On Model Predictive Funnel Control with Equilibrium Endpoint Constraints*

Jens Göbel[1,4] Dario Dennstädt[2,3] Lukas Lanza[2] Karl Worthmann[2] Thomas Berger[3] Tobias Damm[1,4]

***Abstract*— We propose *model predictive funnel control*, a novel model predictive control (MPC) scheme building upon recent results in funnel control. The latter is a high-gain feedback methodology that achieves evolution of the measured output within predefined error margins. The proposed method dynamically optimizes a parameter-dependent error boundary in a receding-horizon manner, thereby combining prescribed error guarantees from funnel control with the predictive advantages of MPC. This approach promises faster optimization times due to a reduced number of decision variables, whose number does not depend on the horizon length, as well as improved robustness due to a continuous feedback law to deal with the intersampling behavior. In this paper, we focus on proving stability by leveraging results from MPC stability theory with terminal equality constraints. Moreover, we rigorously show initial and recursive feasibility.**

Keywords: Adaptive control, Funnel control, Nonlinear output feedback, Predictive control for nonlinear systems, Prescribed transient behavior

I. Introduction

We address the problem of stabilizing a nonlinear dynamical control system subject to time-varying output constraints defined by user-specific performance bounds (funnel boundaries). To achieve this, we propose a novel bi-level control framework that synergies *funnel control* and *model predictive control* (MPC). The architecture comprises:

- A lower-level fixed funnel control law ensuring constraint satisfaction.
- An upper-level MPC-based optimizer tuning the funnel control parameters (see Figure 1).

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[1]Fraunhofer Institute for Industrial Mathematics (ITWM), 67663 Kaiserslautern, Germany, jens.goebel@itwm.fraunhofer.de

[2]Technische Universität Ilmenau, Optimization-based Control Group, Institute of Mathematics, 98693 Ilmenau, Germany, {lukas.lanza, karl.worthmann}@tu-ilmenau.de

[3]Universität Paderborn, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany, thomas.berger@math.upb.de, dario.dennstaedt@uni-paderborn.de

[4]Rheinland-Pfälzische Technische Universität Kaiserslautern-Landau, 67663 Kaiserslautern, Germany, tdamm@rptu.de

Funnel control is a model-free controller that achieves output reference tracking, first described in [1]. It has since garnered significant attention in academic research, see, e.g. [2], [3] and references therein. A related concept is the so-called prescribed performance control (PPC) methodology, first proposed in [4] and further developed in, e.g., [5], [6]. Recently, in [7], PPC has been connected to control barrier functions [8] to find optimization-based reactive feedback laws. Both concepts (funnel control and PPC) are high-gain adaptive feedback control methodologies, where the gain grows towards infinity as the tracking error approaches a predefined error boundary. This means, near the error boundary an inward pointing condition is active, i.e., the error is being pushed back from the boundary. Thus, under structural assumptions on the system, such as a globally well-defined relative degree and a high-gain property, it can be shown that the tracking error evolves within the boundaries. While early results in funnel control rely on the diameter of the funnel being bounded away from zero, in [9] asymptotic tracking is achieved for systems with relative degree one. These results have been extended to systems with arbitrary relative degree in [3]. In the current paper, we will use the results presented in [10], where it was proven that it is possible to achieve exact tracking in prescribed finite time with funnel control.

MPC is a well-established and versatile control methodology that has been widely applied both in research and industry since the 1980s [11]. Unlike the model-free funnel control, MPC relies on a model of the controlled process. It optimizes the predicted system behavior over a discretized input signal according to a pre-defined cost function. While powerful, the computational complexity of solving this optimization problem within strict sampling time constraints grows with the horizon length, posing a significant difficulty in real-time applications [12], [13]. Beyond that, the open-loop application of the control signal between optimization steps limits its robustness against disturbances. One approach to ensure constraint satisfaction under external disturbances or even under model mismatch is tube-based MPC [14], [15]. These methods construct tubes that encompass the uncertainties of the system and always contain the actual system trajectory. However, these tubes are not free of choice.

Our integrated method, termed *model predictive funnel control* (MPFC), merges MPC and funnel control by optimizing a parameter-dependent funnel shape using an MPC-like strategy. At each MPFC step, the funnel shape parameters are optimized over the current prediction

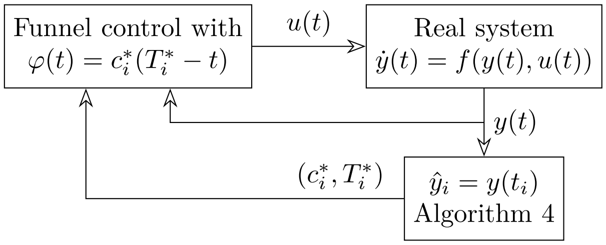


Fig. 1: Feedback control for unknown real system with optimal funnel function parameters. The model is initialized with system data $y(t_i)$. Then, Algorithm 4 provides parameters (c_i^*, T_i^*) to the funnel controller (4). The latter is applied continuously to the system.

horizon for the funnel-controlled closed-loop system. This approach significantly reduces the number of optimization variables compared to classical MPC, where the decision vector comprises the entire discretized input trajectory and grows with the horizon length. Moreover, MPFC provides a stabilizing closed-loop control law even between two recalculation steps, thereby enhancing its robustness relative to traditional MPC, where the computed control is applied in an open-loop fashion until the algorithm's next iteration. Since the actual control of the system is given by a funnel control law, the prescribed performance is maintained even if the model used in the optimization step does not fully match the real system.

