Funnel control for the monodomain equations with the FitzHugh-Nagumo model

Thomas Berger

Institut für Mathematik
Universität Paderborn
Warburger Str. 100
33098 Paderborn
Germany

Tobias Breiten

Institute of Mathematics
Technical University of Berlin
Straße des 17. Juni 136
10623 Berlin
Germany

Marc Puche

Fachbereich Mathematik
Universität Hamburg, Bundesstraße 55
20146 Hamburg
Germany

Timo Reis

Fachbereich Mathematik
Universität Hamburg, Bundesstraße 55
20146 Hamburg
Germany

Abstract

We consider a nonlinear reaction diffusion system of parabolic type known as the monodomain equations, which model the interaction of the electric current in a cell. Together with the FitzHugh-Nagumo model for the nonlinearity they represent defibrillation processes of the human heart. We study a fairly general type with co-located inputs and outputs describing both boundary and distributed control and observation. The control objective is output trajectory tracking with prescribed performance. To achieve this we employ the funnel con-

*Thomas Berger acknowledges support by the German Research Foundation (Deutsche Forschungsgemeinschaft) via the grant BE 6263/1-1.

Email addresses: `thomas.berger@math.upb.de` (Thomas Berger), `tobias.breiten@tu-berlin.de` (Tobias Breiten), `marc.puche@uni-hamburg.de` (Marc Puche), `timo.reis@uni-hamburg.de` (Timo Reis)

troller, which is model-free and of low complexity. The controller introduces a nonlinear and time-varying term in the closed-loop system, for which we prove existence and uniqueness of solutions. Additionally, exploiting the parabolic nature of the problem, we obtain Hölder continuity of the state, inputs and outputs. We illustrate our results by a simulation of a standard test example for the termination of reentry waves.

Keywords: Adaptive control, funnel control, monodomain equations, FitzHugh–Nagumo model
2010 MSC: 35K55, 93C40

---

1. Introduction

We study output trajectory tracking for a class of nonlinear reaction diffusion equations such that a prescribed performance of the tracking error is achieved. To this end, we use the method of funnel control which was developed in [2], see also the survey [3]. The funnel controller is a model-free output-error feedback of high-gain type. Therefore, it is inherently robust and of striking simplicity. The funnel controller has been successfully applied e.g. in temperature control of chemical reactor models [4], control of industrial servo-systems [5] and underactuated multibody systems [6], speed control of wind turbine systems [7, 8, 5], current control for synchronous machines [9, 5], DC-link power flow control [10], voltage and current control of electrical circuits [11], oxygenation control during artificial ventilation therapy [12], control of peak inspiratory pressure [13] and adaptive cruise control [14].

A funnel controller for a large class of systems described by functional differential equations with arbitrary (well-defined) relative degree (see [15] for a definition in the context of nonlinear systems) has been developed in [16]. It is shown in [17] that this abstract class indeed allows for fairly general infinite-dimensional systems, where the internal dynamics are modeled by a (PDE). In particular, it was shown in [18] that the linearized model of a moving water tank, where sloshing effects appear, belongs to the aforementioned system class. On the other hand, not even every linear, infinite-dimensional system has a well-defined relative degree, in which case the results as in [16, 2] cannot be applied. Instead, the feasibility of funnel control has to be investigated directly for the (nonlinear) closed-loop system, see [19] for a boundary controlled heat equation, [20] for a general class of boundary control systems and [21] for the Fokker-Planck equation corresponding to a multi-dimensional Ornstein-Uhlenbeck process.

The nonlinear reaction diffusion system that we consider in the present paper is known as the monodomain model and represents defibrillation processes of the human heart [22]. The monodomain equations are a reasonable simplification of the well accepted bidomain equations, which arise in cardiac electrophysiology [23]. In the monodomain model the dynamics are governed by a parabolic reaction diffusion equation which is coupled with a linear ordinary differential equation that models the ionic current.

It is discussed in [24] that, under certain initial conditions, reentry phenomena and spiral waves may occur. From a medical point of view, these situations can be interpreted as fibrillation processes of the heart that should be terminated by an external control, for instance by applying an external stimulus to the heart tissue, see [25].

The present paper is organized as follows: In Section 2 we introduce the mathematical framework, which strongly relies on preliminaries on Robin elliptic operators. The control objective is presented in Section 3, where we also state the main result on the feasibility of the proposed controller design in Theorem 3.3. The proof of this result is given in Section 4 and it uses several auxiliary results derived in Appendices A and B. We illustrate our result by a simulation in Section 5.

Nomenclature. The set of bounded operators from X to Y is denoted by $\mathcal{L}(X,Y)$, X' stands for the dual of a Banach space X, and B' is the dual of an operator B.
For a bounded and measurable set $\Omega \subset \mathbb{R}^d$, $p \in [1,\infty]$ and $k \in \mathbb{N}_0$, $W^{k,p}(\Omega;\mathbb{R}^n)$ denotes the Sobolev space of equivalence classes of p-integrable and k-times weakly differentiable functions $f : \Omega \to \mathbb{R}^n$, $W^{k,p}(\Omega;\mathbb{R}^n) \cong (W^{k,p}(\Omega))^n$, and the Lebesgue space of equivalence classes of p-integrable functions is $L^p(\Omega) = W^{0,p}(\Omega)$. For $r \in (0,1)$ we further set

$$W^{r,p}(\Omega) := \left\{ \ f \in L^p(\Omega) \ \middle| \ \left((x,y) \mapsto \frac{|f(x)-f(y)|}{|x-y|^{d/p+r}} \right) \in L^p(\Omega \times \Omega) \ \right\}.$$

For a domain Ω with smooth boundary, $W^{k,p}(\partial\Omega)$ denotes the Sobolev space at the boundary.
We identify functions with their restrictions, that is, for instance, if $f \in L^p(\Omega)$ $\Omega_0 \subset \Omega$, then the restriction $f|_{\Omega_0} \in L^p(\Omega_0)$ is again denoted by f. For an interval $J \subset \mathbb{R}$, a Banach space X and $p \in [1,\infty]$, we denote by $L^p(J;X)$ the vector space of equivalence classes of strongly measurable functions $f : J \to X$ such that $\|f(\cdot)\|_X \in L^p(J)$. Note that if $J = (a,b)$ for $a,b \in \mathbb{R}$, the spaces $L^p((a,b);X)$, $L^p([a,b];X)$, $L^p([a,b);X)$ and $L^p((a,b];X)$ coincide, since the points at the boundary have measure zero. We will simply write $L^p(a,b;X)$, also for the case $a = -\infty$ or $b = \infty$. We refer to [26] for further details on Sobolev and Lebesgue spaces.
In the following, let $J \subset \mathbb{R}$ be an interval, X be a Banach space and $k \in \mathbb{N}_0$. Then $C^k(J;X)$ is defined as the space of k-times continuously differentiable functions $f : J \to X$. The space of bounded k-times continuously differentiable functions with bounded first k derivatives is denoted by $BC^k(J;X)$, and it is a Banach space endowed with the usual supremum norm. The space of bounded and uniformly continuous functions will be denoted by $BUC(J;X)$. The Banach space of Hölder continuous functions $C^{0,r}(J;X)$ with $r \in (0,1)$ is given by

$$C^{0,r}(J;X) := \left\{ \ f \in BC(J;X) \ \middle| \ [f]_r := \sup_{t,s \in J, s<t} \frac{\|f(t)-f(s)\|}{(t-s)^r} < \infty \ \right\},$$
$$\|f\|_r := \|f\|_\infty + [f]_r,$$

see [27, Chap. 0]. We like to note that for all $0 < r < q < 1$ we have that

$$C^{0,q}(J;X) \subseteq C^{0,r}(J;X) \subseteq BUC(J;X).$$

For $p \in [1,\infty]$, the symbol $W^{1,p}(J;X)$ stands for the Sobolev space of X-valued equivalance classes of weakly differentiable and p-integrable functions $f : J \to X$ with p-integrable weak derivative, i.e., $f, \dot{f} \in L^p(J;X)$. Thereby, integration (and thus weak differentiation) has to be understood in the Bochner sense, see [28, Sec. 5.9.2]. The spaces $L^p_{\mathrm{loc}}(J;X)$ and $W^{1,p}_{\mathrm{loc}}(J;X)$ consist of all f whose restriction to any compact interval $K \subset J$ are in $L^p(K;X)$ or $W^{1,p}(K;X)$, respectively.

2. The FitzHugh-Nagumo model

Throughout this paper we will frequently use the following assumption. For $d \in \mathbb{N}$ we denote the scalar product in $L^2(\Omega;\mathbb{R}^d)$ by $\langle \cdot,\cdot \rangle$ and the norm in $L^2(\Omega)$ by $\|\cdot\|$.

Assumption 2.1. *Let $d \leq 3$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Further, let $a \in L^\infty(\partial\Omega)$ be nonnegative and $D \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be symmetric-valued and satisfy the* ellipticity condition

$$\exists\, \beta > 0 : \quad \text{for a.a.}\, \zeta \in \Omega\ \forall \xi \in \mathbb{R}^d : \ \xi^\top D(\zeta)\xi = \sum_{i,j=1}^{d} D_{ij}(\zeta)\xi_i\xi_j \geq \beta\|\xi\|^2_{\mathbb{R}^d}. \quad (1)$$

To formulate the model of interest, we consider the Robin elliptic operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ with

$$\begin{aligned} &\mathcal{D}(\mathcal{A}) = \left\{ z \in W^{1,2}(\Omega) \,\middle|\, \mathrm{div} D\nabla z \in L^2(\Omega) \,\wedge\, (\nu^\top \cdot D\nabla z + az)|_{\partial\Omega} = 0 \right\}, \\ &\mathcal{A}z = \mathrm{div} D\nabla z, \qquad z \in \mathcal{D}(\mathcal{A}), \end{aligned} \quad (2)$$

where $\nu : \partial\Omega \to \mathbb{R}^d$ is the outward normal unit vector. The domain of $\mathcal{A}$ is well-defined, since any vector field with square integrable weak divergence has a well-defined normal trace in $W^{1/2,2}(\partial\Omega)'$ (see [50, Lem. 20.2.]) and any element of $W^{1,2}(\Omega)$ has a well-defined trace in $W^{1/2,2}(\partial\Omega)$. Note that the Neumann elliptic operator is a special case of $\mathcal{A}$ emerging from setting $a = 0$.

The model for the interaction of the electric current in a cell is

$$\begin{aligned} \dot{v}(t) &= \mathcal{A}v(t) + p_3(v(t)) - u(t) + I_{s,i}(t) + \mathcal{B}I_{s,e}(t), \qquad && v(0) = v_0, \\ \dot{u}(t) &= c_5 v(t) - c_4 u(t), && u(0) = u_0, \\ y(t) &= \mathcal{B}'v(t), \end{aligned} \quad (3)$$

where

$$p_3(v) := -c_1 v + c_2 v^2 - c_3 v^3,$$

with constants $c_i > 0$ for $i = 1,\dots,5$, initial values $v_0, u_0 \in L^2(\Omega)$, the Robin elliptic operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ and control operator

$\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega)')$, where $W^{1,2}(\Omega)'$ is the dual of $W^{1,2}(\Omega)$ with respect to the pivot space $L^2(\Omega)$; consequently, $\mathcal{B}' \in \mathcal{L}(W^{1,2}(\Omega), \mathbb{R}^m)$.

System (3) is known as the FitzHugh-Nagumo model for the ionic current [30], where

$$I_{ion}(u, v) = p_3(v) - u.$$

The functions $I_{s,i} \in L^2_{\mathrm{loc}}(0, T; L^2(\Omega))$, $I_{s,e} \in L^2_{\mathrm{loc}}(0, T; \mathbb{R}^m)$ are the intracellular and extracellular stimulation currents, respectively. In particular, $I_{s,e}$ is the control input of the system, whereas y is the output.

Next we record some properties of the Robin elliptic operator which are frequently used throughout this article.

Remark 2.2. *If Assumption 2.1 holds, then the Robin elliptic operator $\mathcal{A}$ on Ω has the following properties:*

a) *It follows from [43, Prop. 3.10] that there exists some $\nu \in (0, 1)$ with $\mathcal{D}(\mathcal{A}) \subset C^{0,\nu}(\Omega)$. In particular, $\mathcal{D}(\mathcal{A}) \subset L^\infty(\Omega)$.*

b) *With $\mathcal{A}$ we may associate the bilinear form*

$$\mathfrak{a} : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{R},\ (z_1, z_2) \mapsto \langle \nabla z_1, D\nabla z_2\rangle + \langle z_1, a z_2\rangle_{L^2(\partial\Omega)}, \tag{4}$$

where $\langle\cdot,\cdot\rangle_{L^2(\partial\Omega)}$ denotes the standard inner product in $L^2(\partial\Omega)$. The relation between the operator $\mathcal{A}$ and the form $\mathfrak{a}$ is revealed by the properties

$$\begin{aligned} &\bullet\ \mathcal{D}(\mathcal{A}) = \left\{ z_2 \in W^{1,2}(\Omega) \,\middle|\, \exists\, y_2 \in L^2(\Omega)\ \forall\, z_1 \in W^{1,2}(\Omega) :\ \mathfrak{a}(z_1, z_2) = -\langle z_1, y_2\rangle \right\} \\ &\bullet\ \forall\, z_1 \in W^{1,2}(\Omega)\ \forall\, z_2 \in \mathcal{D}(\mathcal{A}) :\ \mathfrak{a}(z_1, z_2) = -\langle z_1, \mathcal{A} z_2\rangle. \end{aligned} \tag{5}$$

Furthermore, it follows from Kato's first representation theorem [29, Sec. VI.2, Thm 2.1] that $\mathcal{A}$ is uniquely determined by the properties (5)*, and it is moreover closed and densely defined. The property $\mathfrak{a}(z_1, z_2) = \mathfrak{a}(z_2, z_1)$ for all $z_1, z_2 \in W^{1,2}(\Omega)$ further implies that $\mathcal{A}$ is self-adjoint.*

c) *The Rellich-Kondrachov theorem [26, Thm. 6.3] implies that $\mathcal{A}$ has compact resolvent. Moreover, we have $\langle z, \mathcal{A}z\rangle \le 0$ for all $z \in \mathcal{D}(\mathcal{A})$. Combining these findings with self-adjointness of $\mathcal{A}$ we may infer that there exists a real-valued nonnegative and monotonically increasing sequence $(\alpha_j)_{j\in\mathbb{N}_0}$ without accumulation points, such that the spectrum of $\mathcal{A}$ reads $\sigma(\mathcal{A}) = \{\ -\alpha_j \mid j \in \mathbb{N}_0\ \}$, and there exists an an orthonormal basis $(\theta_j)_{j\in\mathbb{N}_0}$ of $L^2(\Omega)$, such that*

$$\forall\, x \in \mathcal{D}(\mathcal{A}) :\ \mathcal{A}x = -\sum_{j=0}^{\infty} \alpha_j \langle x, \theta_j\rangle\, \theta_j. \tag{6}$$

By [38, Prop. 3.2.9] the domain of $\mathcal{A}$ reads

$$\mathcal{D}(\mathcal{A}) = \left\{ \sum_{j=0}^{\infty} \lambda_j \theta_j \,\middle|\, (\lambda_j)_{j\in\mathbb{N}_0} \text{ with } \sum_{j=0}^{\infty} (1 + \alpha_j^2)|\lambda_j|^2 < \infty \right\}. \tag{7}$$

Next we introduce the solution concept.

Definition 2.3. *Let Assumption 2.1 hold and $\mathcal{A}$ be the Robin elliptic operator as in* (2)*, let $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega)')$, and $u_0, v_0 \in L^2(\Omega)$ be given. Further, let $T \in (0, \infty]$ and $I_{s,i} \in L^2_{\text{loc}}(0,T; L^2(\Omega))$, $I_{s,e} \in L^2_{\text{loc}}(0,T; \mathbb{R}^m)$. A triple of functions (u, v, y) is called* solution *of* (3) *on $[0,T)$, if*

(i) *$v \in L^2_{\text{loc}}(0,T; W^{1,2}(\Omega)) \cap C([0,T); L^2(\Omega))$ with $v(0) = v_0$;*
(ii) *$u \in C([0,T); L^2(\Omega))$ with $u(0) = u_0$;*
(iii) *for all $\chi \in L^2(\Omega)$, $\theta \in W^{1,2}(\Omega)$, the scalar functions $t \mapsto \langle u(t), \chi\rangle$, $t \mapsto \langle v(t), \theta\rangle$ are weakly differentiable on $[0,T)$, and for almost all $t \in (0,T)$ we have*

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} \langle v(t), \theta\rangle &= -\mathfrak{a}(v(t), \theta) + \langle p_3(v(t)) - u(t) + I_{s,i}(t), \theta\rangle + \langle I_{s,e}(t), \mathcal{B}'\theta\rangle_{\mathbb{R}^m}\,, \\
\tfrac{\mathrm{d}}{\mathrm{d}t} \langle u(t), \chi\rangle &= \langle c_5 v(t) - c_4 u(t), \chi\rangle\,, \\
y(t) &= \mathcal{B}' v(t),
\end{aligned} \tag{8}$$

where $\mathfrak{a} : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{R}$ is the bilinear form defined in (4)*.*

Remark 2.4.

a) *Weak differentiability of $t \mapsto \langle u(t), \chi\rangle$, $t \mapsto \langle v(t), \theta\rangle$ for all $\chi \in L^2(\Omega)$, $\theta \in W^{1,2}(\Omega)$ on $(0,T)$ further leads to $v \in W^{1,2}_{\text{loc}}(0,T; W^{1,2}(\Omega)')$ and $u \in W^{1,2}_{\text{loc}}(0,T; L^2(\Omega))$.*
b) *The Sobolev embedding theorem [26, Thm. 5.4] implies that the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ is bounded. This guarantees that $p_3(v) \in L^2_{\text{loc}}(0,T; L^2(\Omega))$, whence the first equation in* (8) *is well-defined.*
c) *For later use we investigate when the operator $\mathcal{B}$ has trivial kernel. By $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega)')$, this operator can be regarded to be composed of m bounded linear functionals $b_1, \ldots, b_m$. More precisely, for $e_i \in \mathbb{R}^m$ being the i-th unit vector, we set $b_i := \mathcal{B}e_i \in W^{1,2}(\Omega)'$. Then $\mathcal{B}$ has the representation*

$$\forall\, J_1, \ldots, J_m \in \mathbb{R} :\ \mathcal{B}(J_1, \ldots, J_m) = J_1 b_1 + \ldots + J_m b_m.$$

As a consequence, we have that $\ker \mathcal{B} = \{0\}$ if, and only if, the functionals $b_1, \ldots, b_m$ are linearly independent.

In the following subsections we discuss two important control frameworks, which can be modelled by a suitable choice of an operator $\mathcal{B}$.

