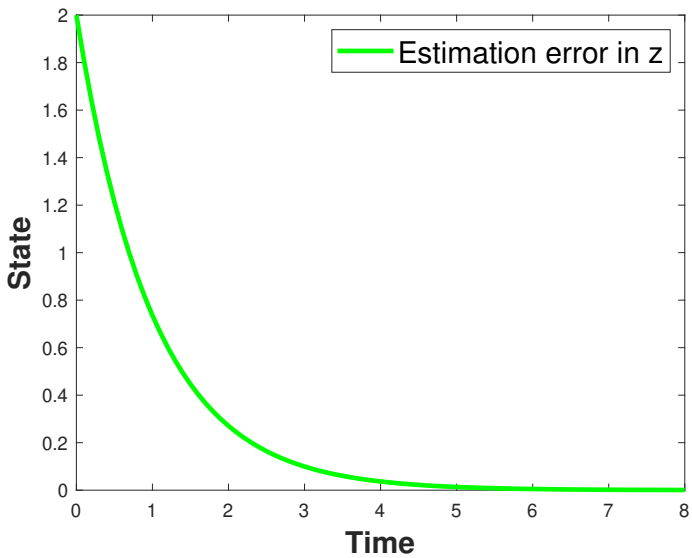

Figure 2: Time response of observer error $e(t)$.

Figure 1 and 2 show the responses of the true and estimated values of $z(t)$ and the time response of the estimation error, respectively. The simulation is realized in MATLAB R2020a and the reduced system (15) and the observer (16) are solved by using the ode15s solver with relative tolerance 10^{-6}. For the simulation, we used the parameters $\begin{bmatrix} x_{f_2}(0) \\ x_\eta(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $w(0) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, and $\bar{u}(t) = \begin{bmatrix} \sin(t) \\ -\sin(t) \end{bmatrix}$.

7. Conclusion

In the present paper, we established a precise mathematical definition and the algebraic characterization of partial detectability for LTI descriptor systems. In addition to this, incorrect results for the algebraic characterization of partial detectability of state space systems in the literature have been pointed out and corrected. It turned out that the proposed concept of partial detectability is equivalent to the existence of a generalized functional estimator, and it is necessary for the existence of a generalized functional observer for LTI descriptor systems. Together with a new rank condition, it is also sufficient and a step-by-step observer (estimator) design procedure was presented. Future research will focus on closing the gap between the system classes discussed in Remark 5 by additional conditions and finding a characterization of descriptor systems for which a generalized functional observer exists.

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