To prove the stability of MPFC, we utilize the well-known concept of *equilibrium endpoint constraints* from classical MPC stability theory, as covered e.g. in [16, Sec. 5.2]. This type of stability proof relies on MPC predictions reaching the system's equilibrium by the end of the prediction horizon ensured by additional constraints. This enables a cost-neutral extension of the horizon, which is exploited to prove the boundedness of the closed-loop costs. In case of the presented version of MPFC, the results from [10] ensure reaching the equilibrium during the prediction horizon quite naturally, without additional constraints.

While prior work has explored combinations of funnel control and MPC under the name *funnel MPC* in a series of papers, see [17], [18] and the references therein, our approach differs fundamentally in both methodology and scope. Existing funnel MPC implementations retain the classical funnel control paradigm of a static, user-defined funnel shape and instead optimize only the applied control values. This penalizes proximity of the system output to the given boundaries via rising costs but does not adapt the funnel geometry itself. Other MPC approaches that consider prescribing output constraints usually focus on discrete-time systems instead [19].

The funnel shape considered in this work, as well as in [10], differs from classical funnel control in that it reaches zero in prescribed finite time $T > 0$. Due to this 'singularity' in T, the closed-loop ODE defined by (7) requires infinitely small step sizes in the numerical solution. The results developed in this brief paper prove the feasibility of the proposed control algorithm from a theoretical perspective. Relaxing the choice of the performance funnel to more general functions such as in [3] and facilitating the numerical implementation is topic of near future research.

Nomenclature. For $x, y \in \mathbb{R}^n$, we use $x^\top y =: \langle x, y\rangle$, and $\|x\| = \sqrt{\langle x, x\rangle}$. For $V \subseteq \mathbb{R}^m$, we denote by $\mathcal{C}^k(V;\mathbb{R}^n)$ the set of k-times continuously differentiable functions $f : V \to \mathbb{R}^n$. For an interval $I \subseteq \mathbb{R}$ and $k \in \mathbb{N}$, $L^\infty(I;\mathbb{R}^n)$ is the Lebesgue space of measurable, essentially bounded functions $f : I \to \mathbb{R}^n$ with norm $\|f\|_\infty = \operatorname{esssup}_{t\in I}\|f(t)\|$.

II. System class and control objective

We introduce the system class under consideration and formulate the control objective precisely. Consider a non-linear multi-input multi-output control system

$$\dot{y}(t) = f(y(t), u(t)), \qquad y(0) = \hat{y}_0 \in \mathbb{R}^m, \tag{1}$$

where $y(t) \in \mathbb{R}^m$ denotes the output, and $u(t) \in \mathbb{R}^m$ denotes the input of the system at time $t \geq 0$, respectively. Note that the dimension of the input and output coincide. The function $f \in \mathcal{C}(\mathbb{R}^m \times \mathbb{R}^m;\mathbb{R}^m)$ is assumed to satisfy the following high-gain property:

Definition 1 (High-gain property [3, Def. 1.2]): For $m \in \mathbb{N}$, a function $f \in \mathcal{C}(\mathbb{R}^m \times \mathbb{R}^m;\mathbb{R}^m)$ satisfies the *high-gain property*, if there exists $\nu \in (0, 1)$ such that, for every compact $K \subset \mathbb{R}^m$, the continuous function

$$\chi : \mathbb{R} \to \mathbb{R},$$
$$s \mapsto \min\left\{\langle v, f(z, -sv)\rangle \,|\, z \in K, v \in \mathbb{R}^m, \nu \leq \|v\| \leq 1\right\}$$

satisfies $\sup_{s\in\mathbb{R}} \chi(s) = \infty$.

Intuitively, the high-gain property states that the system reacts fast, if the input is large. It is therefore a core ingredient when aiming for tracking with prescribed performance via reactive (non-predictive) feedback. Using the high-gain property, we define the class of systems under consideration in this paper.

Definition 2 (System class $\mathcal{S}$): A system (1) belongs to the system class $\mathcal{S}$, if f satisfies the high-gain property (Definition 1) and $f(0, 0) = 0$.

We aim to design a prediction-based feedback controller to stabilize system (1), i.e., $y(t) \to 0$. Moreover, the output should evolve within prescribed margins, i.e.,

$$\forall\, t \geq 0 : \ \|y(t)\| \leq \psi(t) \tag{2}$$

for some positive function ψ given by the control engineer. This predefined performance funnel can be chosen as $\psi \in \mathcal{C}^1(\mathbb{R}_{\geq 0};\mathbb{R})$, with $\psi(t) > 0$ for all $t \geq 0$, or as $\psi(\cdot) \equiv \infty$ if no restrictions are posed on the transient behavior of the system.

In the this paper, we say a (locally) absolutely continuous function $y : [0, \omega) \to \mathbb{R}^m$, $\omega \in (0, \infty]$, with $y(0) = y^0 \in \mathbb{R}^m$ is a solution (in the sense of Carathéodory) to (1), if y satisfies (1) for almost all $t \in [0, \omega)$. A solution y is maximal, if it has no right extension that is also a solution; and global, if $\omega = \infty$.

III. Control Methodology

We introduce the control methodology to achieve the control objective (2) for systems in $\mathcal{S}$ by combining model-free funnel control with model-based optimization. To this end, we first recall the specific version of funnel control at play, before proposing model predictive funnel control.

A. Exact Funnel Control

To utilize well established arguments from MPC stability analysis (continuation with zero costs), we aim to ensure $y(T) = 0$ for some $T < H$, where $H > 0$ is the prediction horizon. To this end, we employ results from [10], where a feedback controller has been proposed, which achieves exact tracking in predefined finite time. We briefly recap the controller design [10]. For $c, T > 0$, define the *funnel boundary* by

$$\varphi(t; c, T) := c(T - t), \quad t \in [0, T). \tag{3}$$

Choose a continuous bijection $\alpha_c : [0, 1) \to [2c, \infty)$, and a continuous surjection $N : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Feasible choices are, e.g., $\alpha_c(s) = 2c/(1 - s)$ and $N(s) = s \cos(s)$. Typically, the control direction is known ($\pm$). In this case, the choice of N simplifies to $N(s) = \pm s$, cf. [3, Rem. 1.8] With these functions and parameters, we recall [10, Thm. 3.1], adapted to the current setting.