2.1. Distributed control

Finite-dimensional distributed control typically means that each point in space is influenced by the same value of the temporal control function according to some predefined spatial shape function. Hence, in the context of the model (3), $\mathcal{B}I_{s,e}(t)$ should be a function of the control input $I_{s,e}(t)$. That is, for instance, the case, if $\operatorname{im} B \subset L^2(\Omega)$. In the context of Remark 2.4 c), this means that for all $i = 1, \ldots, m$ there exist $w_i \in L^2(\Omega)$ such that $b_i = \mathcal{B}e_i = w_i$. In this case, $\mathcal{B}$ has the form

$$\forall\, J_1, \ldots, J_m \in \mathbb{R} :\ \mathcal{B}(J_1, \ldots, J_m) = J_1 w_1 + \ldots + J_m w_m.$$

By setting $w = (w_1, \dots, w_m) \in L^2(\Omega)^m$, the output is given by

$$y(t) = \mathcal{B}' v(t) = \int_\Omega (v(t))(\xi) \cdot w(\xi) \,\mathrm{d}\xi.$$

A typical situation is that $w_1, \dots, w_m$ are indicator functions on some subsets of Ω; such choices have been considered in [31] for instance. Note that $\ker \mathcal{B} = \{0\}$ if, and only if, the functions $w_1, \dots, w_m$ are linearly independent in $L^2(\Omega)$, cf. Remark 2.4 c). For instance, this is satisfied, if $w_1, \dots, w_m$ are indicator functions on disjoint subsets of Ω.

2.2. Boundary control

Boundary control means that the value at the boundary is determined by the control function. In this case, $\mathcal{B}$ takes values in a distribution space. More precisely, $\mathcal{B}$ corresponds to a tuple of linear functionals which assign to $z \in W^{1,2}(\Omega)$ a value which is defined in terms of the trace of z on the boundary $\partial\Omega$. That is, for $\ell \in (0, 1/2]$, $w_1, \dots, w_m \in W^{\ell,2}(\partial\Omega)'$ and $J_1, \dots, J_m \in \mathbb{R}$ we consider the linear functional

$$\mathcal{B}(J_1, \dots, J_m) \in W^{\ell+1/2,2}(\Omega)', \quad z \mapsto \sum_{i=1}^m J_i \, \langle w_i, \mathrm{tr}(z) \rangle_{W^{\ell,2}(\partial\Omega)', W^{\ell,2}(\partial\Omega)} ,$$

where

$$\mathrm{tr}: \; z \mapsto z\big|_{\partial\Omega}$$

is the trace operator, that satisfies $\mathrm{tr} \in \mathcal{L}(W^{\ell+1/2,2}(\Omega), W^{\ell,2}(\partial\Omega))$ for all $\ell \in (0, 1/2]$ by the trace theorem [1, Thm. 9.2.1]. In particular, by the continuity of the embedding $W^{1,2}(\Omega) \subset W^{\ell+1/2,2}(\Omega)$, the mapping

$$\mathcal{B} : (J_1, \dots, J_m) \mapsto \mathcal{B}(J_1, \dots, J_m)$$

is linear and bounded from $\mathbb{R}^m$ to $W^{1,2}(\Omega)'$. Note that the kernel of $\mathcal{B}$ is trivial if, and only if, the boundary functionals $w_1, \dots, w_m$ are linearly independent, cf. Remark 2.4 c).

In the context of the model (3), the operator $\mathcal{B}$ corresponds to a Robin boundary control

$$\nu^\top \cdot D\nabla v(t) + a\, v(t)\big|_{\partial\Omega} = \langle w, I_{s,e}(t) \rangle_{\mathbb{R}^m} ,$$

where $w = (w_1, \dots, w_m)$. In this case, the output is given by the evaluation of the Dirichlet boundary values of $v(t)$ at $w_1, \dots, w_m$. More precisely,

$$y(t) = \mathcal{B}' v(t) = \begin{pmatrix} \langle w_1, \mathrm{tr}(v(t)) \rangle_{W^{\ell,2}(\partial\Omega)', W^{\ell,2}(\partial\Omega)} \\ \vdots \\ \langle w_m, \mathrm{tr}(v(t)) \rangle_{W^{\ell,2}(\partial\Omega)', W^{\ell,2}(\partial\Omega)} \end{pmatrix} .$$

By taking into account that $W^{\ell,2}(\partial\Omega) \subset L^2(\partial\Omega)$ via a canonical embedding, a special case is where $w = (w_1, \dots, w_m) \in L^2(\partial\Omega)^m$, in which, for $J \in \mathbb{R}^m$, $\mathcal{B}$ reads

$$\mathcal{B}J \in W^{1,2}(\Omega)', \; z \mapsto \int_{\partial\Omega} \langle w(\xi), J \rangle_{\mathbb{R}^m} \,\mathrm{tr}(z(\xi)) \,\mathrm{d}\sigma,$$

and the output is given by the weighted integral of the Dirichlet boundary values

$$y(t) = \int_{\partial\Omega} w(\xi) \cdot (v(t))(\xi)\, \mathrm{d}\sigma.$$

Again, $\ker \mathcal{B} = \{0\}$ if, and only if, the boundary functionals $w_1, \dots, w_m$ are linearly independent. This is satisfied, for instance, if $w_1, \dots, w_m$ are indicator functions on disjoint subsets of $\partial\Omega$.

3. Control objective

The objective is that the output y of the system (3) tracks a given reference signal which is $y_{\mathrm{ref}} \in W^{1,\infty}(0,\infty;\mathbb{R}^m)$ with a prescribed performance of the tracking error $e := y - y_{\mathrm{ref}}$, that is e evolves within the performance funnel

$$\mathcal{F}_\varphi := \{\ (t,e) \in [0,\infty) \times \mathbb{R}^m \ \mid \varphi(t)\|e\|_{\mathbb{R}^m} < 1\ \}$$

defined by a function φ belonging to

$$\Phi_\gamma := \left\{\ \varphi \in W^{1,\infty}(0,\infty;\mathbb{R})\ \middle|\ \varphi|_{[0,\gamma]} \equiv 0,\ \forall \delta > 0,\ \inf_{t > \gamma+\delta} \varphi(t) > 0\ \right\},$$

for some $\gamma > 0$.

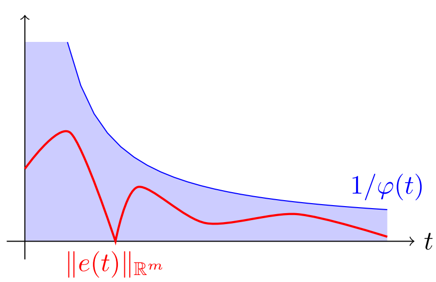

Figure 1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $1/\varphi(t)$.

The situation is illustrated in Fig. 1. The funnel boundary given by $1/\varphi$ is unbounded in a small interval $[0,\gamma]$ to allow for an arbitrary initial tracking error. Since φ is bounded there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. Thus, we seek practical tracking with arbitrary small accuracy $\lambda > 0$, but asymptotic tracking is not required in general.

The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. Sometimes, widening the funnel over some later time interval might be beneficial, for instance in the presence of periodic disturbances or strongly varying reference signals. For typical choices of funnel boundaries see e.g. [33, Sec. 3.2].
A controller which achieves the above described control objective is the funnel controller. In the present paper, it suffices to restrict ourselves to the simple version developed in [2], which is the feedback law

$$I_{s,e}(t) = -\frac{k_0}{1 - \varphi(t)^2 \|\mathcal{B}' v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}} (\mathcal{B}' v(t) - y_{\mathrm{ref}}(t)), \tag{9}$$

where $k_0 > 0$ is some constant used for scaling and agreement of physical units. Note that, by $\varphi|_{[0,\gamma]} \equiv 0$, the controller satisfies

$$\forall\, t \in [0, \gamma] : \ I_{s,e}(t) = -k_0(\mathcal{B}'v(t) - y_{\text{ref}}(t)).$$

Inserting the feedback law (9) into the system (3), we obtain the closed-loop system

$$\begin{aligned} \dot v(t) &= \mathcal{A}v(t) + p_3(v)(t) - u(t) + I_{s,i}(t) - \frac{k_0\mathcal{B}(\mathcal{B}'v(t) - y_{\text{ref}}(t))}{1 - \varphi(t)^2\|\mathcal{B}'v(t) - y_{\text{ref}}(t)\|^2_{\mathbb{R}^m}}, \\ \dot u(t) &= c_5 v(t) - c_4 u(t), \end{aligned} \tag{10}$$

for which we seek to show existence and uniqueness of global solutions – this is the subject of the main result Theorem 3.3 below. Note that the system (10) is a nonlinear and non-autonomous PDE and any solution needs to satisfy that the tracking error evolves in the prescribed performance funnel $\mathcal{F}_\varphi$. Therefore, existence and uniqueness of solutions is a nontrivial problem and even if a solution exists on a finite time interval $[0, T)$, it is not clear that it can be extended to a global solution.

We introduce the following weak solution framework.

Definition 3.1. *Use the assumptions from Definition 2.3. Furthermore, let $k_0 > 0$, $y_{\text{ref}} \in W^{1,\infty}(0, \infty; \mathbb{R}^m)$, $\gamma > 0$ and $\varphi \in \Phi_\gamma$. A triple of functions (u, v, y) is called* solution *of system* (10) *on* $[0, T)$*, if (u, v, y) satisfies the conditions (i)–(iii) from Definition 2.3 with $I_{s,e}$ as in* (9)*.*

Remark 3.2.

a) *To be precise, (u, v, y) is a solution of* (10) *on* $[0, T)$ *if, and only if,*
   (i) *$v \in L^2_{\text{loc}}(0, T; W^{1,2}(\Omega)) \cap C([0, T); L^2(\Omega)))$ with $v(0) = v_0$;*
   (ii) *$u \in C([0, T); L^2(\Omega))$ with $u(0) = u_0$;*
   (iii) *for all $\chi \in L^2(\Omega)$, $\theta \in W^{1,2}(\Omega)$, the scalar functions $t \mapsto \langle u(t), \chi\rangle$, $t \mapsto \langle v(t), \theta\rangle$ are weakly differentiable on* $[0, T)$*, and it holds that, for almost all* $t \in (0, T)$*,*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \langle v(t), \theta\rangle &= -\mathfrak{a}(v(t), \theta) + \langle p_3(v(t)) - u(t) + I_{s,i}(t), \theta\rangle \\ &\qquad - \frac{k_0 \langle \mathcal{B}'v(t) - y_{\text{ref}}(t), \mathcal{B}'\theta\rangle_{\mathbb{R}^m}}{1 - \varphi(t)^2\|\mathcal{B}'v(t) - y_{\text{ref}}(t)\|^2_{\mathbb{R}^m}}, \\ \tfrac{\mathrm{d}}{\mathrm{d}t} \langle u(t), \chi\rangle &= \langle c_5 v(t) - c_4 u(t), \chi\rangle, \\ y(t) &= \mathcal{B}'v(t). \end{aligned} \tag{11}$$

b) *For global solutions it is desirable that $I_{s,e} \in L^\infty(\delta, \infty; \mathbb{R}^m)$ for all $\delta > 0$. Note that this is equivalent to*

$$\limsup_{t\to\infty} \varphi(t)^2\|\mathcal{B}'v(t) - y_{\text{ref}}(t)\|^2_{\mathbb{R}^m} < 1.$$

Furthermore, the output y is a signal that is measured and the control input $I_{s,e}$ is a signal used to manipulate the system, which hence must be generated by a certain device. For both measurement and generation of signals to be feasible it is desirable to have a certain regularity.

In the following we state the main result of the present paper. We will show that the closed-loop system (10) has a unique global solution so that all signals remain bounded. Furthermore, the tracking error stays uniformly away from the funnel boundary. We further show that we gain more regularity of the solution, if $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{r,2}(\Omega)')$ for some $r \in [0,1)$ or even $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega))$. Recall that $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{r,2}(\Omega)')$ if, and only if, $\mathcal{B}' \in \mathcal{L}(W^{r,2}(\Omega), \mathbb{R}^m)$. Furthermore, for any $r \in (0,1)$ we have the inclusions

$$\mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega)) \subset \mathcal{L}(\mathbb{R}^m, L^2(\Omega)) \subset \mathcal{L}(\mathbb{R}^m, W^{r,2}(\Omega)') \subset \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega)').$$

Theorem 3.3. *Use the assumptions from Definition 3.1. Furthermore, assume that* $\ker \mathcal{B} = \{0\}$ *and* $I_{s,i} \in L^\infty(0,\infty; L^2(\Omega))$*. Then there exists a unique solution of* (10) *on* $[0,\infty)$ *and we have*

(i) $u, \dot{u}, v \in BC([0,\infty); L^2(\Omega))$;

(ii) *for all* $\delta > 0$ *we have*

$$\begin{aligned} v &\in BUC([\delta,\infty); W^{1,2}(\Omega)) \cap C^{0,1/2}([\delta,\infty); L^2(\Omega)), \\ y, I_{s,e} &\in BUC([\delta,\infty); \mathbb{R}^m); \end{aligned}$$

(iii) $\exists\, \varepsilon_0 > 0\ \forall\, \delta > 0\ \forall\, t \geq \delta:\ \varphi(t)^2 \|\mathcal{B}' v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m} \leq 1 - \varepsilon_0.$

Furthermore,

a) *if additionally* $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{r,2}(\Omega)')$ *for some* $r \in (0,1)$*, then for all* $\delta > 0$ *we have that*

$$v \in C^{0,1-r/2}([\delta,\infty); L^2(\Omega)), \quad y, I_{s,e} \in C^{0,1-r}([\delta,\infty); \mathbb{R}^m).$$

b) *if additionally* $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, L^2(\Omega))$*, then for all* $\delta > 0$ *and all* $\lambda \in (0,1)$ *we have*

$$v \in C^{0,\lambda}([\delta,\infty); L^2(\Omega)), \quad y, I_{s,e} \in C^{0,\lambda}([\delta,\infty); \mathbb{R}^m).$$

c) *if additionally* $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega))$*, then for all* $\delta > 0$ *we have* $y, I_{s,e} \in W^{1,\infty}([\delta,\infty); \mathbb{R}^m)$.

Remark 3.4.

a) *The condition* $\ker \mathcal{B} = \{0\}$ *is equivalent to* $\operatorname{im} \mathcal{B}'$ *being dense in* $\mathbb{R}^m$*. The latter is equivalent to* $\operatorname{im} \mathcal{B}' = \mathbb{R}^m$ *by the finite-dimensionality of* $\mathbb{R}^m$.
Note that surjectivity of $\mathcal{B}'$ *is mandatory for tracking control, since it is necessary that any reference signal* $y_{\mathrm{ref}} \in W^{1,\infty}(0,\infty; \mathbb{R}^m)$ *can actually be generated by the output* $y(t) = \mathcal{B}' v$*. This property is sometimes called* right-invertibility*, see e.g. [34, Sec. 8.2].*

b) *If the input operator corresponds to Robin boundary control as discussed in Section 2.2, then* $\ker \mathcal{B} = \{0\}$ *and* $\mathcal{B} \in \mathcal{L}(\mathbb{R}, W^{\ell+1/2,2}(\Omega)')$ *for some* $\ell \in (0, 1/2]$*, thus the assertions of Theorem 3.3 a) hold.*

c) *If the input operator corresponds to distributed control as discussed in Section 2.1, then* $\ker \mathcal{B} = \{0\}$ *and* $\mathcal{B} \in \mathcal{L}(\mathbb{R}, L^2(\Omega))$*, thus the assertions of Theorem 3.3 b) hold.*

d) *The proof of Theorem 3.3, carried out in Section 4, exploits the properties of the Robin elliptic operator $\mathcal{A}$, which is self-adjoint, nonpositive, has compact resolvent and satisfies $\mathcal{D}(\mathcal{A}) \subset L^\infty(\Omega)$ (see Remark 2.2 a). Furthermore, the associated bilinear form $\mathfrak{a}$ is defined on a subset of $L^6(\Omega)$ (see Remark 2.4 b). In principle, the Robin elliptic operator can be replaced by an arbitrary operator $\mathcal{A}$ with the aforementioned properties. In this case, Theorem 3.3 is still valid – with the slight modification that the expression $W^{1,2}(\Omega)$ in* (ii) *has to be replaced with the interpolation space $(L^2(\Omega), \mathcal{D}(\mathcal{A}))_{1/2}$ (see Definition A.1).*
For instance, an elliptic operator $\operatorname{div} D\nabla$ *with domain including homogeneous Dirichlet boundary conditions exhibits the above mentioned properties as long as $\partial\Omega$ and D are sufficiently smooth.*

4. Proof of Theorem 3.3

The proof is inspired by the results of [35] on existence and uniqueness of (non-controlled) FitzHugh-Nagamo equations, which is based on a spectral approximation and subsequent convergence proofs by using arguments from [36]. We divide the proof in two major parts. First, we show that there exists a unique solution on the interval $[0, \gamma]$. After that we show that the solution also exists on (γ, ∞), is continuous at $t = \gamma$ and has the desired properties.

4.1. Solution on $[0, \gamma]$

Assuming that $t \in [0, \gamma]$, we have that $\varphi(t) \equiv 0$ so that we need to show existence of a pair of functions (v, u) with the properties as in Definition 2.3 (i)–(iii), where (8) simplifies to

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \langle v(t), \theta \rangle &= -\mathfrak{a}(v(t), \theta) + \langle p_3(v(t)) - u(t) + I_{s,i}(t), \theta \rangle + \langle I_{s,e}(t), \mathcal{B}'\theta \rangle_{\mathbb{R}^m} , \\ \tfrac{\mathrm{d}}{\mathrm{d}t} \langle u(t), \chi \rangle &= \langle c_5 v(t) - c_4 u(t), \chi \rangle , \\ I_{s,e}(t) &= -k_0(\mathcal{B}'v(t) - y_{\mathrm{ref}}(t)), \\ y(t) &= \mathcal{B}'v(t). \end{aligned} \tag{12}$$

Recall that $\mathfrak{a} : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{R}$ is the bilinear form (4).

Step 1: We show existence and uniqueness of a solution.
Step 1a: We show existence of a local solution on $[0, \gamma]$. To this end, let $(\theta_i)_{i\in\mathbb{N}_0}$ be the eigenfunctions of $-\mathcal{A}$ and α_i be the corresponding eigenvalues, with $\alpha_i \geq 0$ for all $i \in \mathbb{N}_0$. Recall that $(\theta_i)_{i\in\mathbb{N}_0}$ forms an orthonormal basis of $L^2(\Omega)$ by Remark 2.2 c). Hence, with $a_i := \langle v_0, \theta_i \rangle$ and $b_i := \langle u_0, \theta_i \rangle$ for $i \in \mathbb{N}_0$ and

$$v_0^n := \sum_{i=0}^n a_i\theta_i, \quad u_0^n := \sum_{i=0}^n b_i\theta_i, \quad n \in \mathbb{N},$$

we have that $v_0^n \to v_0$ and $u_0^n \to u_0$ strongly in $L^2(\Omega)$.
Fix $n \in \mathbb{N}_0$ and let $\gamma_i := \mathcal{B}'\theta_i$ for $i = 0, \dots, n$. Consider, for $j = 0, \dots, n$, the

differential equations

$$
\begin{aligned}
\dot{\mu}_j(t) &= -\alpha_j \mu_j(t) - \nu_j(t) - \left\langle k_0 \left(\sum_{i=0}^{n} \gamma_i \mu_i(t) - y_{\mathrm{ref}}(t) \right), \gamma_j \right\rangle_{\mathbb{R}^m} + \langle I_{s,i}(t), \theta_j \rangle \\
&\qquad + \left\langle p_3 \left(\sum_{i=0}^{n} \mu_i(t) \theta_i \right), \theta_j \right\rangle, \\
\dot{\nu}_j(t) &= -c_4 \nu_j(t) + c_5 \mu_j(t), \qquad\qquad \text{with } \mu_j(0) = a_j,\ \nu_j(0) = b_j,
\end{aligned}
\tag{13}
$$

defined on $\mathfrak{D} := [0, \infty) \times \mathbb{R}^{2(n+1)}$. Given that the functions defining the system of ODEs (13) are continuous, it follows from ODE theory, see e.g. [37, § 10, Thm. XX], that there exists a weakly differentiable solution $(\mu^n, \nu^n) = (\mu_0, \ldots, \mu_n, \nu_0, \ldots, \nu_n) : [0, T_n) \to \mathbb{R}^{2(n+1)}$ of (13) such that $T_n \in (0, \infty]$ is maximal. Furthermore, the closure of the graph of (μ^n, ν^n) is not a compact subset of $\mathfrak{D}$.