Lemma 3 ([10], Thm. 3.1): For φ given in (3), consider a system (1) contained in $\mathcal{S}$ with $\|\hat{y}_0\| < \varphi(0) = cT$. Then, the application of the feedback law

$$u(t) = (N \circ \alpha_c)(\|y(t)\|^2/\varphi(t)^2)\tfrac{y(t)}{\varphi(t)} \tag{4}$$

to system (1) yields a closed-loop initial value problem which has an absolutely continuous maximal solution $y\colon [0, T) \to \mathbb{R}^m$. Moreover, the solution satisfies $\|y(t)\| < \varphi(t)$ for all $t \in [0, T)$, $\lim_{t\to T} \|y(t)\| = 0$, and $u \in L^\infty([0, T]; \mathbb{R}^m)$.

Note that, [10, Thm. 3.1] states $u \in L^\infty([0, T); \mathbb{R}^m)$ but the control can be extended to the closed interval. We will uses this fact in the analysis of the proposed algorithm.

B. Model Predictive Funnel Control

We now present model predictive funnel control, which combines the funnel control scheme depicted in the previous section with model-based predictions to optimize a parameter-dependent funnel shape on a receding horizon. For symmetric positive semi-definite matrices $Q, R \in \mathbb{R}^{m\times m}$, define the *stage costs* by

$$\ell : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}_{\geq 0},\ (y, u) \mapsto \langle y, Qy\rangle + \langle u, Ru\rangle. \tag{5}$$

In addition, choose an *optimality tolerance* $\varepsilon_{\text{opt}} > 0$, a *step size* $h > 0$, and *horizon length* $H := nh$ for fixed $n \in \mathbb{N}_{\geq 2}$ and define the resampling times $t_i = ih$, $i \in \mathbb{N}_0$.

As in Section III-A, define the funnel boundary by (3) and choose a surjection $N \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$ and a bijection $\alpha_c \in \mathcal{C}([0, 1); [2c, \infty))$, depending on the parameter c. These functions define the funnel feedback law (4) up to a specific funnel shape parameterization, which is defined by a choice from the following set: For $t \geq 0$, $\hat{y} \in \mathbb{R}^m$, and an *outer funnel function* ψ, define the *feasible set* by

$$\begin{aligned}\mathcal{F}_H(t, \hat{y}) := \big\{(c, T) \in \mathbb{R}_{>0} \times (0, H] \;\big|\; \|\hat{y}\| < \varphi(0; c, T),\\ \forall\, \tau \in [0, T)\colon\ \varphi(\tau; c, T) \leq \psi(t + \tau)\big\}.\end{aligned} \tag{6}$$

Consider a system (1) of class $\mathcal{S}$. For $(c, T) \in \mathcal{F}_H(t, \hat{y})$, denote the *cost function* by

$$J_H(\hat{y}, c, T) := \int_0^H \ell(y(\tau), u(\tau))\, \mathrm{d}\tau + c \tag{7a}$$

$$\text{s.t. } \dot{y}(\tau) = f(y(\tau), u(\tau)), \quad y(0) = \hat{y}, \tag{7b}$$

$$\text{and } u(\tau) = \begin{cases}(N\circ\alpha_c)\Big(\Big\|\frac{y(\tau)}{\varphi(\tau;c,T)}\Big\|^2\Big)\frac{y(\tau)}{\varphi(\tau;c,T)}, & \tau < T\\ 0, & \text{else.}\end{cases} \tag{7c}$$

Define the *optimal value function* as $V_H(t, \hat{y}) := \inf_{(c,T)\in\mathcal{F}_H(t,\hat{y})} J_H(\hat{y}, c, T)$. For $T \leq 0$ and any $c > 0$, set $J_H(\hat{y}, c, T) := 0$. With this, we may now define the control algorithm.

Algorithm 4 (Model Predictive Funnel Control):

Input: A system (1) of class $\mathcal{S}$, outer funnel boundary $\psi \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R})$ with $\psi(t) > 0$ for all $t \geq 0$, or $\psi \equiv \infty$, stage cost ℓ as in (5), $\varepsilon_{\text{opt}} > 0$, $h > 0$, $H := nh$ for $n \in \mathbb{N}_{\geq 2}$, surjection $N \in \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$, bijections $\alpha_c \in \mathcal{C}([0, 1); [2c, \infty))$ for $c > 0$.

$i \leftarrow 0$

loop

At time t_i, measure the output $y(t_i)$ of system (1) and set $\hat{y}_i = y(t_i) \in \mathbb{R}^m$.

if $i = 0$ **then**

Find $(c_0^*, T_0^*) \in \mathcal{F}_H(0, \hat{y}_0)$ such that

$$J_H(\hat{y}_0, c_0^*, T_0^*) \leq V_H(0, \hat{y}_0) + \varepsilon_{\text{opt}}$$

else

Find $(c_i^*, T_i^*) \in \mathcal{F}_H(t_i, \hat{y}_i)$ such that

$$\begin{aligned}J_H(\hat{y}_i, c_i^*, T_i^*) \leq \min\big\{J_H(\hat{y}_i, c_{i-1}^*, T_{i-1}^* - h),\\ V_H(t_i, \hat{y}_i) + \varepsilon_{\text{opt}}\big\}\end{aligned} \tag{8}$$

On the interval $[t_i, t_i + h)$ apply (7c) with $\alpha_{c_i^*}$ and $\varphi(t - t_i; c_i^*, T_i^*)$ to system (1)

$i \leftarrow i + 1$

IV. Main Results

We present the two main results. In Theorem 5, we prove that Algorithm 4 is initially and recursively feasible for all initial values $\hat{y}_0$ with $\|\hat{y}_0\| < \psi(0)$. Utilizing well-known results of stability via equilibrium terminal constraints, cf. [16, Sec. 5.2], Theorem 8 shows that the closed-loop application of the algorithm is stabilizing.

Theorem 5: Let $\psi \in \mathcal{C}^1(\mathbb{R}_{\geq 0}; \mathbb{R}_{>0})$ with $\psi(t) > 0$ for all $t \geq 0$, or $\psi = \infty$, be given. Let $y(0)$ in (1) such that $\|y(0)\| < \psi(0)$. Then, Algorithm 4 is initially feasible, i.e., there exist $(c_0^*, T_0^*) \in \mathcal{F}_H(0, y(0))$ such that $J_H(y(0), c_0^*, T_0^*) \leq V_H(0, y(0)) + \varepsilon_{\text{opt}}$. Moreover, Algorithm 4 is recursively feasible, meaning the solvability of the optimization problem in (8) at time $t_i, i \in \mathbb{N}_0$ implies its solvability at the next time step t_{i+1}. Additionally, every maximal solution $y_i : [t_i, t_{i+1}] \to \mathbb{R}^m$ of the initial value problem (1) with initial value $y_i(t_i) = y_{i-1}(t_i)$ on the interval $[t_i, t_{i+1}]$ stays inside the funnel boundaries, meaning, for all $i \in \mathbb{N}$ and $t \in [t_i, t_{i+1}]$, we have

$$\|y_i(t)\| \le \max\{0, \varphi(t - t_i; c_i^*, T_i^*)\} \le \psi(t). \tag{9}$$

Proof: *Step 1:* Note the following observation. Given $(\hat{t}, \hat{y}) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^m$ with $\|\hat{y}\| < \psi(\hat{t})$, then $\mathcal{F}(\hat{t}, \hat{y}) \neq \emptyset$. To see this, consider the following two cases. If $\psi \equiv \infty$, then $\left(\frac{\|\hat{y}\|+1}{H}, H\right) \in \mathcal{F}(\hat{t}, \hat{y})$. Otherwise, invoking $\psi \in \mathcal{C}^1(\mathbb{R}_{\ge 0}; \mathbb{R})$ with $\psi(t) > 0$ for all $t \ge 0$, we get that the candidates

$$\hat{T} := \min\left\{\frac{\psi(\hat{t}) + \|\hat{y}\|}{2\|\dot{\psi}|_{[\hat{t}, \hat{t}+H]}\|_\infty}, H\right\}, \quad \hat{c} := \frac{\psi(\hat{t}) + \|\hat{y}\|}{2\hat{T}}$$

fulfill $(\hat{c}, \hat{T}) \in \mathcal{F}_H(\hat{t}, \hat{y}) \neq \emptyset$ by checking the set predicates. $\hat{T} \in (0, H]$ and $\hat{c} > 0$ follow directly. Furthermore,

$$\|\hat{y}\| < \varphi(0; \hat{c}, \hat{T}) = \hat{c}\hat{T} = \tfrac{1}{2}(\psi(\hat{t}) + \|\hat{y}\|) < \psi(\hat{t}),$$

where we used $\|\hat{y}\| < \psi(\hat{t})$ and, since for $\tau \in [0, \hat{T})$ we have $\frac{\mathrm{d}}{\mathrm{d}\tau}\varphi(\tau; \hat{c}, \hat{T}) = -\hat{c} \le -\|\dot{\psi}|_{[\hat{t}, \hat{t}+H]}\|_\infty$, we get that

$$\begin{aligned}\varphi(\tau; \hat{c}, \hat{T}) &= \varphi(0; \hat{c}, \hat{T}) + \textstyle\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}s}\varphi(s; \hat{c}, \hat{T})\, ds \\ &< \psi(\hat{t}) + \textstyle\int_{\hat{t}}^{\hat{t}+\tau} \dot{\psi}(s)\, ds = \psi(\hat{t} + \tau).\end{aligned}$$

Step 2: Given $(t_i, \hat{y}_i) \in \mathbb{R}_{\ge 0} \times \mathbb{R}^m$ with $\|\hat{y}_i\| < \psi(t_i)$ for $i \in \mathbb{N}_0$, we show $V_H(t_i, \hat{y}_i) \in \mathbb{R}$. According to Step 1, there exist $c, T \in \mathbb{R}$ such that $\varphi(t - t_i; c, T) < \psi(t)$ for all $t \in [t_i, t_i + T]$. Thus, $\|\hat{y}_i\| < \varphi(0; c, T)$. Then, Lemma 3 yields an input signal $u_i \in L^\infty([t_i, t_i + T]; \mathbb{R}^m)$ such that, for $(c, T) \in \mathcal{F}_H(t_i, \hat{y}_i)$, the initial value problem (1) with initial value $y(t_i) = \hat{y}_i$ has a solution $\tilde{y}_i : [t_i, t_i + T] \to \mathbb{R}^m$ with $\lim_{t\to T} \|\tilde{y}_i(t_i + t)\| = 0$ and

$$\forall\, t \in [t_i, t_i + T) \colon \|\tilde{y}_i(t)\| < \varphi(t - t_i; c, T) \le \psi(t), \tag{10}$$

where $cT < \infty$. On $[t_i + T, t_i + H]$, the solution $\tilde{y}_i$ can be continued with $u = 0$. Since $f(0, 0) = 0$ by assumption, we get $\tilde{y}_i|_{[t_i+T, t_i+H]} \equiv 0$. Then, $\tilde{y}_i, u_i \in L^\infty([t_i, t_i + H]; \mathbb{R}^m)$ implies that $J_H(\hat{y}_i, c, T)$ in (7) is finite. Thus, $V_H(t_i, \hat{y}_i)$ is finite.