Now, set $v_n(t) := \sum_{i=0}^{n} \mu_i(t) \theta_i$ and $u_n(t) := \sum_{i=0}^{n} \nu_i(t) \theta_i$. We intend to show that (v_n) and (u_n) have subsequences which weakly converge to solutions of (10) on $[0, \gamma]$. Invoking (13) and using the functions θ_j we have that for $j = 0, \ldots, n$ the functions (v_n, u_n) satisfy

$$
\begin{aligned}
\langle \dot{v}_n(t), \theta_j \rangle &= -\mathfrak{a}(v_n(t), \theta_j) - \langle u_n(t), \theta_j \rangle + \langle p_3(v_n(t)), \theta_j \rangle + \langle I_{s,i}(t), \theta_j \rangle \\
&\qquad - \langle k_0(\mathcal{B}' v_n(t) - y_{\mathrm{ref}}(t)), \mathcal{B}' \theta_j \rangle_{\mathbb{R}^m}, \\
\langle \dot{u}_n(t), \theta_j \rangle &= -c_4 \langle u_n(t), \theta_j \rangle + c_5 \langle v_n(t), \theta_j \rangle.
\end{aligned}
\tag{14}
$$

Step 1b: We show the boundedness of (v_n, u_n). Consider the Lyapunov function candidate

$$
V : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R},\ (v, u) \mapsto \frac{1}{2}(c_5 \|v\|^2 + \|u\|^2). \tag{15}
$$

Observe that, since the family $(\theta_i)_{i \in \mathbb{N}_0}$ is orthonormal, we have $\|v_n\|^2 = \sum_{j=0}^{n} \mu_j^2$ and $\|u_n\|^2 = \sum_{j=0}^{n} \nu_j^2$. Hence we find that, for all $t \in [0, T_n)$,

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} V(v_n(t), u_n(t)) &\overset{(13)}{=} c_5 \sum_{j=0}^{n} \mu_j(t) \dot{\mu}_j(t) + \sum_{j=0}^{n} \nu_j(t) \dot{\nu}_j(t) \\
&= -c_5 \sum_{j=0}^{n} \alpha_j \mu_j(t)^2 - c_4 \sum_{j=0}^{n} \nu_j(t)^2 \\
&\quad - c_5 \left\langle k_0 \left(\sum_{i=0}^{n} \gamma_i \mu_i(t) - y_{\mathrm{ref}}(t) \right), \sum_{j=0}^{n} \gamma_j \mu_j(t) \right\rangle_{\mathbb{R}^m} \\
&\quad + c_5 \langle p_3(v_n(t)), v_n(t) \rangle + c_5 \langle I_{s,i}(t), v_n(t) \rangle
\end{aligned}
$$

hence, omitting the argument t for brevity in the following,

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} V(v_n, u_n) = &- c_5 \mathfrak{a}(v_n, v_n) - c_4 \|u_n\|^2 + c_5 \langle I_{s,i}, v_n \rangle \\
&- c_5 k_0 \|\overline{e}_n\|^2_{\mathbb{R}^m} + c_5 k_0 \langle \overline{e}_n, y_{\mathrm{ref}} \rangle_{\mathbb{R}^m} + c_5 \langle p_3(v_n), v_n \rangle,
\end{aligned}
\tag{16}
$$

where

$$\overline{e}_n(t) := \sum_{i=0}^{n} \gamma_i \mu_i(t) - y_{\mathrm{ref}}(t) = \mathcal{B}' v_n(t) - y_{\mathrm{ref}}(t).$$

Before proceeding, recall Young's inequality for products, i.e., for $a, b \geq 0$ and $p, q \geq 1$ such that $1/p + 1/q = 1$ we have that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

which will be frequently used in the following. Note that

$$\begin{aligned} \langle p_3(v_n), v_n \rangle &= -c_1 \|v_n\|^2 + c_2 \left\langle v_n^2, v_n \right\rangle - c_3 \|v_n\|_{L^4}^4, \\ c_2 |\left\langle v_n^2, v_n \right\rangle| &= |\left\langle \epsilon v_n^3, \epsilon^{-1} c_2 \right\rangle| \leq \frac{3\epsilon^{4/3}}{4} \|v_n\|_{L^4}^4 + \frac{c_2^4}{4\epsilon^4} |\Omega|, \end{aligned}$$

where the latter follows from Young's inequality with $p = \frac{4}{3}$ and $q = 4$. Choosing $\epsilon = \left(\frac{2}{3} c_3\right)^{\frac{3}{4}}$ we obtain

$$\langle p_3(v_n), v_n \rangle \leq \frac{27 c_2^4}{32 c_3^3} |\Omega| - c_1 \|v_n\|^2 - \frac{c_3}{2} \|v_n\|_{L^4}^4.$$

Moreover,

$$\langle \overline{e}_n, y_{\mathrm{ref}} \rangle_{\mathbb{R}^m} \leq \frac{1}{2} \|\overline{e}_n\|_{\mathbb{R}^m}^2 + \frac{1}{2} \|y_{\mathrm{ref}}\|_{\mathbb{R}^m}^2 \leq \frac{1}{2} \|\overline{e}_n\|_{\mathbb{R}^m}^2 + \frac{1}{2} \|y_{\mathrm{ref}}\|_{\infty}^2$$

and

$$\langle I_{s,i}, v_n \rangle \leq \frac{c_1}{2} \|v_n\|^2 + \frac{1}{2c_1} \|I_{s,i}\|^2 \leq \frac{c_1}{2} \|v_n\|^2 + \frac{1}{2c_1} \|I_{s,i}\|_{2,\infty}^2,$$

where $\|I_{s,i}\|_{2,\infty} = \operatorname{ess\,sup}_{t \geq 0} \left(\int_\Omega |I_{s,i}(\zeta, t)|^2 \, \mathrm{d}\zeta \right)^{1/2}$, so that with

$$C_\infty := \frac{k_0 c_5}{2} \|y_{\mathrm{ref}}\|_\infty^2 + \frac{c_5}{2c_1} \|I_{s,i}\|_{2,\infty}^2 + \frac{27 c_2^4}{32 c_3^3} |\Omega|,$$

we may further estimate (16) by

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} V(v_n, u_n) \leq &- c_5 \mathfrak{a}(v_n, v_n) - c_4 \|u_n\|^2 - \frac{c_1 c_5}{2} \|v_n\|^2 - \frac{c_5 k_0}{2} \|\overline{e}_n\|_{\mathbb{R}^m}^2 \\ &- \frac{c_3 c_5}{2} \|v_n\|_{L^4}^4 + C_\infty \\ \leq &- c_5 \mathfrak{a}(v_n, v_n) - \frac{c_1 c_5}{2} \|v_n\|^2 - \frac{c_5 k_0}{2} \|\overline{e}_n\|_{\mathbb{R}^m}^2 - \frac{c_3 c_5}{2} \|v_n\|_{L^4}^4 + C_\infty. \end{aligned}$$

Then we obtain that, for all $t \in [0, T_n)$,

$$\begin{aligned} &V(v_n(t), u_n(t)) + c_5 \int_0^t \mathfrak{a}(v_n(s), v_n(s)) \, \mathrm{d}s + \frac{c_1 c_5}{2} \int_0^t \|v_n(s)\|^2 \, \mathrm{d}s \\ &+ \frac{c_5 k_0}{2} \int_0^t \|\overline{e}_n(s)\|_{\mathbb{R}^m}^2 \, \mathrm{d}s + \frac{c_3 c_5}{2} \int_0^t \|v_n(s)\|_{L^4}^4 \, \mathrm{d}s \leq V(v_0^n, u_0^n) + C_\infty t. \end{aligned}$$

Since $(u_n^0, v_n^0) \to (u_0, v_0)$ strongly in $L^2(\Omega)$ and since we have for all $p \in L^2(\Omega)$ that $\|\sum_{i=0}^n \langle p, \theta_i\rangle \theta_i\|^2 \le \|p\|^2$, it follows that, for all $t \in [0, T_n)$,

$$\begin{aligned} &c_5\|v_n(t)\|^2 + \|u_n(t)\|^2 + 2c_5 \int_0^t \mathfrak{a}(v_n(s), v_n(s))\,\mathrm{d}s \,+ c_1c_5 \int_0^t \|v_n(s)\|^2\,\mathrm{d}s \\ &\quad + c_5k_0 \int_0^t \|\overline{e}_n(s)\|^2_{\mathbb{R}^m}\,\mathrm{d}s \,+ c_3c_5 \int_0^t \|v_n(s)\|^4_{L^4}\,\mathrm{d}s \,\le 2C_\infty t + c_5\|u_0\|^2 + \|v_0\|^2. \end{aligned} \tag{17}$$

Step 1c: We show that $T_n = \infty$. Assume that $T_n < \infty$, then it follows from (17) together with (1) and (4) that (v_n, u_n) is bounded, thus the solution (μ^n, ν^n) of (13) is bounded on $[0, T_n)$. But this implies that the closure of the graph of (μ^n, ν^n) is a compact subset of $\mathfrak{D}$, a contradiction. Therefore, $T_n = \infty$ and in particular the solution is defined for all $t \in [0, \gamma]$.

Step 1d: We show convergence of (v_n, u_n) *to a solution of* (12) *on* $[0, \gamma]$. First note that it follows from (17) that

$$\forall\, t \in [0, \gamma]: \; \|v_n(t)\|^2 \le C_v, \quad \|u_n(t)\|^2 \le C_u \tag{18}$$

for some $C_v, C_u > 0$. From (17) and condition (1) in Assumption 2.1 it follows that there is a constant $C_\delta > 0$ such that

$$\int_0^\gamma \|\nabla v_n(t)\|^2\,\mathrm{d}t \,\le \delta^{-1} \int_0^\gamma \mathfrak{a}(v_n(t), v_n(t))\,\mathrm{d}t \,\le C_\delta.$$

This together with (17) and (18) implies that there exist constants $C_1, C_2 > 0$ with

$$\|v_n\|^4_{L^4(0,\gamma;L^4(\Omega))} \le C_1, \quad \|v_n\|_{L^2(0,\gamma;W^{1,2}(\Omega))} \le C_2. \tag{19}$$

Note that (19) directly implies that

$$\begin{aligned} \|v_n^2\|^2_{L^2(0,\gamma;L^2(\Omega))} &= \|v_n\|^4_{L^4(0,\gamma;L^4(\Omega))} \le C_1, \\ \|v_n^3\|_{L^{4/3}(0,\gamma;L^{4/3}(\Omega))} &= \left(\|v_n^2\|^2_{L^2(0,\gamma;L^2(\Omega))}\right)^{3/4} \le C_1^{3/4}. \end{aligned} \tag{20}$$

Multiplying the second equation in (14) by $\dot{\nu}_j$ and summing over $j \in \{0, \dots, n\}$ leads to

$$\begin{aligned} \|\dot{u}_n\|^2 &= -\frac{c_4}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2 + c_5\,\langle v_n, \dot{u}_n\rangle \\ &\le -\frac{c_4}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2 + \frac{c_5^2}{2}\|v_n\|^2 + \frac{1}{2}\|\dot{u}_n\|^2, \end{aligned}$$

thus

$$\|\dot{u}_n\|^2 \le -c_4\tfrac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2 + c_5^2\|v_n\|^2.$$

Upon integration over $[0, \gamma]$ and using (18) this yields that

$$\int_0^\gamma \|\dot{u}_n(t)\|^2\,\mathrm{d}t \,\le c_4C_u + c_5^2 \int_0^\gamma \|v_n(t)\|^2\,\mathrm{d}t \,\le \hat{C}_3$$

for some $\hat{C}_3 > 0$, where the last inequality is a consequence of (17). This together with (18) implies that there is $C_3 > 0$ such that $\|u_n\|_{W^{1,2}(0,\gamma;L^2(\Omega))} \leq C_3$.

Now, let P_n be the orthogonal projection of $L^2(\Omega)$ onto the subspace generated by the set $\{\ \theta_i\ \mid i = 1, \dots, n\ \}$. Consider

$$\|v\|_{W^{1,2}} = \big(\|v\|^2 + \mathfrak{a}(v,v)\big)^{1/2} = \left(\sum_{i=0}^{n}(1+\alpha_i)|\,\langle v, \theta_i\rangle\,|^2\right)^{1/2}, \quad v \in P_n\big(L^2(\Omega)\big),$$

which is – by Remark A.3 – indeed a norm on (the projection of) $W^{1,2}(\Omega)$ which is equivalent to the standard norm on $W^{1,2}(\Omega)$ by the properties of D and a in Assumption 2.1. By duality we have that

$$\|\hat{v}\|_{(W^{1,2})'} = \left(\sum_{i=0}^{n}(1+\alpha_i)^{-1}|\,\langle \hat{v}, \theta_i\rangle\,|^2\right)^{1/2}$$

is a norm on (the projection of) $W^{1,2}(\Omega)'$, cf. [38, Prop. 3.4.8]. Note that we can consider $P_n : W^{1,2}(\Omega)' \to W^{1,2}(\Omega)'$, which is a bounded linear operator with norm one, independent of n. Using this together with the fact that the injection from $L^2(\Omega)$ into $W^{1,2}(\Omega)'$ is continuous and $\mathcal{A} \in \mathcal{L}(W^{1,2}(\Omega), W^{1,2}(\Omega)')$, we can rewrite the weak formulation (14) as

$$\dot{v}_n = P_n\mathcal{A}v_n + P_n p_3(v_n) - P_n u_n + P_n I_{s,i} - P_n\mathcal{B}k_0(\mathcal{B}'v_n - y_{\mathrm{ref}}). \tag{21}$$

Since $v_n \in L^2(0,\gamma;W^{1,2}(\Omega))$ and hence, by the Sobolev embedding theorem, $v_n \in L^2(0,\gamma;L^p(\Omega))$ for all $2 \leq p \leq 6$, we find that $p_3(v_n) \in L^2(0,\gamma;L^2(\Omega))$. We also have $\mathcal{A}v_n \in L^2(0,\gamma;W^{1,2}(\Omega)')$ and $\mathcal{B}k_0(\mathcal{B}'v_n - y_{\mathrm{ref}}) \in L^2(0,\gamma;W^{1,2}(\Omega)')$ so that by using the estimates (17)–(20) together with (21), there exists $C_4 > 0$ independent of n and t with

$$\|\dot{v}_n\|_{L^2(0,\gamma;W^{1,2}(\Omega)')} \leq C_4.$$

Now, by Lemma B.6 we have that there exist subsequences of (u_n), (v_n) and $(\dot{v}_n)$, resp., again denoted in the same way, for which

$$\begin{aligned}
u_n &\to u \in W^{1,2}(0,\gamma;L^2(\Omega)) \text{ weakly},\\
u_n &\to u \in W^{1,\infty}(0,\gamma;L^2(\Omega)) \text{ weak}^\star,\\
v_n &\to v \in L^2(0,\gamma;W^{1,2}(\Omega)) \text{ weakly},\\
v_n &\to v \in L^\infty(0,\gamma;L^2(\Omega)) \text{ weak}^\star,\\
v_n &\to v \in L^4(0,\gamma;L^4(\Omega)) \text{ weakly},\\
\dot{v}_n &\to \dot{v} \in L^2(0,\gamma;W^{1,2}(\Omega)') \text{ weakly}.
\end{aligned} \tag{22}$$

Moreover, let $p_0 = p_1 = 2$ and $X = W^{1,2}(\Omega)$, $Y = L^2(\Omega)$, $Z = W^{1,2}(\Omega)'$. Then, [36, Chap. 1, Thm. 5.1] implies that

$$W := \{\ u \in L^{p_0}(0,\gamma;X)\ \mid \dot{u} \in L^{p_1}(0,\gamma;Z)\ \}$$

with norm $\|u\|_{L^{p_0}(0,\gamma;X)}+\|\dot{u}\|_{L^{p_1}(0,\gamma;Z)}$ has a compact injection into $L^{p_0}(0,\gamma;Y)$, so that the weakly convergent sequence $v_n \to v \in W$ converges strongly in $L^2(0,\gamma;L^2(\Omega))$ by [39, Lem. 1.6]. Further, $(u(0),v(0)) = (u_0,v_0)$ and by $v \in W^{1,2}(0,\gamma;L^2(\Omega))$, $v \in L^2(0,\gamma;W^{1,2}(\Omega))$ and $\dot{v} \in L^2(0,\gamma;W^{1,2}(\Omega)')$ it follows that $u,v \in C([0,\gamma];L^2(\Omega))$, see for instance [39, Thm. 1.32]. Moreover, note that $\mathcal{B}'v - y_{\mathrm{ref}} \in L^2(0,\gamma;\mathbb{R}^m)$. Hence, (u,v) is a solution of (10) in $[0,\gamma]$ and

$$\begin{aligned} \dot{v}(t) &= \mathcal{A}v(t) + p_3(v(t)) - u(t) + I_{s,i}(t) - \mathcal{B}k_0(\mathcal{B}'v(t) - y_{\mathrm{ref}}(t)), && v(0) = v_0,\\ \dot{u}(t) &= c_5 v(t) - c_4 u(t), && u(0) = u_0, \end{aligned} \tag{23}$$

is satisfied in $W^{1,2}(\Omega)'$. Moreover, by (20), [36, Chap. 1, Lem. 1.3] and $v_n \to v$ in $L^4(0,\gamma;L^4(\Omega))$ we have that $v_n^3 \to v^3$ weakly in $L^{4/3}(0,\gamma;L^{4/3}(\Omega))$ and $v_n^2 \to v^2$ weakly in $L^2(0,\gamma;L^2(\Omega))$.