Step 3: We show initial and recursive feasibility of Algorithm 4.

Step 3.1: According to *Step 2*, $V_H(0, y(0))$ is finite. Thus, there exist $(c_0^*, T_0^*) \in \mathcal{F}_H(0, y(0))$ such that $J_H(y(0), c_0^*, T_0^*) \le V_H(0, y(0)) + \varepsilon_{\text{opt}}$. This is, Algorithm 4 is initially feasible.

Step 3.2 We show recursive feasibility. For $i \in \mathbb{N}_0$ let $y_i : [t_i, t_{i+1}] \to \mathbb{R}^m$ be the solution of the initial value problem (1) with initial value $y_i(t_i) = y_{i-1}(t_i)$ and control u_i. Further, let $(c_i^*, T_i^*) \in \mathcal{F}_H(t_i, y_{i-1}(t_i))$ be the associated funnel parameters. Define $\hat{y}_{i+1} := y_i(t_{i+1})$. We have to show that there exist $(c_{i+1}^*, T_{i+1}^*) \in \mathcal{F}_H(t_{i+1}, \hat{y}_{i+1})$ satisfying (8) for index $i + 1$. According to *Step 2*, we know $V_H(t_{i+1}, \hat{y}_{i+1}) \in \mathbb{R}$. If $V_H(t_{i+1}, \hat{y}_{i+1}) + \varepsilon_{\text{opt}} \le J_H(\hat{y}_{i+1}, c_i^*, T_i^* - h)$, then the assertion follows as in *Step 3.1*. Otherwise, note that $y_i = \tilde{y}_i|_{[t_i, t_{i+1}]}$ from *Step 2* where $\tilde{y}_i : [t_i, t_i + T_i^*] \to \mathbb{R}^m$ is a solution of the initial value problem (1) with initial value $\hat{y}_i = y_{i-1}(t_i)$. As discussed in *Step 2*, $\tilde{y}_i$ can be extended by zero on the entire interval $[t_i, t_{i+1} + H]$. As $\hat{y}_{i+1} = \tilde{y}_i(t_{i+1})$, $(c_{i+1}^*, T_{i+1}^*) := (c_i^*, T_i^* - h) \in \mathcal{F}_H(t_{i+1}, \hat{y}_{i+1})$ satisfies (8).

Step 4: As a direct consequence of (10) in *Step 2*, every solution y_i satisfies (9). This completes the proof. ∎

Next, we prove boundedness of the closed-loop costs.

Lemma 6: Let the assumptions of Theorem 5 be fulfilled. Then, the input signals $u_i(\cdot)$ and the output signals $y_i(\cdot)$ resulting from Algorithm 4 fulfill

$$\begin{aligned}J_\infty^{cl}(H, y_0(0)) &:= \sum_{i=0}^{\infty} \int_{t_i}^{t_{i+1}} \ell(y_i(t), u_i(t))dt \\ &\le V_H(0, y_0(0)) + \varepsilon_{\text{opt}} < \infty.\end{aligned}$$

Proof: Continuing the considerations of the proof of Theorem 5, it follows that for all $\hat{t} \ge 0$ and $\hat{y} \in \mathbb{R}^m$ we have $\mathcal{F}_{H-h}(\hat{t}, \hat{y}) \subseteq \mathcal{F}_H(\hat{t}, \hat{y})$. Moreover, the horizon can be extended with zero cost:

$$\forall\, (c, T) \in \mathcal{F}_{H-h}(\hat{t}, \hat{y}) : \; J_{H-h}(\hat{y}, c, T) = J_H(\hat{y}, c, T).$$

Therefore, $V_{H-h}(\hat{t}, \hat{y}) \ge V_H(\hat{t}, \hat{y})$. We calculate, for $i \in \mathbb{N}$:

$$\begin{aligned}&J_H(y_i(t_i), c_i^*, T_i^*) \\ &= \textstyle\int_{t_i}^{t_i+H} \ell(y_i(t), u_i(t))\, dt + c_i^* \qquad (11)\\ &= \textstyle\int_{t_i}^{t_{i+1}} \ell(y_i(t), u_i(t))\, dt + \int_{t_{i+1}}^{t_i+H} \ell(y_i(t), u_i(t))\, dt + c_i^* \\ &\overset{(7)}{=} \textstyle\int_{t_i}^{t_{i+1}} \ell(y_i(t), u_i(t))\, dt + J_{H-h}(y_{i+1}(t_{i+1}), c_i^*, T_i^* - h) \\ &= \textstyle\int_{t_i}^{t_{i+1}} \ell(y_i(t), u_i(t))\, dt + J_H(y_{i+1}(t_{i+1}), c_i^*, T_i^* - h) \\ &\overset{(8)}{\ge} \textstyle\int_{t_i}^{t_{i+1}} \ell(y_i(t), u_i(t))\, dt + J_H(y_{i+1}(t_{i+1}), c_{i+1}^*, T_{i+1}^*).\end{aligned}$$

If $T_i^* - h \le 0$, then y_i and u_i are already zero on the interval $[t_{i+1}, t_i + H]$. Thus, the second summand vanishes in this case. Rearranging gives

$$\begin{aligned}&\int_{t_i}^{t_{i+1}} \ell(y_i(t), u_i(t))dt \\ &\quad \le J_H(y_i(t_i), c_i^*, T_i^*) - J_H(y_{i+1}(t_{i+1}), c_{i+1}^*, T_{i+1}^*).\end{aligned}$$

As this inequality holds for all $i \in \mathbb{N}_0$, summing up to $K \in \mathbb{N}$ gives

$$\begin{aligned}&\sum_{i=0}^{K} \int_{t_i}^{t_{i+1}} \ell(y_i(t), u_i(t))dt \\ &\quad \le J_H(y_0(0), c_0^*, T_0^*) - J_H(y_{K+1}(t_{K+1}), c_{K+1}^*, T_{K+1}^*) \\ &\quad \le J_H(y_0(0), c_0^*, T_0^*) \le V_H(0, y_0(0)) + \varepsilon_{\text{opt}}.\end{aligned}$$

Since the costs are nonnegative, the left-hand-side is monotonically increasing and bounded for $K \to \infty$. It hence converges. ∎

We prove boundedness of the optimized parameters.

Corollary 7: Let the assumptions of Theorem 5 be fulfilled. Then, the set $\{c_i^* \mid i \in \mathbb{N}_0\} \subset \mathbb{R}$ resulting from Algorithm 4 is bounded.

Proof: Observe that for all $i \in \mathbb{N}$ we have $c_i^* \leq J_H(y_i(t_i), c_i^*, T_i^*) \leq J_H(y_{i-1}(t_{i-1}), c_{i-1}^*, T_{i-1}^*)$ according to (7) and (11). Thus, $c_i^* \leq J_H(y_0(0), c_0^*, T_0^*)$ for all $i \in \mathbb{N}$. ■

Next, we show that the boundedness of the closed-loop costs implies convergence of the closed-loop solution, if Q is positive definite. We thereby show that Algorithm 4 fulfills the control objective (2) in Section II. In the context of Algorithm 4, define the *closed-loop solution* as

$$y^{cl}(t) := \sum_{i=1}^{\infty} \chi_{[t_i,t_{i+1})}(t)\, y_i(t),$$
$$u^{cl}(t) := \sum_{i=1}^{\infty} \chi_{[t_i,t_{i+1})}(t)\, u_i(t),$$

where $\chi_{[t_i,t_{i+1})}$ is the indicator function on $[t_i, t_{i+1})$.

Theorem 8: Let the assumptions of Theorem 5 be fulfilled. In addition, assume Q to be positive definite. Then, Algorithm 4 asymptotically stabilizes system (1), i.e. it holds $\lim_{t\to\infty} y^{cl}(t) = 0$.

Proof: Seeking a contradiction, we assume

$$\exists\, \varepsilon > 0\ \forall\, t_0 \geq 0\ \exists\, t > t_0\colon\ \|y^{cl}(t)\| > \varepsilon. \tag{14}$$

Step 1: We define some constants for later use. Let $h > 0$ be the equidistant step size $h = t_{i+1} - t_i$. Further, since Q is symmetric positive definite, there exists a constant $q > 0$ such that $y^\top Q y \geq q\|y\|^2$ for all $y \in \mathbb{R}^m$. For $t \geq 0$, define the closed-loop funnel function as $\varphi^{cl}(t) := \sum_{i=1}^{\infty} \chi_{[t_i,t_{i+1})}(t)\, \varphi(t - t_i; c_i^*, T_i^*)$. For $x \in \mathbb{R}_{\geq 0}$, define $\hat{N}(x) := \max_{\xi\in[0,x]} |N(\xi)|$.

Step 2: We show that, given the assumption (14), the control input u^{cl} is unbounded. According to Lemma 6, $J_\infty^{cl}(H, y_0(0)) < \infty$. This implies

$$\forall\, \varepsilon_c > 0\ \exists\, t_{\varepsilon_c} \in \mathbb{R}\colon \int_{t_{\varepsilon_c}}^{\infty} \ell(y^{cl}(t), u^{cl}(t))\ \mathrm{d}t < \varepsilon_c \tag{15}$$

and, since Q is positive definite

$$\forall\, t \geq 0\ \exists\, \tau_{\varepsilon/3}(t) > t\colon\ \|y^{cl}(\tau_{\varepsilon/3}(t))\| < \tfrac{\varepsilon}{3}. \tag{16}$$

So, combining (14) – (16), for any given $\varepsilon_c \in (0, \frac{qh\varepsilon^2}{36})$, we find $t_\varepsilon > \tau_{\varepsilon/3}(t_{\varepsilon_c})$ with $\|y^{cl}(t_\varepsilon)\| > \varepsilon$. There exist

$$t_{1/3} := \max\{t \in \mathbb{R}_{\geq 0} \mid \|y^{cl}(t)\| = \tfrac{1}{3}\varepsilon\ \wedge\ t < t_\varepsilon\},$$
$$t_{2/3} := \min\{t \in \mathbb{R}_{\geq 0} \mid \|y^{cl}(t)\| = \tfrac{2}{3}\varepsilon\ \wedge\ t > t_{1/3}\}$$

because y^{cl} is continuous and $t_\varepsilon > \tau_{\varepsilon/3}(t_{\varepsilon_c})$. We estimate

$$\varepsilon_c \geq \int_{t_{1/3}}^{t_{2/3}} \ell(y^{cl}(t), u^{cl}(t))\ \mathrm{d}t \geq \int_{t_{1/3}}^{t_{2/3}} y^{cl}(t)^\top Q y^{cl}(t)\ \mathrm{d}t$$
$$\geq \int_{t_{1/3}}^{t_{2/3}} q\|y^{cl}(t)\|^2\ \mathrm{d}t \geq q\left(\frac{\varepsilon}{3}\right)^2 (t_{2/3} - t_{1/3}).$$