Step 1e: We show uniqueness of the solution (v,u)*.* To this end, we separate the linear part of p_3 so that

$$p_3(v) = -c_1 v - c_3 \hat{p}_3(v), \quad \hat{p}_3(v) := v^2\,(v-c)\,, \quad c := c_2/c_3.$$

Assume that (v_1,u_1) and (v_2,u_2) are two solutions of (10) on $[0,\gamma]$ with the same initial values, $v_1(0) = v_2(0) = v_0$ and $u_1(0) = u_2(0) = u_0$. Let $t_0 \in (0,\gamma]$ be given. Let $Q_0 := (0,t_0)\times\Omega$. Define

$$\Sigma(t,\zeta) := |v_1(t,\zeta)| + |v_2(t,\zeta)|,$$

and let

$$Q^\Lambda := \{(t,\zeta) \in Q_0 \mid \Sigma(t,\zeta) \le \Lambda\}, \quad \Lambda > 0.$$

Note that, by convexity of the map $x \mapsto x^p$ on $[0,\infty)$ for $p > 1$, we have that

$$\forall\, a,b \ge 0:\ \left(\tfrac{1}{2}a + \tfrac{1}{2}b\right)^p \le \tfrac{1}{2}a^p + \tfrac{1}{2}b^p.$$

Therefore, since $v_1,v_2 \in L^4(0,\gamma;L^4(\Omega))$, we find that $\Sigma \in L^4(0,\gamma;L^4(\Omega))$. Hence, by the monotone convergence theorem, for all $\epsilon > 0$ we may choose Λ large enough such that

$$\int_{Q_0\setminus Q^\Lambda} |\Sigma(\zeta,t)|^4 \,\mathrm{d}\zeta\,\mathrm{d}t < \epsilon.$$

Note that without loss of generality we may assume that $\Lambda > \frac{c}{3}$. Let $V := v_2 - v_1$ and $U := u_2 - u_1$, then, by (10),

$$\begin{aligned} \dot{V} &= (\mathcal{A} - c_1 I)V - c_3(\hat{p}_3(v_2) - \hat{p}_3(v_1)) - U - k_0\mathcal{B}\mathcal{B}'V,\\ \dot{U} &= c_5 V - c_4 U. \end{aligned}$$

By [39, Thm. 1.32], we have for all $t \in (0,\gamma)$ that

$$\begin{aligned} \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|^2 &= \left\langle \dot{V}(t), V(t)\right\rangle_{W^{1,2}(\Omega)',W^{1,2}(\Omega)},\\ \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|^2 &= \left\langle \dot{U}(t), U(t)\right\rangle_{W^{1,2}(\Omega)',W^{1,2}(\Omega)}, \end{aligned}$$

thus we may compute that

$$\begin{aligned}
\tfrac{c_5}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|V\|^2 + \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 &= \langle(\mathcal{A}-c_1 I)V - U - k_0\mathcal{B}\mathcal{B}'V, c_5 V\rangle_{W^{1,2}(\Omega)',W^{1,2}(\Omega)} \\
&\quad - c_4\|U\|^2 + c_5\langle U, V\rangle - c_5 c_3\left\langle \hat{p}_3(v_2) - \hat{p}_3(v_1), V\right\rangle \\
&= c_5\left\langle(\mathcal{A}-c_1 I)V, V\right\rangle_{W^{1,2}(\Omega)',W^{1,2}(\Omega)} - c_5 k_0\left\langle \mathcal{B}'V, \mathcal{B}'V\right\rangle_{\mathbb{R}^m} \\
&\quad - c_4\|U\|^2 - c_5 c_3\left\langle \hat{p}_3(v_2) - \hat{p}_3(v_1), V\right\rangle \\
&\le -c_5 c_3\left\langle \hat{p}_3(v_2) - \hat{p}_3(v_1), V\right\rangle.
\end{aligned}$$

Integration over $[0, t_0]$ and using $(U(0), V(0)) = (0, 0)$ leads to

$$\begin{aligned}
\tfrac{c_5}{2}\|V(t_0)\|^2 + \tfrac{1}{2}\|U(t_0)\|^2 &\le -c_5 c_3\int_0^{t_0}\int_\Omega (\hat{p}_3(v_2(\zeta,t)) - \hat{p}_3(v_1(\zeta,t)))V(\zeta,t)\,\mathrm{d}\zeta\,\mathrm{d}t \\
&= -c_5 c_3\int_{Q^\Lambda}(\hat{p}_3(v_2(\zeta,t)) - \hat{p}_3(v_1(\zeta,t)))V(\zeta,t)\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\quad - c_5 c_3\int_{Q_0\setminus Q^\Lambda}(\hat{p}_3(v_2(\zeta,t)) - \hat{p}_3(v_1(\zeta,t)))V(\zeta,t)\,\mathrm{d}\zeta\,\mathrm{d}t\,.
\end{aligned} \tag{24}$$

Note that on Q^Λ we have $-\Lambda \le v_1 \le \Lambda$ and $-\Lambda \le v_2 \le \Lambda$. Let $a, b \in [-\Lambda, \Lambda]$, then the mean value theorem implies

$$(\hat{p}_3(b) - \hat{p}_3(a))(b-a) = \hat{p}_3'(\xi)(b-a)^2$$

for some $\xi \in (-\Lambda, \Lambda)$. Since $\hat{p}_3'(\xi) = 3\xi^2 - 2c\xi$ has a minimum at

$$\xi^* = \frac{c}{3}$$

we have that

$$(\hat{p}_3(b) - \hat{p}_3(a))(b-a) = \hat{p}_3'(\xi)(b-a)^2 \ge -\frac{c^2}{3}(b-a)^2.$$

Using that in inequality (24) leads to

$$\begin{aligned}
\tfrac{c_5}{2}\|V(t_0)\|^2 + \tfrac{1}{2}\|U(t_0)\|^2 &\le c_5 c_3\frac{c^2}{3}\int_{Q^\Lambda} V(\zeta,t)^2\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\quad - c_5 c_3\int_{Q_0\setminus Q^\Lambda}(\hat{p}_3(v_2(\zeta,t)) - \hat{p}_3(v_1(\zeta,t)))V(\zeta,t)\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\le c_5 c_3\frac{c^2}{3}\int_{Q_0} V(\zeta,t)^2\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\quad + c_5 c_3\int_{Q_0\setminus Q^\Lambda}|\hat{p}_3(v_2(\zeta,t)) - \hat{p}_3(v_1(\zeta,t))||V(\zeta,t)|\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\le c_5 c_3\frac{c^2}{3}\int_0^{t_0}\|V(t)\|^2\,\mathrm{d}t + 2c_5 c_3\int_{Q_0\setminus Q^\Lambda}|\Sigma(\zeta,t)|^4\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\le c_3\frac{c^2}{3}\int_0^{t_0} c_5\|V(t)\|^2 + \|U(t)\|^2\,\mathrm{d}t + 2c_5 c_3\epsilon.
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary we may infer that

$$\tfrac{c_5}{2}\|V(t_0)\|^2 + \tfrac{1}{2}\|U(t_0)\|^2 \le \frac{2c_3c^2}{3}\int_0^{t_0} \tfrac{c_5}{2}\|V(t)\|^2 + \tfrac{1}{2}\|U(t)\|^2 \,\mathrm{d}t\,.$$

Hence, by Gronwall's lemma and $U(0) = 0, V(0) = 0$ it follows that $U(t_0) = 0$ and $V(t_0) = 0$. Since t_0 was arbitrary, this shows that $v_1 = v_2$ and $u_1 = u_2$ on $[0, \gamma]$.

Step 2: We show that for all $\epsilon \in (0, \gamma)$ and all $t \in [\epsilon, \gamma]$ we have $v(t) \in W^{1,2}(\Omega)$. In particular, this guarantees $v(\gamma) \in W^{1,2}(\Omega)$, which is required as an initial condition in the second part of the proof in Section 4.2.

Fix $\epsilon \in (0, \gamma)$. First we show that $v \in BUC([\epsilon, \gamma]; W^{1,2}(\Omega))$. Multiplying the first equation in (14) by $\dot{\mu}_j$ and summing over $j \in \{0, \ldots, n\}$ we obtain

$$\begin{aligned}
\|\dot{v}_n\|^2 &= -\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(v_n, v_n) - \langle u_n, \dot{v}_n\rangle + \langle p_3(v_n), \dot{v}_n\rangle + \langle I_{s,i}, \dot{v}_n\rangle \\
&\quad - k_0\, \langle \mathcal{B}' v_n - y_{\mathrm{ref}}, \mathcal{B}'\dot{v}_n\rangle_{\mathbb{R}^m} \\
&= -\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(v_n, v_n) - \langle u_n, \dot{v}_n\rangle + \langle p_3(v_n), \dot{v}_n\rangle + \langle I_{s,i}, \dot{v}_n\rangle \\
&\quad - k_0\, \langle \mathcal{B}' v_n - y_{\mathrm{ref}}, \mathcal{B}'\dot{v}_n - \dot{y}_{\mathrm{ref}}\rangle_{\mathbb{R}^m} - k_0\, \langle \mathcal{B}' v_n - y_{\mathrm{ref}}, \dot{y}_{\mathrm{ref}}\rangle_{\mathbb{R}^m}
\end{aligned}$$

Furthermore, we may derive that

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} v_n^4 &= 4v_n^3\dot{v}_n = -\frac{4}{c_3}\big(p_3(v_n) - c_2 v_n^2 + c_1 v_n\big)\dot{v}_n, \quad \text{thus} \\
p_3(v_n)\dot{v}_n &= -\frac{c_3}{4}\tfrac{\mathrm{d}}{\mathrm{d}t} v_n^4 + c_2 v_n^2 \dot{v}_n - c_1 v_n \dot{v}_n,
\end{aligned}$$

and this implies, for any $\eta > 0$,

$$\begin{aligned}
\langle p_3(v_n), \dot{v}_n\rangle &\le -\frac{c_3}{4}\tfrac{\mathrm{d}}{\mathrm{d}t}\|v_n\|_{L^4}^4 + c_2\,\langle v_n^2, \dot{v}_n\rangle - c_1\,\langle v_n, \dot{v}_n\rangle \\
&\le -\frac{c_3}{4}\tfrac{\mathrm{d}}{\mathrm{d}t}\|v_n\|_{L^4}^4 + \frac{c_2}{2}\left(\eta\|v_n\|_{L^4}^4 + \frac{1}{\eta}\|\dot{v}_n\|^2\right) + \frac{c_1}{2}\left(\eta\|v_n\|^2 + \frac{1}{\eta}\|\dot{v}_n\|^2\right) \\
&\overset{(18)}{\le} -\frac{c_3}{4}\tfrac{\mathrm{d}}{\mathrm{d}t}\|v_n\|_{L^4}^4 + \frac{c_2}{2}\left(\eta\|v_n\|_{L^4}^4 + \frac{1}{\eta}\|\dot{v}_n\|^2\right) + \frac{c_1}{2}\left(\eta C_v + \frac{1}{\eta}\|\dot{v}_n\|^2\right).
\end{aligned}$$

Moreover, we find that, recalling $\overline{e}_n = \mathcal{B}' v_n - y_{\mathrm{ref}}$

$$\begin{aligned}
\langle u_n, \dot{v}_n\rangle &\le \frac{\eta}{2}\|u_n\|^2 + \frac{1}{\eta}\|\dot{v}_n\|^2 \overset{(18)}{\le} \frac{\eta C_u}{2} + \frac{1}{\eta}\|\dot{v}_n\|^2, \\
\langle I_{s,i}, \dot{v}_n\rangle &\le \frac{\eta}{2}\|I_{s,i}\|_{2,\infty}^2 + \frac{1}{\eta}\|\dot{v}_n\|^2, \\
\langle \overline{e}_n, \dot{y}_{\mathrm{ref}}\rangle_{\mathbb{R}^m} &\le \frac{1}{2}\|\overline{e}_n\|_{\mathbb{R}^m}^2 + \frac{1}{2}\|\dot{y}_{\mathrm{ref}}\|_\infty^2.
\end{aligned}$$

Therefore, choosing η large enough, we obtain that there exist constants $Q_1, Q_2 > 0$ independent of n such that

$$\begin{aligned}
\|\dot{v}_n\|^2 \le &-\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(v_n, v_n) - \frac{c_3}{4}\tfrac{\mathrm{d}}{\mathrm{d}t}\|v_n\|_{L^4}^4 - \frac{k_0}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|\overline{e}_n\|_{\mathbb{R}^m}^2 + \frac{1}{2}\|\dot{v}_n\|^2 \\
&+ Q_1\|v_n\|_{L^4}^4 + Q_2 + \frac{k_0}{2}\|\overline{e}_n\|_{\mathbb{R}^m}^2,
\end{aligned}$$

thus,

$$\begin{aligned}\|\dot{v}_n\|^2 + \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{a}(v_n, v_n) + \frac{c_3}{2}\|v_n\|_{L^4}^4 + k_0\|\overline{e}_n\|_{\mathbb{R}^m}^2\right)&\\ \leq 2Q_1\|v_n\|_{L^4}^4 + 2Q_2 + k_0\|\overline{e}_n\|_{\mathbb{R}^m}^2.&\end{aligned} \tag{25}$$

As a consequence, we find that for all $t \in [0, \gamma]$ we have

$$\begin{aligned} t\|\dot{v}_n(t)\|^2 &+ \tfrac{\mathrm{d}}{\mathrm{d}t}\left(t\mathfrak{a}(v_n(t), v_n(t)) + \frac{c_3 t}{2}\|v_n(t)\|_{L^4}^4 + k_0 t\|\overline{e}_n(t)\|_{\mathbb{R}^m}^2\right)\\ &\overset{(25)}{\leq} \left(2Q_1 t + \frac{c_3}{2}\right)\|v_n(t)\|_{L^4}^4 + \mathfrak{a}(v_n(t), v_n(t)) + 2Q_2 t + k_0(t+1)\|\overline{e}_n(t)\|_{\mathbb{R}^m}^2. \end{aligned}$$

Since $t\|\dot{v}_n(t)\|^2 \geq 0$ and $t \leq \gamma$ for all $t \in [0, \gamma]$, it follows that

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\left(t\mathfrak{a}(v_n(t), v_n(t)) + \frac{c_3 t}{2}\|v_n(t)\|_{L^4}^4 + k_0 t\|\overline{e}_n(t)\|_{\mathbb{R}^m}^2\right)\\ &\leq \left(2Q_1\gamma + \frac{c_3}{2}\right)\|v_n(t)\|_{L^4}^4 + \mathfrak{a}(v_n(t), v_n(t)) + 2Q_2\gamma + k_0(\gamma+1)\|\overline{e}_n(t)\|_{\mathbb{R}^m}^2. \end{aligned}$$

Integrating the former and using (17), there exist $P_1, P_2 > 0$ independent of n such that for $t \in [0, \gamma]$ we have

$$t\mathfrak{a}(v_n(t), v_n(t)) + \frac{c_3 t}{2}\|v_n(t)\|_{L^4}^4 + k_0 t\|\overline{e}_n(t)\|_{\mathbb{R}^m}^2 \leq P_1 + P_2 t.$$

Thus, there exist constants $C_5, C_6 > 0$ independent of n such that

$$\forall\, t \in [0, \gamma]:\ t\mathfrak{a}(v_n(t), v_n(t)) \leq C_5 \ \wedge\ t\|\overline{e}_n(t)\|_{\mathbb{R}^m} \leq C_6.$$

Hence, for all $\epsilon \in (0, \gamma)$, it follows from the above estimates together with (17) that $v_n \in L^\infty(\epsilon, \gamma; W^{1,2}(\Omega))$ and $\overline{e}_n \in L^\infty(\epsilon, \gamma; \mathbb{R}^m)$, so that in addition to (22), from Lemma B.6 we further have that there exists a subsequence such that

$$v_n \to v \in L^\infty(\epsilon, \gamma; W^{1,2}(\Omega)) \text{ weak}^\star$$

and $\mathcal{B}'v \in L^\infty(\epsilon, \gamma; \mathbb{R}^m)$ for all $\epsilon \in (0, \gamma)$, hence $I_{s,e} \in L^2(0, \gamma; \mathbb{R}^m) \cap L^\infty(\epsilon, \gamma; \mathbb{R}^m)$. By the Sobolev embedding theorem, $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $2 \leq p \leq 6$ we have that $p_3(v) \in L^\infty(\epsilon, \gamma; L^2(\Omega))$. Moreover, since (23) holds, we can rewrite it as

$$\dot{v}(t) = (\mathcal{A} - c_1 I)v(t) + I_r(t) + \mathcal{B}I_{s,e}(t),$$

where $I_r := c_2 v^2 - c_3 v^3 - u + I_{s,i} \in L^2(0, \gamma; L^2(\Omega)) \cap L^\infty(\epsilon, \gamma; L^2(\Omega))$ and Proposition B.5 (recall that $W^{1,2}(\Omega)' = X_{-1/2}$ and hence $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, X_{-1/2})$) with $c = c_1$ implies that $v \in BUC([\epsilon, \gamma]; W^{1,2}(\Omega))$. Hence, for all $\epsilon \in (0, \gamma)$, $v(t) \in W^{1,2}(\Omega)$ for $t \in [\epsilon, \gamma]$, so that in particular $v(\gamma) \in W^{1,2}(\Omega)$.

4.2. Solution on (γ, ∞)

The crucial step in this part of the proof is to show that the error remains uniformly bounded away from the funnel boundary while $v \in L^\infty(\gamma, \infty; W^{1,2}(\Omega))$. The proof is divided into several steps.

Step 1: We show existence of an approximate solution by means of a time-varying state-space transformation.
Again, let $(\theta_i)_{i\in\mathbb{N}_0}$ be the eigenfunctions of $-\mathcal{A}$ and let α_i be the corresponding eigenvalues, with $\alpha_i \geq 0$ for all $i \in \mathbb{N}_0$. Recall that $(\theta_i)_{i\in\mathbb{N}_0}$ forms an orthonormal basis of $L^2(\Omega)$ by Remark 2.2 c). Let $(u_\gamma, v_\gamma) := (u(\gamma), v(\gamma))$, $a_i := \langle v_\gamma, \theta_i\rangle$ and $b_i := \langle u_\gamma, \theta_i\rangle$ for $i \in \mathbb{N}_0$ and

$$v_\gamma^n := \sum_{i=0}^n a_i\theta_i, \quad u_\gamma^n := \sum_{i=0}^n b_i\theta_i, \quad n \in \mathbb{N}.$$

Then we have that $v_\gamma^n \to v_\gamma$ strongly in $W^{1,2}(\Omega)$ and $u_\gamma^n \to u_\gamma$ strongly in $L^2(\Omega)$. As stated in Remark 3.4 a) we have that $\ker \mathcal{B} = \{0\}$ implies $\mathcal{B}'\mathcal{D}(\mathcal{A}) = \mathbb{R}^m$. As a consequence, there exist $q_1, \ldots, q_m \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{B}'q_k = e_k$, where e_k denotes the k-th unit vector in $\mathbb{R}^m$ for $k = 1, \ldots, m$. By Remark 2.2 a), we further have $q_k \in C^{0,\nu}(\Omega)$ for some $\nu \in (0, 1)$.
Note that $U := \bigcup_{n\in\mathbb{N}} U_n$, where $U_n = \operatorname{span}\{\theta_i\}_{i=0}^n$, satisfies $\overline{U} = W^{1,2}(\Omega)$ with the respective norm. Moreover, $\overline{\mathcal{B}'U} = \mathbb{R}^m$. Since $\mathbb{R}^m$ is complete and finite dimensional and $\mathcal{B}'$ is linear and continuous it follows that $\mathcal{B}'U = \mathbb{R}^m$. By the surjectivity of $\mathcal{B}'$ we have that for all $k \in \{1, \ldots, m\}$ there exist $n_k \in \mathbb{N}$ and $q_k \in U_{n_k}$ such that $\mathcal{B}'q_k = e_k$. Thus, there exists $n_0 \in \mathbb{N}$ with $q_k \in U_{n_0}$ for all $k = \{1, \ldots, m\}$, hence the q_k are a (finite) linear combination of the eigenfunctions θ_i.
Define $q \in W^{1,2}(\Omega; \mathbb{R}^m) \cap C^{0,\nu}(\Omega; \mathbb{R}^m)$ by $q(\zeta) = \big(q_1(\zeta), \ldots, q_m(\zeta)\big)^\top$ and $q \cdot y_{\mathrm{ref}}$ by

$$(q \cdot y_{\mathrm{ref}})(t, \zeta) := \sum_{k=1}^m q_k(\zeta) y_{\mathrm{ref},k}(t), \quad \zeta \in \Omega,\, t \geq 0.$$

We may define $q \cdot \dot{y}_{\mathrm{ref}}$ analogously. Note that we have $(q \cdot y_{\mathrm{ref}}) \in BC([0, \infty) \times \Omega)$, because

$$|(q \cdot y_{\mathrm{ref}})(t, \zeta)| \leq \sum_{k=1}^m \|q_k\|_\infty \|y_{\mathrm{ref},k}\|_\infty$$

for all $\zeta \in \Omega$ and $t \geq 0$, where we write $\|\cdot\|_\infty$ for the supremum norm. We define $q_{k,j} := \langle q_k, \theta_j\rangle$ for $k = 1, \ldots, m$, $j \in \mathbb{N}_0$ and $q_k^n := \sum_{j=0}^n q_{k,j}$ for $n \in \mathbb{N}_0$. Similarly, $q^n := (q_1^n, \ldots, q_m^n)^\top$, $n \in \mathbb{N}$, satisfies $q^n = q$ for all $n \geq n_0$ since $q_k \in U_{n_0}$ for all $k = 1, \ldots, m$, thus $q^n \to q$ strongly in $W^{1,2}(\Omega)$.
Since $\mathcal{B}' : W^{r,2}(\Omega) \to \mathbb{R}^m$ is continuous for some $r \in [0, 1]$, it follows that for all $\theta \in W^{r,2}(\Omega)$ there exists $\Gamma_r > 0$ such that

$$\|\mathcal{B}'\theta\|_{\mathbb{R}^m} \leq \Gamma_r \|\theta\|_{W^{r,2}}.$$

For $n \in \mathbb{N}_0$, let

$$\kappa_n := \big((n + 1)\Gamma_r(1 + \|v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma)\|_{W^{r,2}}^2)\big)^{-1}.$$

Note that for $v_\gamma \in W^{1,2}(\Omega)$ it holds that $\kappa_n > 0$ for all $n \in \mathbb{N}_0$, $(\kappa_n)_{n\in\mathbb{N}_0}$ is bounded by Γ_r^{-1} (and monotonically decreasing) and $\kappa_n \to 0$ as $n \to \infty$ and by construction

$$\forall\, n \in \mathbb{N}_0 :\ \kappa_n \|\mathcal{B}'(v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma))\|_{\mathbb{R}^m} < 1.$$

Consider a modification of φ induced by κ_n, namely

$$\varphi_n := \varphi + \kappa_n, \quad n \in \mathbb{N}_0.$$

It is clear that for each $n \in \mathbb{N}_0$ we have $\varphi_n \in W^{1,\infty}([\gamma,\infty);\mathbb{R})$, the estimates $\|\varphi_n\|_\infty \le \|\varphi\|_\infty + \Gamma_r^{-1}$ and $\|\dot\varphi_n\|_\infty = \|\dot\varphi\|_\infty$ are independent of n, and $\varphi_n \to \varphi \in \Phi_\gamma$ uniformly. Moreover, $\inf_{t>\gamma} \varphi_n(t) > 0$.
Now, fix $n \in \mathbb{N}_0$. For $t \ge \gamma$, define, in the respective spaces,

$$\begin{aligned}
\phi(e) &:= \frac{k_0}{1 - \|e\|_{\mathbb{R}^m}^2} e, \quad e \in \mathbb{R}^m,\ \|e\|_{\mathbb{R}^m} < 1,\\
\omega_0(t) &:= \dot\varphi_n(t)\varphi_n(t)^{-1},\\
F(t,z) &:= \varphi_n(t) f_{-1}(t) + \varphi_n(t) f_0(t) + f_1(t) z + \varphi_n(t)^{-1} f_2(t) z^2\\
&\quad - c_3 \varphi_n(t)^{-2} z^3, \quad z \in \mathbb{R},\\
f_{-1}(t) &:= I_{s,i}(t) + \sum_{k=1}^m y_{\mathrm{ref},k}(t)\mathcal{A} q_k,\\
f_0(t) &:= -q \cdot (\dot y_{\mathrm{ref}}(t) + c_1 y_{\mathrm{ref}}(t)) + c_2 (q \cdot y_{\mathrm{ref}}(t))^2 - c_3 (q \cdot y_{\mathrm{ref}}(t))^3,\\
f_1(t) &:= (q \cdot y_{\mathrm{ref}}(t))(2c_2 - 3c_3(q \cdot y_{\mathrm{ref}}(t))),\\
f_2(t) &:= c_2 - 3c_3(q \cdot y_{\mathrm{ref}}(t)),\\
g(t) &:= c_5 (q \cdot y_{\mathrm{ref}}(t)).
\end{aligned}$$

We have that $f_{-1} \in L^\infty(\gamma,\infty; L^2(\Omega))$, since

$$\begin{aligned}
\|f_{-1}\|_{2,\infty} &:= \operatorname{ess\,sup}_{t\ge\gamma} \left(\int_\Omega f_{-1}(\zeta,t)^2 \,\mathrm{d}\lambda \right)^{1/2}\\
&\le \|I_{s,i}\|_{2,\infty} + \sum_{k=1}^m \|y_{\mathrm{ref},k}\|_\infty \, \|Aq_k\|_{L^2} < \infty.
\end{aligned}$$

Furthermore, we have that $f_0 \in L^\infty((\gamma,\infty)\times\Omega)$, because

$$\begin{aligned}
|f_0(\zeta,t)| \le\ & (\|\dot y_{\mathrm{ref}}\|_\infty + c_1 \|y_{\mathrm{ref}}\|_\infty) \sum_{k=1}^m \|q_k\|_\infty + c_2 \|y_{\mathrm{ref}}\|_\infty^2 \left(\sum_{k=1}^m \|q_k\|_\infty \right)^2\\
&+ c_3 \|y_{\mathrm{ref}}\|_\infty^3 \left(\sum_{k=1}^m \|q_k\|_\infty \right)^3 \ \text{for a.a. } (\zeta,t) \in \Omega \times [\gamma,\infty),
\end{aligned}$$

whence

$$\|f_0\|_{\infty,\infty} := \operatorname{ess\,sup}_{t\ge\gamma,\zeta\in\Omega} |f_0(\zeta,t)| < \infty.$$

Similarly $\|f_1\|_{\infty,\infty} < \infty$, $\|f_2\|_{\infty,\infty} < \infty$ and $\|g\|_{\infty,\infty} < \infty$.
Consider the system of $2(n+1)$ ODEs

$$
\begin{aligned}
\dot{\mu}_j(t) &= -\alpha_j \mu_j(t) - (c_1 - \omega_0(t))\mu_j(t) - \nu_j(t) - \left\langle \phi\left(\sum_{i=0}^{n} \mathcal{B}'\theta_i \mu_i(t)\right), \mathcal{B}'\theta_j \right\rangle_{\mathbb{R}^m} \\
&\qquad + \left\langle F\left(t, \sum_{i=0}^{n} \mu_i(t)\theta_i\right), \theta_j \right\rangle, \\
\dot{\nu}_j(t) &= -(c_4 - \omega_0(t))\nu_j(t) + c_5 \mu_j(t) + \varphi_n(t) \langle g(t), \theta_j \rangle
\end{aligned}
\tag{26}
$$

defined on

$$
\mathfrak{D} := \left\{ (t, \mu_0, \ldots, \mu_n, \nu_0, \ldots, \nu_n) \in [\gamma, \infty) \times \mathbb{R}^{2(n+1)} \;\middle|\; \left\| \sum_{i=0}^{n} \gamma_i \mu_i \right\|_{\mathbb{R}^m} < 1 \right\},
$$

with initial value

$$
\mu_j(\gamma) = \kappa_n \left(a_j - \sum_{k=1}^{m} q_{k,j} y_{\mathrm{ref},k}(\gamma) \right), \quad \nu_j(\gamma) = \kappa_n b_j, \quad j \in \mathbb{N}_0.
$$

Given that the functions defining the system of ODEs (26) are continuous, the set $\mathfrak{D}$ is relatively open in $[\gamma, \infty) \times \mathbb{R}^{2(n+1)}$ and by construction the initial condition satisfies $(\gamma, \mu_0(\gamma), \ldots, \mu_n(\gamma), \nu_0(\gamma), \ldots, \nu_n(\gamma)) \in \mathfrak{D}$ it follows from ODE theory, see e.g. [37, § 10, Thm. XX], that there exists a weakly differentiable solution

$$
(\mu^n, \nu^n) = (\mu_0, \ldots, \mu_n, \nu_0, \ldots, \nu_n) : [\gamma, T_n) \to \mathbb{R}^{2(n+1)}
$$

such that $T_n \in (\gamma, \infty]$ is maximal. Furthermore, the closure of the graph of (μ^n, ν^n) is not a compact subset of $\mathfrak{D}$.
With that, we may define

$$
z_n(t) := \sum_{i=0}^{n} \mu_i(t)\theta_i, \quad w_n(t) := \sum_{i=0}^{n} \nu_i(t)\theta_i, \quad e_n(t) := \sum_{i=0}^{n} \mathcal{B}'\theta_i \mu_i(t), \quad t \in [\gamma, T_n)
$$

and note that

$$
z_\gamma^n := z_n(\gamma) = \kappa_n (v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma)), \quad w_\gamma^n := w_n(\gamma) = \kappa_n u_\gamma^n.
$$

From the orthonormality of the θ_i we have that

$$
\begin{aligned}
\langle \dot{z}_n(t), \theta_j \rangle &= -\mathfrak{a}(z_n(t), \theta_j) - (c_1 - \omega_0(t)) \langle z_n(t), \theta_j \rangle - \langle w_n(t), \theta_j \rangle \\
&\qquad - \langle \phi(\mathcal{B}' z_n(t)), \mathcal{B}'\theta_j \rangle_{\mathbb{R}^m} + \langle F(t, z_n(t)), \theta_j \rangle, \\
\langle \dot{w}_n(t), \theta_j \rangle &= -(c_4 - \omega_0(t)) \langle w_n(t), \theta_j \rangle + c_5 \langle z_n(t), \theta_j \rangle + \varphi_n \langle g(t), \theta_j \rangle.
\end{aligned}
\tag{27}
$$

Define now

$$\begin{aligned}
v_n(t) &:= \varphi_n(t)^{-1} z_n(t) + q^n \cdot y_{\mathrm{ref}}(t), \\
u_n(t) &:= \varphi_n(t)^{-1} w_n(t), \\
\tilde{\mu}_i(t) &:= \varphi_n(t)^{-1} \mu_i(t) + \sum_{k=1}^{m} q_{k,i} y_{\mathrm{ref},k}(t), \\
\tilde{\nu}_i(t) &:= \varphi_n(t)^{-1} \nu_i(t),
\end{aligned} \tag{28}$$

then $v_n(t) = \sum_{i=0}^{n} \tilde{\mu}_i(t)\theta_i$ and $u_n(t) = \sum_{i=0}^{n} \tilde{\nu}_i(t)\theta_i$. With this transformation we obtain that (v_n, u_n) satisfies, for all $\theta \in W^{1,2}(\Omega)$, $\chi \in L^2(\Omega)$ and all $t \in [\gamma, T_n)$ that

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} \langle v_n(t), \theta \rangle =& - \mathfrak{a}(v_n(t), \theta) + \langle p_3(v_n(t) + (q - q^n) \cdot y_{\mathrm{ref}}(t)) - u_n(t), \theta \rangle \\
&+ \left\langle I_{s,i}(t) - (q - q^n) \cdot \dot{y}_{\mathrm{ref}}(t) + \sum_{k=1}^{m} y_{\mathrm{ref},k}(t) \mathcal{A}(q_k - q_k^n), \theta \right\rangle \\
&+ \left\langle I_{s,e}^n(t), \mathcal{B}'\theta \right\rangle_{\mathbb{R}^m}, \\
\tfrac{\mathrm{d}}{\mathrm{d}t} \langle u_n(t), \chi \rangle =& \langle c_5(v_n(t) + (q - q^n) \cdot y_{\mathrm{ref}}(t)) - c_4 u_n(t), \chi \rangle, \\
I_{s,e}^n(t) =& - \frac{k_0}{1 - \varphi_n(t)^2 \|\mathcal{B}'(v_n(t) - q^n \cdot y_{\mathrm{ref}}(t))\|_{\mathbb{R}^m}^2} (\mathcal{B}'(v_n(t) - q^n \cdot y_{\mathrm{ref}}(t))),
\end{aligned}$$

with $(u_n(\gamma), v_n(\gamma)) = (u_\gamma, v_\gamma)$. Since there exists some $n_0 \in \mathbb{N}$ with $q^n = q$ for all $n \geq n_0$, we have for all $n \geq n_0$, $\theta \in W^{1,2}(\Omega)$ and $\chi \in L^2(\Omega)$ that

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} \langle v_n(t), \theta \rangle =& - \mathfrak{a}(v_n(t), \theta) + \langle p_3(v_n(t)) - u_n(t), \theta \rangle \\
&+ \langle I_{s,i}(t), \theta \rangle + \left\langle I_{s,e}^n(t), \mathcal{B}'\theta \right\rangle_{\mathbb{R}^m}, \\
\tfrac{\mathrm{d}}{\mathrm{d}t} \langle u_n(t), \chi \rangle =& \langle c_5 v_n(t) - c_4 u_n(t), \chi \rangle, \\
I_{s,e}^n(t) =& - \frac{k_0}{1 - \varphi_n(t)^2 \|\mathcal{B}' v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m}^2} (\mathcal{B}' v_n(t) - y_{\mathrm{ref}}(t)),
\end{aligned} \tag{29}$$

Step 2: We show boundedness of (z_n, w_n) in terms of φ_n.
Consider again the Lyapunov function (15) and observe that $\|z_n(t)\|^2 = \sum_{j=0}^{n} \mu_j(t)^2$

and $\|w_n(t)\|^2 = \sum_{j=0}^{n} \nu_j(t)^2$. We find that, for all $t \in [\gamma, T_n)$,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} V(z_n(t), w_n(t)) &= c_5 \sum_{j=0}^{n} \mu_j(t)\dot{\mu}_j(t) + \sum_{j=0}^{n} \nu_j(t)\dot{\nu}_j(t) \\
&= -c_5 \sum_{j=0}^{n} \alpha_j \mu_j(t)^2 - c_5(c_1 - \omega_0(t)) \sum_{j=0}^{n} \mu_j(t)^2 \\
&\quad - (c_4 - \omega_0(t)) \sum_{j=0}^{n} \nu_j(t)^2 - c_5 \left\langle \phi(e_n(t)), e_n(t) \right\rangle_{\mathbb{R}^m} \\
&\quad + \varphi_n(t) \left\langle g(t), \sum_{i=0}^{n} \nu_i(t)\theta_i \right\rangle \\
&\quad + c_5 \left\langle F\left(t, \sum_{i=0}^{n} \mu_i(t)\theta_i\right), \sum_{i=0}^{n} \mu_i(t)\theta_i \right\rangle,
\end{aligned}$$

hence, omitting the argument t for brevity in the following,

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} V(z_n, w_n) = &- c_5 \mathfrak{a}(z_n, z_n) - c_5(c_1 - \omega_0)\|z_n\|^2 - (c_4 - \omega_0)\|w_n\|^2 \\
&- c_5 \frac{k_0 \|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2} + c_5 \left\langle F(t, z_n), z_n \right\rangle + \varphi_n \left\langle g, w_n \right\rangle.
\end{aligned} \tag{30}$$

Next we use some Young and Hölder inequalities to estimate the term

$$\begin{aligned}
\langle F(t, z_n), z_n \rangle = &\underbrace{\varphi_n(t) \langle f_{-1}(t), z_n \rangle}_{I_{-1}} + \underbrace{\varphi_n(t) \langle f_0(t), z_n \rangle}_{I_0} + \underbrace{\langle f_1(t) z_n, z_n \rangle}_{I_1} \\
&+ \underbrace{\varphi_n(t)^{-1} \left\langle f_2(t) z_n^2, z_n \right\rangle}_{I_2} - c_3 \varphi_n(t)^{-2} \underbrace{\left\langle z_n^3, z_n \right\rangle}_{=\|z_n\|_{L^4}^4}.
\end{aligned}$$

For the first term we derive, using Young's inequality for products with $p = 4/3$ and $q = 4$, that

$$\begin{aligned}
I_{-1} &\le \left\langle \frac{2^{1/2} \varphi_n^{3/2} |I_{s,i}|}{c_3^{1/4}}, \frac{c_3^{1/4} |z_n|}{2^{1/2} \varphi_n^{1/2}} \right\rangle + \sum_{k=1}^{m} \left\langle \frac{(4m)^{1/4} \varphi_n^{3/2} \|y_{\mathrm{ref}}\|_\infty |A q_k|}{c_3^{1/4}}, \frac{c_3^{1/4} |z_n|}{(4m)^{1/4} \varphi_n^{1/2}} \right\rangle \\
&\le \frac{2^{2/3} 3 \varphi_n^2 \|I_{s,i}\|_{2,\infty}^{4/3} |\Omega|^{1/3}}{4 c_3^{1/3}} + \sum_{k=1}^{m} \frac{3 (4m)^{1/3} \varphi_n^2 \|y_{\mathrm{ref}}\|_\infty^{4/3} \|A q_k\|^{4/3} |\Omega|^{1/3}}{4 c_3^{1/3}} + \frac{c_3 \|z_n\|_{L^4}^4}{8 \varphi_n^2}
\end{aligned}$$

and with the same choice we obtain for the second term

$$I_0 \le \left\langle \frac{2^{1/4} \varphi_n^{3/2} \|f_0\|_{\infty,\infty}}{c_3^{1/4}}, \frac{c_3^{1/4} |z_n|}{2^{1/4} \varphi_n^{1/2}} \right\rangle \le \frac{2^{1/3} 3 \varphi_n^2 \|f_0\|_{\infty,\infty}^{4/3} |\Omega|}{4 c_3^{1/3}} + \frac{c_3 \|z_n\|_{L^4}^4}{8 \varphi_n^2}.$$