Since y^{cl} is piecewise differentiable, the mean value theorem yields the existence of $t_{\mathrm{crit}} \in [t_{1/3}, t_{2/3}]$ with

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|y^{cl}(t_{\mathrm{crit}})\| \geq \tfrac{\frac{2}{3}\varepsilon - \frac{1}{3}\varepsilon}{t_{2/3} - t_{1/3}} = \tfrac{\varepsilon}{3(t_{2/3} - t_{1/3})} \geq \tfrac{3\varepsilon^3 q}{\varepsilon_c} \xrightarrow[\varepsilon_c\to 0]{} \infty.$$

Since f is continuous and $\|y^{cl}(t_{\mathrm{crit}})\| \leq \frac{2}{3}\varepsilon < \infty$, this implies

$$\|u^{cl}(t_{\mathrm{crit}})\| \xrightarrow[\varepsilon_c\to 0]{} \infty. \tag{17}$$

Step 3: We will lead statement (17) to a contradiction by establishing a bound on $\|u^{cl}(t_{\mathrm{crit}})\|$. To this end, we distinguish two cases.

Case 1: $\varphi^{cl}(t_{\mathrm{crit}}) > \varepsilon$. Since $\|y^{cl}(t_{\mathrm{crit}})\| \leq \frac{2}{3}\varepsilon$ and $\varphi^{cl}(t_{\mathrm{crit}}) > \varepsilon$, we have $\frac{\|y^{cl}(t_{\mathrm{crit}})\|}{\varphi^{cl}(t_{\mathrm{crit}})} < \frac{2}{3}$. Then,

$$\|u^{cl}(t_{\mathrm{crit}})\| = \left\| N\left(\alpha\left(\tfrac{\|y^{cl}(t_{\mathrm{crit}})\|^2}{\varphi^{cl}(t_{\mathrm{crit}})^2}\right)\right) \tfrac{y^{cl}(t_{\mathrm{crit}})}{\varphi^{cl}(t_{\mathrm{crit}})} \right\|$$
$$\leq \hat{N}\left(\alpha\left(\tfrac{4}{9}\right)\right) \tfrac{2}{3} < \infty$$

contradicting (17).

Case 2: $\varphi^{cl}(t_{\mathrm{crit}}) \leq \varepsilon$. In this case, a similar chain of arguments can be performed. Define $\hat{t}_{1/3} := \min\{t \in \mathbb{R}_{\geq 0} \mid \|y^{cl}(t)\| = \frac{1}{3}\varepsilon\ \wedge\ t > t_\varepsilon\}$, and $\hat{t}_{2/3} := \max\{t \in \mathbb{R}_{\geq 0} \mid \|y^{cl}(t)\| = \frac{2}{3}\varepsilon\ \wedge\ t < \hat{t}_{1/3}\}$, where $t_{1/3}$ exists due to (16), and $t_{2/3}$ due to the intermediate value theorem. We estimate

$$\varepsilon_c \geq \int_{\hat{t}_{2/3}}^{\hat{t}_{1/3}} \ell(y^{cl}(t), u^{cl}(t))\ \mathrm{d}t$$
$$\geq \int_{\hat{t}_{2/3}}^{\hat{t}_{1/3}} y^{cl}(t)^\top Q y^{cl}(t)\ \mathrm{d}t \geq q\left(\tfrac{\varepsilon}{3}\right)^2 (\hat{t}_{1/3} - \hat{t}_{2/3}).$$

The mean value theorem yields the existence of $\hat{t}_{\mathrm{crit}} \in [\hat{t}_{2/3}, \hat{t}_{1/3}]$ with

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\|y^{cl}(t_{\mathrm{crit}})\| \leq \tfrac{\frac{1}{3}\varepsilon - \frac{2}{3}\varepsilon}{\hat{t}_{1/3} - \hat{t}_{2/3}} = \tfrac{-\varepsilon}{3(\hat{t}_{1/3} - \hat{t}_{2/3})} \leq \tfrac{-\varepsilon^3 q}{\varepsilon_c} \xrightarrow[\varepsilon_c\to 0]{} -\infty.$$

Since f is continuous and $\|y^{cl}(\hat{t}_{\mathrm{crit}})\| \leq \frac{2}{3}\varepsilon < \infty$, this implies

$$\|u^{cl}(\hat{t}_{\mathrm{crit}})\| \xrightarrow[\varepsilon_c\to 0]{} \infty. \tag{18}$$

Step 3.1: We show $\varphi^{cl}(\hat{t}_{\mathrm{crit}}) > \frac{3}{4}\varepsilon$. Seeking a contradiction, assume $\varphi^{cl}(\hat{t}_{\mathrm{crit}}) \leq \frac{3}{4}\varepsilon$. Since $\varphi^{cl}(t_{\mathrm{crit}}) \leq \varepsilon$ and $\varphi^{cl}(t_\varepsilon) > \varepsilon$, there must be a resampling time between t_{crit} and t_ε, i.e., there exists $i^* \in \mathbb{N}$ such that $t_{i^*} \in [t_{\mathrm{crit}}, t_\varepsilon)$. W.l.o.g. $T_i^* > h$ (otherwise we have convergence, since $y^{cl}(t) = 0$ for $t \geq t_{i^*} + h$). Since $\varphi^{cl}(t_{i^*}) > \varepsilon$,

$$\varphi^{cl}(t_{i^*} + \tau) > \varepsilon\left(1 - \tfrac{\tau}{h}\right) \quad \text{for } \tau \in [0, h).$$