Using $p = q = 2$ we find that the third term satisfies

$$I_1 \le \left\langle \frac{2 \varphi_n \|f_1\|_{\infty,\infty}}{\sqrt{c_3}}, \frac{\sqrt{c_3} |z_n|^2}{2 \varphi_n} \right\rangle \le \frac{2 \varphi_n^2 \|f_1\|_{\infty,\infty}^2 |\Omega|}{c_3} + \frac{c_3 \|z_n\|_{L^4}^4}{8 \varphi_n^2},$$

and finally, with $p = 4$ and $q = 4/3$,

$$
\begin{aligned}
I_2 &\le \left\langle \varphi_n^{-1} \|f_2\|_{\infty,\infty}, |z_n|^3 \right\rangle = \left\langle \frac{3^{3/2}\varphi_n^{1/2}\|f_2\|_{\infty,\infty}}{c_3^{3/4}}, \left| \frac{c_3^{1/4} z_n}{\varphi_n^{1/2}\sqrt{3}} \right|^3 \right\rangle \\
&\le \frac{9^3 \varphi_n^2 \|f_2\|_{\infty,\infty}^4 |\Omega|}{4c_3^3} + \frac{c_3}{12\varphi_n^2}\|z_n\|_{L^4}^4.
\end{aligned}
$$

Summarizing, we have shown that

$$
\langle F(t, z_n), z_n \rangle \le K_0\varphi_n^2 - \frac{13c_3}{24\varphi_n^2}\|z_n\|_{L^4}^4 \le K_0\varphi_n^2 - \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4,
$$

where

$$
\begin{aligned}
K_0 \coloneqq \; & \frac{2^{2/3} 3 \|I_{s,i}\|_{2,\infty}^{4/3} |\Omega|^{1/3}}{4c_3^{1/3}} + \sum_{k=1}^{m} \frac{3(4m)^{1/3} \|y_{\mathrm{ref}}\|_\infty^{4/3} \|Aq_k\|^{4/3} |\Omega|^{1/3}}{4c_3^{1/3}} \\
& + \frac{2^{1/3} 3 \|f_0\|_{\infty,\infty}^{4/3} |\Omega|}{4c_3^{1/3}} + \frac{2\|f_1\|_{\infty,\infty}^2 |\Omega|}{c_3} + \frac{9^3 \|f_2\|_{\infty,\infty}^4 |\Omega|}{4c_3^3}.
\end{aligned}
$$

Finally, using Young's inequality with $p = q = 2$, we estimate the last term in (30) as follows

$$
\varphi_n \langle g, w_n \rangle \le \frac{\varphi_n^2 \|g\|_{\infty,\infty}^2 |\Omega|}{2c_4} + \frac{c_4}{2}\|w_n\|^2.
$$

We have thus obtained the estimate

$$
\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} V(z_n, w_n) \le & -(\sigma - 2\omega_0) V(z_n, w_n) \\
& - c_5 \mathfrak{a}(z_n, z_n) - c_5 \frac{k_0 \|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2} - \frac{c_3 c_5}{2\varphi_n^2}\|z_n\|_{L^4}^4 + \varphi_n^2 K_1,
\end{aligned} \tag{31}
$$

where

$$
\sigma \coloneqq 2\min\{c_1, c_4\}, \quad K_1 \coloneqq c_5 K_0 + \frac{\|g\|_{\infty,\infty}^2 |\Omega|}{2c_4}.
$$

In particular, we have the estimate

$$
\tfrac{\mathrm{d}}{\mathrm{d}t} V(z_n, w_n) \le -(\sigma - 2\omega_0) V(z_n, w_n) + \varphi_n^2 K_1
$$

on $[\gamma, T_n)$, which implies that

$$
V(z_n(t), w_n(t)) \le \mathrm{e}^{-K(t,\gamma)} V(z_n(\gamma), w_n(\gamma)) + \int_\gamma^t \mathrm{e}^{-K(t,s)} \varphi_n(s)^2 K_1 \,\mathrm{d}s,
$$

where

$$
K(t,s) = \int_s^t \sigma - 2\omega_0(\tau)\,\mathrm{d}\tau = \sigma(t-s) - 2\ln\varphi_n(t) + 2\ln\varphi_n(s), \quad \gamma \le s \le t < T_n.
$$

Therefore, invoking $\varphi_n(\gamma) = \kappa_n$, for all $t \in [\gamma, T_n)$ we have

$$
\begin{aligned}
&c_5\|z_n(t)\|^2 + \|w_n(t)\|^2 = 2V(z_n(t), w_n(t)) \\
&\le 2\mathrm{e}^{-\sigma(t-\gamma)} \frac{\varphi_n(t)^2}{\kappa_n^2} V(z_n(\gamma), w_n(\gamma)) + \frac{2K_1}{\sigma}\varphi_n(t)^2 \\
&= \varphi_n(t)^2 \left((c_5\|v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma)\|^2 + \|u_\gamma^n\|^2)\mathrm{e}^{-\sigma(t-\gamma)} + 2K_1\sigma^{-1} \right) \\
&\le \varphi_n(t)^2 \left(c_5\|v_\gamma - q \cdot y_{\mathrm{ref}}(\gamma)\|^2 + \|u_\gamma\|^2 + 2K_1\sigma^{-1} \right).
\end{aligned}
$$

Thus there exist $M, N > 0$ which are independent of n and t such that

$$
\forall\, t \in [\gamma, T_n): \ \|z_n(t)\|^2 \le M\varphi_n(t)^2 \ \text{ and } \ \|w_n(t)\|^2 \le N\varphi_n(t)^2, \tag{32}
$$

and, as a consequence,

$$
\forall\, t \in [\gamma, T_n): \ \|v_n(t) - q^n \cdot y_{\mathrm{ref}}(t)\|^2 \le M \ \text{ and } \ \|u_n(t)\|^2 \le N. \tag{33}
$$

Step 3: We show $T_n = \infty$ and that e_n is uniformly bounded away from 1 on $[\gamma, \infty)$.
Step 3a: We derive some estimates for $\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2$ and for an integral involving $\|z_n\|_{L^4}^4$. In a similar way in which we have derived (31) we can obtain the estimate

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 \le &- \mathfrak{a}(z_n, z_n) - (c_1 - \omega_0)\|z_n\|^2 + \|z_n\|\|w_n\| \\
&- \frac{k_0\|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2} - \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4 + K_0\varphi_n^2.
\end{aligned} \tag{34}
$$

Using (32) and $-c_1\|z_n\|^2 \le 0$ leads to

$$
\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 \le &- \mathfrak{a}(z_n, z_n) - \frac{k_0\|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2} - \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4 \\
&+ \|\dot{\varphi}\|_\infty M\varphi_n + (K_0 + \sqrt{MN})\varphi_n^2.
\end{aligned}
$$

Hence,

$$
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 \le - \mathfrak{a}(z_n, z_n) - \frac{k_0\|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2} - \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4 + K_1\varphi_n + K_2\varphi_n^2 \tag{35}
$$

on $[\gamma, T_n)$, where $K_1 := M\|\dot{\varphi}\|_\infty$ and $K_2 := K_0 + \sqrt{MN}$. Observe that

$$
\frac{c_3}{2}\varphi_n^{-3}\|z_n\|_{L^4}^4 \le -\frac{\varphi_n^{-1}}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + K_3,
$$

where $K_3 \coloneqq K_1 + K_2\|\varphi\|_\infty$. Therefore,

$$
\begin{aligned}
&\frac{c_3}{2}\int_\gamma^t \mathrm{e}^s\varphi_n(s)^{-3}\|z_n(s)\|_{L^4}^4\,\mathrm{d}s\\
&\le K_3(\mathrm{e}^t-\mathrm{e}^\gamma) - \frac{1}{2}\int_\gamma^t \mathrm{e}^s\varphi_n(s)^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n(s)\|^2\,\mathrm{d}s\\
&= K_3(\mathrm{e}^t-\mathrm{e}^\gamma) - \frac{1}{2}\left(\mathrm{e}^t\varphi_n(t)^{-1}\|z_n(t)\|^2 - \frac{\|z_\gamma^n\|^2}{\kappa_n}\mathrm{e}^\gamma\right)\\
&\quad + \frac{1}{2}\int_\gamma^t \mathrm{e}^s\varphi_n(s)^{-2}(\varphi_n(s)-\dot\varphi_n(s))\|z_n(s)\|^2\,\mathrm{d}s\\
&\le \frac{\mathrm{e}^t}{2}(2K_3 + (\|\varphi\|_\infty + \Gamma_r^{-1} + \|\dot\varphi\|_\infty)M) + \kappa_n\mathrm{e}^\gamma(\|v_\gamma\|^2 + \|q\cdot y_{\mathrm{ref}}(\gamma)\|^2),
\end{aligned}
$$

and hence there exist $D_0, D_1 > 0$ independent of n and t such that

$$
\forall\, t\in[\gamma,T_n):\ \int_\gamma^t \mathrm{e}^s\varphi_n(s)^{-3}\|z_n(s)\|_{L^4}^4\,\mathrm{d}s \le D_1\mathrm{e}^t + \kappa_n D_0. \tag{36}
$$

Step 3b: We derive an estimate for $\|\dot z_n\|^2$. Multiplying the first equation in (27) by $\dot\mu_j$ and summing over $j\in\{0,\dots,n\}$ we obtain

$$
\begin{aligned}
\|\dot z_n\|^2 = &-\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(z_n,z_n) - \frac{c_1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + \frac{k_0}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\ln(1-\|e_n\|_{\mathbb{R}^m}^2)\\
&+ \langle \omega_0 z_n + F(t,z_n) - w_n, \dot z_n\rangle\,.
\end{aligned}
$$

We can estimate the last term above by

$$
\begin{aligned}
\langle \omega_0 z_n, \dot z_n\rangle &\le \frac{7}{2}\|\dot\varphi\|_\infty^2\varphi_n^{-2}\|z_n\|^2 + \frac{1}{14}\|\dot z_n\|^2 \overset{(32)}{\le} \frac{7}{2}\|\dot\varphi\|_\infty^2 M + \frac{1}{14}\|\dot z_n\|^2,\\
\langle -w_n, \dot z_n\rangle &\le \frac{7}{2}\|w_n\|^2 + \frac{1}{14}\|\dot z_n\|^2,\\
\langle F(t,z_n), \dot z_n\rangle &\le \frac{7}{2}\varphi_n^2\left(m\sum_{k=1}^m \|y_{\mathrm{ref},k}\|_\infty^2\|\mathcal{A}q_k\|^2 + \|I_{s,i}\|_{2,\infty}^2 + \|f_0\|_{\infty,\infty}^2|\Omega|\right)\\
&\quad + \frac{7}{2}\|f_1\|_{\infty,\infty}^2\|z_n\|^2 + \frac{7}{2}\varphi_n^{-2}\|f_2\|_{\infty,\infty}^2\|z_n\|_{L^4}^4\\
&\quad + \frac{5}{14}\|\dot z_n\|^2 - \frac{c_3}{4\varphi_n^2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|_{L^4}^4.
\end{aligned}
$$

Inserting these inequalities, substracting $\frac{1}{2}\|\dot z_n\|^2$ and then multiplying by 2 gives

$$
\begin{aligned}
\|\dot z_n\|^2 = &-\tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(z_n,z_n) - c_1\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + k_0\tfrac{\mathrm{d}}{\mathrm{d}t}\ln(1-\|e_n\|_{\mathbb{R}^m}^2) - \frac{c_3}{2\varphi_n^2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|_{L^4}^4\\
&+ 7\varphi_n^2\left(m\sum_{k=1}^m \|y_{\mathrm{ref},k}\|_\infty^2\|\mathcal{A}q_k\|^2 + \|I_{s,i}\|_{2,\infty}^2 + \|f_0\|_{\infty,\infty}^2|\Omega| + \|f_1\|_{\infty,\infty}^2 M + N\right)\\
&+ 7\|\dot\varphi\|_\infty^2 M + 7\varphi_n^{-2}\|f_2\|_{\infty,\infty}^2\|z_n\|_{L^4}^4.
\end{aligned}
$$

Now we add and subtract $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2$, thus we obtain

$$
\begin{aligned}
\|\dot{z}_n\|^2 \le & -\tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(z_n,z_n) - \left(c_1+\frac{1}{2}\right)\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + k_0\tfrac{\mathrm{d}}{\mathrm{d}t}\ln(1-\|e_n\|^2_{\mathbb{R}^m}) - \frac{c_3}{2\varphi_n^2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^4_{L^4} \\
& + 7(\|\varphi\|_\infty+\Gamma_r^{-1})^2\left(m\sum_{k=1}^m \|y_{\mathrm{ref},k}\|^2_\infty\|\mathcal{A}q_k\|^2 + \|I_{s,i}\|^2_{2,\infty} + \|f_0\|^2_{\infty,\infty}|\Omega|\right. \\
& \left. + \|f_1\|^2_{\infty,\infty}M\right) + 7(N(\|\varphi\|_\infty+\Gamma_r^{-1})^2 + \|\dot{\varphi}\|^2_\infty M) + 7\varphi_n^{-2}\|f_2\|^2_{\infty,\infty}\|z_n\|^4_{L^4} \\
& + \frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2.
\end{aligned}
$$

By the product rule we have

$$
-\frac{c_3}{2\varphi_n^2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^4_{L^4} = -\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c_3}{2\varphi_n^2}\|z_n\|^4_{L^4}\right) - c_3\varphi_n^{-3}\dot{\varphi}_n\|z_n\|^4_{L^4},
$$

thus we find that

$$
\begin{aligned}
&\|\dot{z}_n\|^2 + \tfrac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(z_n,z_n) - k_0\tfrac{\mathrm{d}}{\mathrm{d}t}\ln(1-\|e_n\|^2_{\mathbb{R}^m}) + \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c_3}{2\varphi_n^2}\|z_n\|^4_{L^4}\right) \\
&\quad \le -\left(c_1+\frac{1}{2}\right)\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_1 + E_2\varphi_n^{-3}\|z_n\|^4_{L^4} + \frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2,
\end{aligned}
\tag{37}
$$

where

$$
\begin{aligned}
E_1 :=\ & 7(\|\varphi\|_\infty+\Gamma_r^{-1})^2\left(m\sum_{k=1}^m \|y_{\mathrm{ref},k}\|^2_\infty\|Aq_k\|^2 + \|I_{s,i}\|^2_{2,\infty} + \|f_0\|^2_{\infty,\infty}|\Omega|\right. \\
& \left. + \|f_1\|^2_{\infty,\infty}M\right) + 7\big(N(\|\varphi\|_\infty+\Gamma_r^{-1})^2 + \|\dot{\varphi}\|^2_\infty M\big), \\
E_2 :=\ & 7\|f_2\|^2_{\infty,\infty}(\|\varphi\|_\infty+\Gamma_r^{-1}) + c_3\|\dot{\varphi}\|_\infty
\end{aligned}
$$

are independent of n and t.

Step 3c: We show uniform boundedness of e_n. Using (35) in (37) we obtain

$$
\begin{aligned}
\|\dot{z}_n\|^2 + \dot{\rho}_n \le & -\left(c_1+\frac{1}{2}\right)\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_1 + E_2\varphi_n^{-3}\|z_n\|^4_{L^4} \\
& - \mathfrak{a}(z_n,z_n) - \frac{k_0\|e_n\|^2_{\mathbb{R}^m}}{1-\|e_n\|^2_{\mathbb{R}^m}} - \frac{c_3}{2\varphi_n^2}\|z_n\|^4_{L^4} + K_1\varphi_n + K_2\varphi_n^2 \\
= & -\left(c_1+\frac{1}{2}\right)\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_2\varphi_n^{-3}\|z_n\|^4_{L^4} \\
& - \mathfrak{a}(z_n,z_n) - \frac{k_0}{1-\|e_n\|^2_{\mathbb{R}^m}} - \frac{c_3}{2\varphi_n^2}\|z_n\|^4_{L^4} + \Lambda,
\end{aligned}
$$

where

$$\rho_n \coloneqq \mathfrak{a}(z_n, z_n) - k_0 \ln(1 - \|e_n\|^2_{\mathbb{R}^m}) + \frac{c_3}{2\varphi_n^2}\|z_n\|^4_{L^4},$$
$$\Lambda \coloneqq E_1 + K_1(\|\varphi\|_\infty + \Gamma_r^{-1}) + K_2(\|\varphi\|_\infty + \Gamma_r^{-1})^2 + k_0,$$

and we have used the equality

$$\frac{\|e_n\|^2_{\mathbb{R}^m}}{1 - \|e_n\|^2_{\mathbb{R}^m}} = -1 + \frac{1}{1 - \|e_n\|^2_{\mathbb{R}^m}}.$$

Adding and subtracting $k_0 \ln(1 - \|e_n\|^2_{\mathbb{R}^m})$ leads to

$$\begin{aligned}
\|\dot{z}_n\|^2 + \dot{\rho}_n \leq& - \rho_n - \left(c_1 + \frac{1}{2}\right) \tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_2\varphi_n^{-3}\|z_n\|^4_{L^4} \\
&- k_0 \left(\frac{1}{1 - \|e_n\|^2_{\mathbb{R}^m}} + \ln(1 - \|e_n\|^2_{\mathbb{R}^m})\right) + \Lambda \\
\leq& - \rho_n - \left(c_1 + \frac{1}{2}\right) \tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_2\varphi_n^{-3}\|z_n\|^4_{L^4} + \Lambda, \qquad (38)
\end{aligned}$$

where for the last inequality we have used that

$$\forall\, p \in (-1, 1):\ \frac{1}{1 - p^2} \geq \ln\left(\frac{1}{1 - p^2}\right) = -\ln(1 - p^2).$$

We may now use the integrating factor e^t to obtain

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t\rho_n\right) = \mathrm{e}^t(\rho_n + \dot{\rho}_n) \leq -\mathrm{e}^t\left(c_1 + \frac{1}{2}\right)\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_2\mathrm{e}^t\varphi_n^{-3}\|z_n\|^4_{L^4} + \Lambda\mathrm{e}^t \underbrace{-\mathrm{e}^t\|\dot{z}_n\|^2}_{\leq 0}.$$

Integrating and using (36) yields that for all $t \in [\gamma, T_n)$ we have

$$\begin{aligned}
\mathrm{e}^t\rho_n(t) - \rho_n(\gamma)\mathrm{e}^\gamma \leq& (E_2D_1 + \Lambda)\mathrm{e}^t + \kappa_n E_2 D_0 - \int_\gamma^t \mathrm{e}^s\left(c_1 + \frac{1}{2}\right)\tfrac{\mathrm{d}}{\mathrm{d}t}\|z_n(s)\|^2\,\mathrm{d}s \\
\leq& (E_2D_1 + \Lambda)\mathrm{e}^t + \kappa_n E_2 D_0 + \left(c_1 + \frac{1}{2}\right)\|z_\gamma^n\|^2\mathrm{e}^\gamma \\
&+ \left(c_1 + \frac{1}{2}\right)\int_\gamma^t \mathrm{e}^s\|z_n(s)\|^2\,\mathrm{d}s \\
\overset{(32)}{\leq}& (E_2D_1 + \Lambda)\mathrm{e}^t + \kappa_n E_2 D_0 + \left(c_1 + \frac{1}{2}\right)\kappa_n^2\mathrm{e}^\gamma(\|v_\gamma - q \cdot y_{\mathrm{ref}}(\gamma)\|^2) \\
&+ \left(c_1 + \frac{1}{2}\right)(\|\varphi\|_\infty + \Gamma_r^{-1})^2 M\mathrm{e}^t.
\end{aligned}$$

Thus, there exit $\Xi_1, \Xi_2, \Xi_3 > 0$ independent of n and t, such that

$$\rho_n(t) \leq \rho_n(\gamma)\mathrm{e}^{-(t-\gamma)} + \Xi_1 + \kappa_n(\Xi_2 + \kappa_n\Xi_3)\mathrm{e}^{-(t-\gamma)}.$$

Invoking the definition of ρ_n and that $\mathrm{e}^{-(t-\gamma)} \le 1$ for $t \ge \gamma$ we find that

$$\forall\, t \in [\gamma, T_n):\ \rho_n(t) \le \rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3, \tag{39}$$

where

$$\begin{aligned}\rho_n^0 \coloneqq{}& \kappa_n^2 \mathfrak{a}(v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma), v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma)) - k_0 \ln(1 - \kappa_n^2 \|\mathcal{B}'(v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma))\|_{\mathbb{R}^m}^2)\\ &+ \kappa_n^2 \|v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma)\|_{L^4}^4 = \rho_n(\gamma).\end{aligned}$$

Note that by construction of κ_n and the Sobolev embedding theorem, $(\rho_n^0)_{n\in\mathbb{N}}$ is bounded, $\rho_n^0 \to 0$ as $n \to \infty$, so that ρ_n^0 can be bounded independently of n. Again using the definition of ρ_n and (39) we find that

$$k_0 \ln\left(\frac{1}{1 - \|e_n\|_{\mathbb{R}^m}^2}\right) = \rho_n - \mathfrak{a}(z_n, z_n) - \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4 \le \rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3,$$

and hence

$$\frac{1}{1 - \|e_n\|_{\mathbb{R}^m}^2} \le \exp\left(\frac{1}{k_0}\left(\rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3\right)\right) =: \varepsilon(n).$$

We may thus conclude that

$$\forall\, t \in [\gamma, T_n):\ \|e_n(t)\|_{\mathbb{R}^m}^2 \le 1 - \varepsilon(n), \tag{40}$$

or, equivalently,

$$\forall\, t \in [\gamma, T_n):\ \varphi_n(t)^2 \|\mathcal{B}'(v_n(t) - q^n \cdot y_{\mathrm{ref}}(t))\|_{\mathbb{R}^m}^2 \le 1 - \varepsilon(n). \tag{41}$$

Moreover, from (39), the definition of ρ, $k_0 \ln(1 - \|e_n\|_{\mathbb{R}^m}^2) \le 0$ and Assumption 2.1 we have that

$$\delta \|\nabla z_n\|^2 + \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4 \le \rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3.$$

Reversing the change of variables leads to

$$\begin{aligned}\forall\, t \in [\gamma, T_n):\ \delta \varphi_n(t)^2 \|\nabla(v_n(t) - q^n \cdot y_{\mathrm{ref}}(t))\|^2 + \varphi_n(t)^2 \|v_n(t) - q^n \cdot y_{\mathrm{ref}}(t)\|_{L^4}^4&\\ \le \rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3,&\end{aligned} \tag{42}$$

which implies that for all $t \in [\gamma, T_n)$ we have $v_n(t) \in W^{1,2}(\Omega)$.
Step 3d: We show that $T_n = \infty$. Assuming $T_n < \infty$ it follows from (40) that the graph of the solution (μ^n, ν^n) from Step 2 would be a compact subset of $\mathfrak{D}$, a contradiction. Therefore, we have $T_n = \infty$.

Step 4: We show convergence of the approximate solution, uniqueness and regularity of the solution in $[\gamma, \infty) \times \Omega$.
Step 4a: we prove some inequalities for later use. From (39) we have that, on $[\gamma, \infty)$,

$$\varphi_n^{-2} \|z_n\|_{L^4}^4 \le \rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3.$$

Using a similar procedure as for the derivation of (36) we may obtain the estimate

$$\forall\, t \geq 0: \int_\gamma^t \varphi_n(s)^{-3} \|z_n(s)\|_{L^4}^4 \,\mathrm{d}s \,\leq \kappa_n d_0 + d_1 t \tag{43}$$

for $d_0, d_1 > 0$ independent of n and t. Further, we can integrate (38) on the interval $[\gamma, t]$ to obtain, invoking $\rho_n(t) \geq 0$ and (43),

$$\int_\gamma^t \|\dot{z}_n(s)\|^2 \,\mathrm{d}s \,\leq \rho_n^0 + \left(c_1 + \frac{1}{2}\right) \kappa_n^2 (\|v_\gamma - q \cdot y_{\mathrm{ref}}(\gamma)\|^2) + E_2(\kappa_n d_0 + d_1 t) + \Lambda t$$

for all $t \geq \gamma$. Hence, there exist $S_0, S_1, S_2 > 0$ independent of n and t such that

$$\forall\, t \geq \gamma: \int_\gamma^t \|\dot{z}_n(s)\|^2 \,\mathrm{d}s \,\leq \rho_n^0 + S_0 \kappa_n + S_1 \kappa_n^2 + S_2 t. \tag{44}$$

This implies existence of $S_3, S_4 > 0$ such that

$$\forall\, t \geq \gamma: \int_\gamma^t \left\| \tfrac{\mathrm{d}}{\mathrm{d}t}(\varphi_n v_n) \right\|^2 \,\mathrm{d}s \,\leq \rho_n^0 + S_0 \kappa_n + S_1 \kappa_n^2 + S_3 t + S_4. \tag{45}$$

In order to improve (43), we observe that from (34) it follows

$$\begin{aligned}
\tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \|z_n\|^2 \leq & - \mathfrak{a}(z_n, z_n) - (c_1 - \omega_0) \|z_n\|^2 + \|z_n\| \|w_n\| \\
& - \frac{k_0 \|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2} - \frac{c_3}{2\varphi_n^2} \|z_n\|_{L^4}^4 + K_0 \varphi_n^2 \\
\leq & \, \omega_0 \|z_n\|^2 - \frac{c_3}{2\varphi_n^2} \|z_n\|_{L^4}^4 + K_2 \varphi_n^2 - \mathfrak{a}(z_n, z_n) - \frac{k_0 \|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2},
\end{aligned}$$

which gives

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \varphi_n^{-2} \|z_n\|^2 \leq 2K_2 - c_3 \varphi_n^{-4} \|z_n\|_{L^4}^4 - 2\varphi_n^{-2} \mathfrak{a}(z_n, z_n) - \frac{2k_0 \varphi_n^{-2} \|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2}.$$

This implies that for all $t \geq \gamma$ we have

$$\begin{aligned}
\int_\gamma^t c_3 \varphi_n(s)^{-4} \|z_n(s)\|_{L^4}^4 + 2\varphi_n(s)^{-2} \mathfrak{a}(z_n(s), z_n(s)) + \frac{2k_0 \varphi_n(s)^{-2} \|e_n(s)\|_{\mathbb{R}^m}^2}{1 - \|e_n(s)\|_{\mathbb{R}^m}^2} \,\mathrm{d}s \\
\leq 2K_2 t + \|v_\gamma - q \cdot y_{\mathrm{ref}}(\gamma)\|^2,
\end{aligned} \tag{46}$$

which is bounded independently of n. This shows that for all $t \geq \gamma$ we have

$$\begin{aligned}
c_3 \int_\gamma^t \|v_n(s) - q^n \cdot y_{\mathrm{ref}}(s)\|_{L^4}^4 \,\mathrm{d}s \,+ \int_\gamma^t 2\mathfrak{a}(v_n(s) - q^n \cdot y_{\mathrm{ref}}(s), v_n(s) - q^n \cdot y_{\mathrm{ref}}(s)) \,\mathrm{d}s \\
+ \int_\gamma^t \frac{2k_0 \|\mathcal{B}'(v_n(s) - q^n \cdot y_{\mathrm{ref}}(s))\|_{\mathbb{R}^m}^2}{1 - \varphi_n(s)^2 \|\mathcal{B}'(v_n(s) - q^n \cdot y_{\mathrm{ref}}(s))\|_{\mathbb{R}^m}^2} \,\mathrm{d}s \,\leq 2K_2 t + \|v_\gamma - q \cdot y_{\mathrm{ref}}(\gamma)\|^2.
\end{aligned} \tag{47}$$

In order to prove that $\|\dot{w}_n\|^2$ is bounded independently of n and t, a last calculation is required. Multiply the second equation in (27) by $\dot{\nu}_j$ and sum over j to obtain

$$\|\dot{w}_n\|^2 = -(c_4-\omega_0)\left\langle w_n, \dot{w}_n\right\rangle + c_5\left\langle z_n, \dot{w}_n\right\rangle + \varphi_n\left\langle g, \dot{w}_n\right\rangle.$$

Using $(\omega_0 - c_4)w_n = (\dot{\varphi}_n - c_4\varphi_n)\varphi_n^{-1}w_n$ and the inequalities

$$\begin{aligned}
-(c_4-\omega_0)\left\langle w_n, \dot{w}_n\right\rangle &\le \frac{3}{2}\|\dot{\varphi}-c_4\varphi\|_\infty^2\varphi_n^{-2}\|w_n\|^2 + \frac{\|\dot{w}_n\|^2}{6}\\
&\le \frac{3}{2}(\|\dot{\varphi}\|_\infty + c_4(\|\varphi\|_\infty+\Gamma_r^{-1}))^2N + \frac{\|\dot{w}_n\|^2}{6},\\
c_5\left\langle z_n, \dot{w}_n\right\rangle &\le \frac{3c_5^2}{2}\|z_n\|^2 + \frac{1}{6}\|\dot{w}_n\|^2\\
&\le \frac{3c_5^2M}{2}(\|\varphi\|_\infty+\Gamma_r^{-1})^2 + \frac{1}{6}\|\dot{w}_n\|^2,\\
\varphi_n\left\langle g, \dot{w}_n\right\rangle &\le \frac{3}{2}(\|\varphi\|_\infty+\Gamma_r^{-1})^2\|g\|_{\infty,\infty}^2|\Omega| + \frac{1}{6}\|\dot{w}_n\|^2,
\end{aligned}$$

it follows that for all $t\ge\gamma$ we have

$$\begin{aligned}
\|\dot{w}_n(t)\|^2 \le& 3\|(\|\dot{\varphi}\|_\infty + c_4(\|\varphi\|_\infty+\Gamma_r^{-1}))^2N\\
&+3c_5^2M(\|\varphi\|_\infty+\Gamma_r^{-1})^2 + 3(\|\varphi\|_\infty+\Gamma_r^{-1})^2\|g\|_{\infty,\infty}^2|\Omega|,
\end{aligned}\tag{48}$$

which is bounded independently of n and t. Multiplying the second equation in (27) by φ_n^{-1} and θ_i and summing over $i\in\{0,\dots,n\}$ leads to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\varphi_n^{-1}w_n) = -\varphi^{-2}\dot{\varphi}_n w_n + \varphi_n^{-1}\dot{w}_n = -c_4\varphi_n^{-1}w_n + c_5\varphi_n^{-1}z_n + g_n,$$

where

$$g_n := \sum_{i=0}^n \left\langle g, \theta_i\right\rangle\theta_i.$$

Taking the norm of the latter gives

$$\begin{aligned}
\left\|\tfrac{\mathrm{d}}{\mathrm{d}t}(\varphi_n^{-1}w_n)\right\| &\le c_4\varphi_n^{-1}\|w_n\| + c_5\varphi_n^{-1}\|z_n\| + \|g_n\|\\
&\le c_4N + c_5M + \|g\|_{\infty,\infty},
\end{aligned}$$

thus

$$\forall\, t\ge\gamma:\ \|\dot{u}_n(t)\| \le c_4N + c_5M + \|g\|_{\infty,\infty}.\tag{49}$$

Step 4b: We show that (v_n, u_n) converges weakly. Let $T>\gamma$ be given. Using a similar argument as in Section 4.1, we have that $v_n\in L^2(\gamma,T;W^{1,2}(\Omega))$ and $\dot{v}_n\in L^2(\gamma,T;W^{1,2}(\Omega)')$, since (47) together with (41) implies that $I_{s,e}^n\in L^2(\gamma,T;\mathbb{R}^m)$ and $v_n\in L^2(\gamma,T;W^{1,2}(\Omega))$.

Furthermore, analogously to Section 4.1, we have that there exist subsequences such that

$$\begin{aligned}
&u_n\to u\in W^{1,2}(\gamma,T;L^2(\Omega)) \text{ weakly},\\
&v_n\to v\in L^2(\gamma,T;W^{1,2}(\Omega)) \text{ weakly},\\
&\dot{v}_n\to\dot{v}\in L^2(\gamma,T;(W^{1,2}(\Omega))') \text{ weakly},
\end{aligned}$$

so that $u, v \in C([\gamma, T]; L^2(\Omega))$. Also $v_n^2 \to v^2$ weakly in $L^2((\gamma, T) \times \Omega)$ and $v_n^3 \to v^3$ weakly in $L^{4/3}((\gamma, T) \times \Omega)$.

We may infer further properties of u and v. By (33), (42), (45) & (49) we have that $u_n, \dot{u}_n$ lie in a bounded subset of $L^\infty(\gamma, \infty; L^2(\Omega))$ and that v_n lie in a bounded subset of $L^\infty(\gamma, \infty; L^2(\Omega))$. Moreover, $\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_n v_n) \in L^2_{\mathrm{loc}}(\gamma, \infty; L^2(\Omega))$. Then, using Lemma B.6, we find a subsequence such that

$$\begin{aligned} u_n &\to u \in L^\infty(\gamma, T; L^2(\Omega)) \text{ weak}^\star, \\ \dot{u}_n &\to \dot{u} \in L^\infty(\gamma, T; L^2(\Omega)) \text{ weak}^\star, \\ v_n &\to v \in L^\infty(\gamma, T; L^2(\Omega)) \text{ weak}^\star, \\ \varphi_n v_n &\to \varphi v \in L^\infty(\gamma, T; W^{1,2}(\Omega)) \text{ weak}^\star, \\ \dot{v}_n &\to \dot{v} \in L^2(\gamma, T; W^{1,2}(\Omega)') \text{ weakly}, \\ \varphi_n \dot{v}_n &\to \varphi \dot{v} \in L^2(\gamma, T; L^2(\Omega)) \text{ weakly}, \end{aligned}$$

since $\varphi_n \to \varphi$ in $BC([\gamma, T]; \mathbb{R})$. Moreover, by $\inf_{t > \gamma + \delta} \varphi(t) > 0$, we also have that $v \in L^\infty(\gamma + \delta, T; W^{1,2}(\Omega))$ and $\dot{v} \in L^2(\gamma + \delta, T; L^2(\Omega))$ for all $\delta > 0$.

Further, $\kappa_n, \rho_n^0 \to 0$ and

$$\varepsilon(n) \underset{n \to \infty}{\to} \varepsilon_0 := \exp\left(-k_0^{-1} \Xi_1\right).$$

Thus, by (33), (41), (42) & (47) we have $v \in L^4((\gamma, T) \times \Omega)$ and for almost all $t \in [\gamma, T)$ the following estimates hold:

$$\begin{aligned} &\|v(t) - q \cdot y_{\mathrm{ref}}(t)\| \le \sqrt{M}, \\ &\|u(t)\| \le \sqrt{N}, \\ &\varphi(t)^2 \|\mathcal{B}' v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m} \le 1 - \varepsilon_0, \\ &\delta \varphi(t)^2 \|\nabla(v(t) - q \cdot y_{\mathrm{ref}}(t))\|^2 + \varphi(t)^2 \|v(t) - q \cdot y_{\mathrm{ref}}(t)\|^4_{L^4} \le \Xi_1, \\ &\int_\gamma^t \|v(s) - q \cdot y_{\mathrm{ref}}(s)\|^4_{L^4} \, \mathrm{d}s \le 2K_2 t + \|v_\gamma - q \cdot y_{\mathrm{ref}}(\gamma)\|^2. \end{aligned} \tag{50}$$

Moreover, as in Section 4.1, $v_n \to v$ strongly in $L^2(\gamma, T; L^2(\Omega))$ and $u, v \in C([\gamma, T); L^2(\Omega))$ with $(u(\gamma), v(\gamma)) = (u_\gamma, v_\gamma)$.

Hence, for $\chi \in L^2(\Omega)$ and $\theta \in W^{1,2}(\Omega)$ we have that (u_n, v_n) satisfy the integrated version of (29), thus we obtain that for $t \in (\gamma, T)$

$$\begin{aligned} \langle v(t), \theta \rangle &= \langle v_\gamma, \theta \rangle + \int_\gamma^t -\mathfrak{a}(v(s), \theta) + \langle p_3(v(s)) - u(s) + I_{s,i}(s), \theta \rangle \, \mathrm{d}s\,, \\ &\quad + \int_\gamma^T \langle I_{s,e}(s), \mathcal{B}'\theta \rangle_{\mathbb{R}^m} \, \mathrm{d}s\,, \\ \langle u(t), \chi \rangle &= \langle u_\gamma, \chi \rangle + \int_\gamma^t \langle c_5 v(s) - c_4 u(s), \chi \rangle \, \mathrm{d}s\,, \\ I_{s,e}(t) &= -\frac{k_0}{1 - \varphi(t)^2 \|\mathcal{B}' v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}} (\mathcal{B}' v(t) - y_{\mathrm{ref}}(t)) \end{aligned}$$

by bounded convergence [40, Thm. II.4.1]. Hence, (u, v) is a solution of (10) in (γ, T). Moreover, (23) also holds in $W^{1,2}(\Omega)'$ for $t \geq \gamma$, that is

$$\dot{v}(t) = \mathcal{A}v(t) + p_3(v(t)) + \mathcal{B}I_{s,e}(t) - u(t) + I_{s,i}(t). \tag{51}$$

Step 5: We show uniqueness of the solution on $[0, \infty)$.
The proof is similar, but, in an essential step, also different from the proof given in Step 1e of Section 4.1. Let $T > 0$ and assume that (v_1, u_1) and (v_2, u_2) are two solutions of (10) on $[\gamma, T)$ with $v_1(\gamma) = v_2(\gamma) = v_\gamma$ and $u_1(\gamma) = u_2(\gamma) = u_\gamma$. Choose $\hat{p}_3$, $Q_\gamma := (\gamma, T) \times \Omega$, Σ, Λ and Q^Λ similar to Step 1e of Section 4.1, where we invoke that $v_1, v_2 \in L^4((\gamma, T) \times \Omega)$. Let $V := v_2 - v_1$ and $U := u_2 - u_1$, then, by (10),

$$\begin{aligned}
\dot{V} &= (\mathcal{A} - c_1 I)V - c_3(\hat{p}_3(v_2) - \hat{p}_3(v_1)) - U \\
&\quad - k_0 \mathcal{B} \left(\frac{\mathcal{B}'v_2 - y_{\mathrm{ref}}}{1 - \varphi^2 \|\mathcal{B}'v_2 - y_{\mathrm{ref}}\|^2_{\mathbb{R}^m}} - \frac{\mathcal{B}'v_1 - y_{\mathrm{ref}}}{1 - \varphi^2 \|\mathcal{B}'v_1 - y_{\mathrm{ref}}\|^2_{\mathbb{R}^m}} \right), \\
\dot{U} &= c_5 V - c_4 U.
\end{aligned}$$