This implies, if $\varphi^{cl}(\hat{t}_{\mathrm{crit}}) \leq \frac{3}{4}\varepsilon$, then $\hat{t}_{\mathrm{crit}} - t_{i^*} > \frac{h}{4}$. But $\|y^{cl}(t)\| \geq \frac{\varepsilon}{3}$ for $t \in [t_{i^*}, \hat{t}_{\mathrm{crit}}] \subseteq [t_{1/3}, \hat{t}_{1/3}]$. This leads to

$$\varepsilon_c > \int_{t_{i^*}}^{\infty} \ell(y^{cl}(t), u^{cl}(t))\ \mathrm{d}t > q\int_{t_{i^*}}^{\hat{t}_{\mathrm{crit}}} \|y^{cl}(t)\|^2\ \mathrm{d}t$$
$$> q\left(\hat{t}_{\mathrm{crit}} - t_{i^*}\right)\left(\tfrac{\varepsilon}{3}\right)^2 > q\,\tfrac{h}{4}\left(\tfrac{\varepsilon}{3}\right)^2 > \varepsilon_c,$$

a contradiction. Therefore, $\|y^{cl}(\hat{t}_{\mathrm{crit}})\| \leq \frac{2}{3}\varepsilon$ and $\varphi^{cl}(\hat{t}_{\mathrm{crit}}) > \frac{3}{4}\varepsilon$. It follows that

$$\|u^{cl}(\hat{t}_{\mathrm{crit}})\| = \left\| N\left(\alpha\left(\tfrac{\|y^{cl}(\hat{t}_{\mathrm{crit}})\|^2}{\varphi^{cl}(\hat{t}_{\mathrm{crit}})^2}\right)\right) \tfrac{y^{cl}(\hat{t}_{\mathrm{crit}})}{\varphi^{cl}(\hat{t}_{\mathrm{crit}})} \right\|$$
$$\leq \hat{N}\left(\alpha\left(\tfrac{8^2}{9^2}\right)\right) \tfrac{8}{9} < \infty,$$

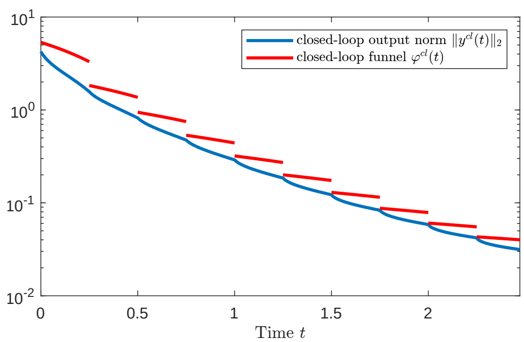
which contradicts (18). This completes the proof. ■

V. Numerical Example

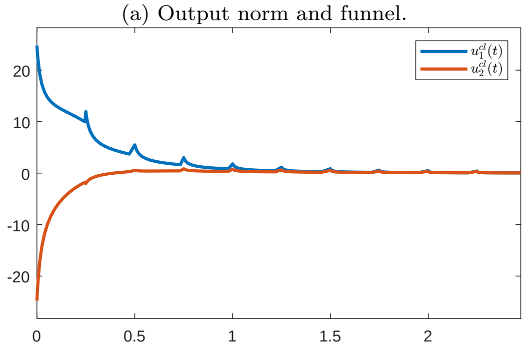
We illustrate Algorithm 4 considering the system

$$\dot{y}(t) = \begin{pmatrix} y_1(t)^2 + y_1(t) \\ y_2(t)^2 + y_1(t) \end{pmatrix} - u(t), \quad y(0) = \begin{pmatrix} 3 \\ -3 \end{pmatrix}. \tag{19}$$

In the stage costs (5), we set $Q = I_2$ and $R = 0.2 \cdot I_2$, where $I_2 \in \mathbb{R}^{2\times 2}$ denotes the identity matrix. As control parameters, we choose $N(s) = s$ (which is possible due to known control direction [10, Rem. 2.5]) and $\alpha_c(s) = \frac{2c}{1-s}$, and set $\psi(t) = \infty$ for no additional output constraints. As described in [10, Sec. 4.1], because $\lim_{t\to T} \varphi(t) = 0$, simulation is only possible on an interval $[0, t_{\max}]$ with $t_{\max} < T$. We choose $t_{\max} = T - 10^{-9}/c$ to guarantee a spatial accuracy of 10^{-9}. The prediction horizon is chosen as $H = 5$ and the step size as $h = 0.25$. This numerical experiment was performed in Matlab R2024b, employing `ode45` as ODE solver ($\texttt{AbsTol} = 10^{-9}$, $\texttt{RelTol} = 10^{-6}$) and `fmincon` as optimization algorithm (using default parameterization). The closed-loop system output norm $\|y^{cl}\|_2$, as well as the closed-loop funnel φ^{cl} is plotted in Figure 2a. The provided input is shown in Figure 2b.



(a) Output norm and funnel.



(b) Control input.

Fig. 2: Simulation of system (19) under the control generated by Algorithm 4.

VI. Conclusion and Outlook

We presented model predictive funnel control, a novel combination of funnel control and model predictive control. It features a constant number of decision variables in the optimization problem and yields a closed-loop feedback law, even on inter-sampling intervals. We rigorously showed boundedness of the closed-loop costs, as well as convergence of the solution. Future work will address systems of higher relative degree and with internal dynamics, relaxing the choice of funnel boundary ($\varphi(T) \neq 0$), considering input constraints in the control algorithm, and performing reference tracking tasks, to name a few of the upcoming topics. We will conduct numerical studies to demonstrate preferable computational performance compared to MPC in specific scenarios.

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