Define

$$\Xi(t) := \frac{\mathcal{B}'v_2(t) - y_{\mathrm{ref}}(t)}{1 - \varphi(t)^2 \|\mathcal{B}'v_2(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}} - \frac{\mathcal{B}'v_1(t) - y_{\mathrm{ref}}(t)}{1 - \varphi(t)^2 \|\mathcal{B}'v_1(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}} \in \mathbb{R}^m$$

for $t \in (\gamma, T)$, then we may compute that

$$\begin{aligned}
\tfrac{c_5}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \|V\|^2 + \tfrac{1}{2} \tfrac{\mathrm{d}}{\mathrm{d}t} \|U\|^2 &= \langle (\mathcal{A} - c_1 I)V - U, c_5 V \rangle_{W^{1,2}(\Omega)', W^{1,2}(\Omega)} - c_4 \|U\|^2 \\
&\quad + c_5 \langle U, V \rangle - c_5 c_3 \langle \hat{p}_3(v_2) - \hat{p}_3(v_1), V \rangle - c_5 k_0 \langle \Xi, \mathcal{B}'V \rangle_{\mathbb{R}^m} \\
&\leq -c_5 c_3 \langle \hat{p}_3(v_2) - \hat{p}_3(v_1), V \rangle - c_5 k_0 \langle \Xi, \mathcal{B}'V \rangle_{\mathbb{R}^m}.
\end{aligned}$$

Define

$$e_i(t) := \mathcal{B}'v_i(t) - y_{\mathrm{ref}}(t), \quad k_i(t) := \frac{1}{1 - \varphi(t)^2 \|e_i(t)\|^2_{\mathbb{R}^m}}, \quad t \in (\gamma, T),\ i = 1, 2$$

and observe that on (γ, T) we have

$$\begin{aligned}
k_1 \geq k_2 \quad &\iff \quad 1 - \varphi^2 \|e_2\|^2_{\mathbb{R}^m} \geq 1 - \varphi^2 \|e_1\|^2_{\mathbb{R}^m} \\
&\iff \quad \|e_1\|^2_{\mathbb{R}^m} \geq \|e_2\|^2_{\mathbb{R}^m},
\end{aligned}$$

by which

$$\forall\, t \in (\gamma, T): \ \big(k_1(t) - k_2(t)\big)\big(\|e_1(t)\|^2_{\mathbb{R}^m} - \|e_2(t)\|^2_{\mathbb{R}^m}\big) \geq 0.$$

Then we may calculate that

$$\begin{aligned}
\langle \Xi, \mathcal{B}'V \rangle_{\mathbb{R}^m} &= \langle k_2 e_2 - k_1 e_1, e_2 - e_1 \rangle_{\mathbb{R}^m} \\
&= k_2 \|e_2\|^2_{\mathbb{R}^m} + k_1 \|e_1\|^2_{\mathbb{R}^m} - (k_1 + k_2) \langle e_2, e_1 \rangle_{\mathbb{R}^m} \\
&\geq k_2 \|e_2\|^2_{\mathbb{R}^m} + k_1 \|e_1\|^2_{\mathbb{R}^m} - \tfrac{1}{2}(k_1 + k_2)(\|e_1\|^2_{\mathbb{R}^m} + \|e_2\|^2_{\mathbb{R}^m}) \\
&= \tfrac{1}{2}(k_2 - k_1)\|e_2\|^2_{\mathbb{R}^m} + \tfrac{1}{2}(k_1 - k_2)\|e_1\|^2_{\mathbb{R}^m} \\
&= \tfrac{1}{2}(k_1 - k_2)\big(\|e_1\|^2_{\mathbb{R}^m} - \|e_2\|^2_{\mathbb{R}^m}\big) \geq 0.
\end{aligned}$$

Therefore, we have that

$$\tfrac{c_5}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|V\|^2 + \tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|U\|^2 \le -c_5c_3\left\langle \hat{p}_3(v_2) - \hat{p}_3(v_1), V\right\rangle .$$

Then the same arguments as in Step 1e of Section 4.1 apply to conclude that $V(t) = 0$ and $U(t) = 0$ for all $t \in (\gamma, T)$, thus $v_1 = v_2$ and $u_1 = u_2$ on (γ, T). Combining this with uniqueness on $[0, \gamma]$ and invoking that $T > 0$ was arbitrary we obtain a unique solution on $[0, \infty)$.

Step 6: We show the regularity properties of the solution.
To this end, note that for all $\delta > 0$ we have that

$$v \in L^2_{\mathrm{loc}}(\gamma, \infty; W^{1,2}(\Omega)) \cap L^\infty(\gamma + \delta, \infty; W^{1,2}(\Omega)),$$

so that $I_r := I_{s,i} + c_2v^2 - c_3v^3 - u \in L^2_{\mathrm{loc}}(\gamma, \infty; L^2(\Omega)) \cap L^\infty(\gamma + \delta, \infty; L^2(\Omega))$, and the application of Proposition B.5 yields that $v \in BC([\gamma, \infty); L^2(\Omega)) \cap BUC((\gamma, \infty); W^{1,2}(\Omega))$. By the uniform continuity of v and the completeness of $W^{1,2}(\Omega)$, v has a limit at $t = \gamma$, see for instance [41, Thm. II.13.D]. Thus, $v \in L^\infty(\gamma, \infty; W^{1,2}(\Omega))$. From Section 4.1 and the latter we have that $v \in L^2_{\mathrm{loc}}(0, \infty; W^{1,2}(\Omega)) \cap L^\infty(\delta, \infty; W^{1,2}(\Omega))$ for all $\delta > 0$, so we have

$$\begin{aligned} I_{s,e} &\in L^2_{\mathrm{loc}}(0, \infty; \mathbb{R}^m) \cap L^\infty(\delta, \infty; \mathbb{R}^m),\\ v &\in L^2_{\mathrm{loc}}(0, \infty; W^{1,2}(\Omega)) \cap L^\infty(\delta, \infty; W^{1,2}(\Omega))\\ &\quad \cap BC([0, \infty); L^2(\Omega)) \cap BUC([\delta, \infty); W^{1,2}(\Omega)),\end{aligned}$$

so that $I_r := I_{s,i} + c_2v^2 - c_3v^3 - u \in L^2_{\mathrm{loc}}(0, \infty; L^2(\Omega)) \cap L^\infty(\delta, \infty; L^2(\Omega))$. Recall that by assumption we have $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{r,2}(\Omega)')$ for some $r \in [0, 1]$. Applying Proposition B.5 we have that for all $\delta > 0$ the unique solution of (51) satisfies

$$\begin{aligned} &\text{if } r = 0: \quad \forall\, \lambda \in (0, 1): \ v \in C^{0,\lambda}([\delta, \infty); L^2(\Omega));\\ &\text{if } r \in (0, 1): \quad v \in C^{0,1-r/2}([\delta, \infty); L^2(\Omega));\\ &\text{if } r = 1: \quad v \in C^{0,1/2}([\delta, \infty); L^2(\Omega)).\end{aligned} \tag{52}$$

Since $u, v \in BC([0, \infty); L^2(\Omega))$ and $\dot{u} = c_4v - c_5u$, we also have $\dot{u} \in BC([0, \infty); L^2(\Omega))$.
Now, from (52) and $\mathcal{B}' \in \mathcal{L}(W^{r,2}(\Omega), \mathbb{R}^m)$ for $r \in [0, 1]$ we obtain that

- for $r = 0$ and $\lambda \in (0, 1)$: $y = \mathcal{B}'v \in C^{0,\lambda}([\delta, \infty); \mathbb{R}^m)$;
- for $r \in (0, 1)$: $y = \mathcal{B}'v \in C^{0,1-r}([\delta, \infty); \mathbb{R}^m)$;
- for $r = 1$: $y = \mathcal{B}'v \in BUC([\delta, \infty); \mathbb{R}^m)$.

Further, from (50) we have

$$\forall\, t \ge \delta: \ \varphi(t)^2\|\mathcal{B}'v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m} \le 1 - \varepsilon_0,$$

hence $I_{s,e} \in L^\infty(\delta, \infty; \mathbb{R}^m)$ and $I_{s,e}$ has the same regularity properties as y, since we have that $\varphi \in \Phi_\gamma$ and $y_{\mathrm{ref}} \in W^{1,\infty}(0, \infty; \mathbb{R}^m)$. Therefore, we have proved statements (i)–(iii) in Theorem 3.3 as well as a) and b).

It remains to show c), for which we additionally require that $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega))$. Then there exist $b_1, \ldots, b_m \in W^{1,2}(\Omega)$ such that $(\mathcal{B}'x)_i = \langle x, b_i \rangle$ for all $i = 1, \ldots, m$ and $x \in L^2(\Omega)$. Using the b_i in the weak formulation for $i = 1, \ldots, m$, we have

$$\tfrac{\mathrm{d}}{\mathrm{d}t} \langle v(t), b_i \rangle = -\mathfrak{a}(v(t), b_i) + \langle p_3(v(t)) - u(t) + I_{s,i}(t), b_i \rangle + \langle I_{s,e}(t), \mathcal{B}'b_i \rangle_{\mathbb{R}^m}.$$

Since $(\mathcal{B}'v(t))_i = \langle v(t), b_i \rangle$, this leads to

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}'v(t)_i) = -\mathfrak{a}(v(t), b_i) + \langle p_3(v(t)) - u(t) + I_{s,i}(t), b_i \rangle + \langle I_{s,e}(t), \mathcal{B}'b_i \rangle_{\mathbb{R}^m}.$$

Taking the absolute value and using the Cauchy-Schwarz inequality yields

$$\begin{aligned} \left| \tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}'v(t))_i \right| \le \|D\|_{L^\infty} \|v(t)\|_{W^{1,2}} \|b_i\|_{W^{1,2}} &+ \|p_3(v(t)) - u(t) + I_{s,i}(t)\|_{L^2} \|b_i\|_{L^2} \\ &+ \|I_{s,e}(t)\|_{\mathbb{R}^m} \|\mathcal{B}'b_i\|_{\mathbb{R}^m}, \end{aligned}$$

and therefore

$$\forall\, i = 1, \ldots, m \ \forall\, \delta >: \ \left\| \tfrac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}'v)_i \right\|_{L^\infty(\delta,\infty;\mathbb{R}^m)} < \infty,$$

by which $y = \mathcal{B}'v \in W^{1,\infty}(\delta, \infty; \mathbb{R}^m)$ as well as $I_{s,e} \in W^{1,\infty}(\delta, \infty; \mathbb{R}^m)$. This completes the proof of the theorem. □

5. A numerical example

In this section, we illustrate the practical applicability of the funnel controller by means of a numerical example. The setup chosen here is a standard test example for termination of reentry waves and has been considered similarly e.g. in [42, 24]. All simulations are generated on an AMD Ryzen 7 1800X @ 3.68 GHz x 16, 64 GB RAM, MATLAB® Version 9.2.0.538062 (R2017a). The solutions of the ODE systems are obtained by the MATLAB® routine `ode23`. The parameters for the FitzHugh-Nagumo model (3) used here are as follows:

$$\Omega = (0,1)^2, \quad D = \begin{bmatrix} 0.015 & 0 \\ 0 & 0.015 \end{bmatrix}, \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \approx \begin{pmatrix} 1.614 \\ 0.1403 \\ 0.012 \\ 0.00015 \\ 0.015 \end{pmatrix}.$$

The spatially discrete system of ODEs corresponds to a finite element discretization with piecewise linear finite elements on a uniform 64×64 mesh. For the control action, we assume that $\mathcal{B} \in \mathcal{L}(\mathbb{R}^4, W^{1,2}(\Omega)')$, where the Robin control operator is defined by

$$\begin{aligned} &\mathcal{B}'z = \left(\textstyle\int_{\Gamma_1} z(\xi)\,\mathrm{d}\sigma, \int_{\Gamma_2} z(\xi)\,\mathrm{d}\sigma, \int_{\Gamma_3} z(\xi)\,\mathrm{d}\sigma, \int_{\Gamma_4} z(\xi)\,\mathrm{d}\sigma \right)^\top, \\ &\Gamma_1 = \{1\} \times [0,1], \quad \Gamma_2 = [0,1] \times \{1\}, \quad \Gamma_3 = \{0\} \times [0,1], \quad \Gamma = [0,1] \times \{0\}. \end{aligned}$$

The purpose of the numerical example is to model a typical defibrillation process as a tracking problem as discussed above. In this context, system (3) is initialized with $(v(0), u(0)) = (v_0^*, u_0^*)$ and $I_{s,i} = 0 = I_{s,e}$, where (v_0^*, u_0^*) is an arbitrary snapshot of a reentry wave. The resulting reentry phenomena are shown in Fig. 2 and resemble a dysfunctional heart rhythm which impedes the intracellular stimulation current $I_{s,i}$. The objective is to design a stimulation current $I_{s,e}$ such that the dynamics return to a natural heart rhythm modeled by a reference trajectory y_{ref}. The trajectory $y_{\mathrm{ref}} = \mathcal{B}' v_{\mathrm{ref}}$ corresponds to a solution $(v_{\mathrm{ref}}, u_{\mathrm{ref}})$ of (3) with $(v_{\mathrm{ref}}(0), u_{\mathrm{ref}}(0)) = (0, 0)$, $I_{s,e} = 0$ and

$$I_{s,i}(t) = 101 \cdot w(\xi)(\chi_{[49,51]}(t) + \chi_{[299,301]}(t)),$$

where the excitation domain of the intracellular stimulation current $I_{s,i}$ is described by

$$w(\xi) = \begin{cases} 1, & \text{if } (\xi_1 - \frac{1}{2})^2 + (\xi_2 - \frac{1}{2})^2 \leq 0.0225, \\ 0, & \text{otherwise.} \end{cases}$$

The smoothness of the signal is guaranteed by convoluting the original signal with a triangular function. The function φ characterizing the performance funnel (see Fig. 3) is chosen as

$$\varphi(t) = \begin{cases} 0, & t \in [0, 0.05], \\ \tanh(\frac{t}{100}), & t > 0.05. \end{cases}$$

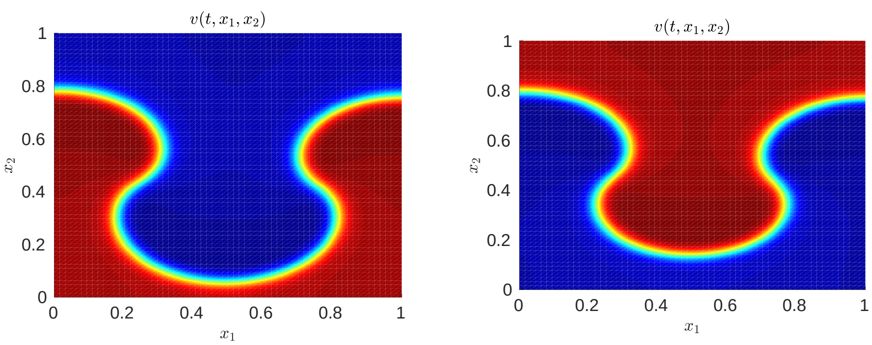

Figure 2: Snapshots of reentry waves for $t = 100$ (left) and $t = 200$ (right).

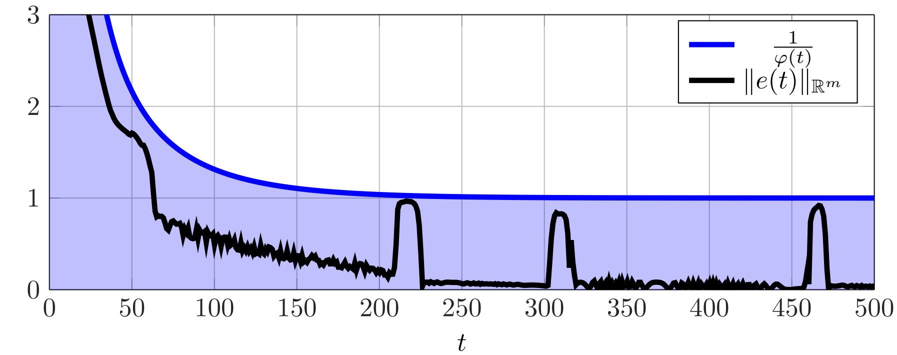

Figure 3: Error dynamics and funnel boundary.

Fig. 4 shows the results of the closed-loop system for $(v(0), u(0)) = (v_0^*, u_0^*)$ and the control law

$$I_{s,e}(t) = -\frac{0.75}{1 - \varphi(t)^2 \|\mathcal{B}' v(t) - y_{\text{ref}}(t)\|_{\mathbb{R}^m}^2} (\mathcal{B}' v(t) - y_{\text{ref}}(t)),$$

which is visualized in Fig. 5. Let us note that the sudden changes in the feedback law are due to the jump discontinuities of the intracellular stimulation current $I_{s,i}$ used for simulating a regular heart beat.

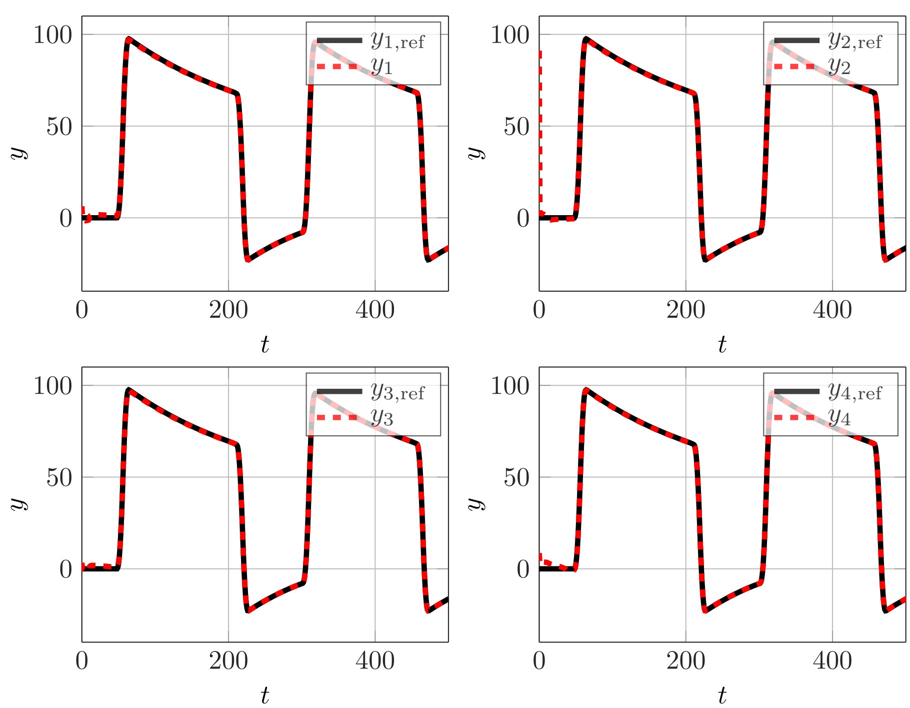

Figure 4: Reference signals and outputs of the funnel controlled system.

We see from Fig. 4 that the controlled system tracks the desired reference signal with the prescribed performance. Also note that the performance constraints are not active on the interval $[0, 0.05]$. Fig. 5 further shows that the tracking is achieved with a comparably small control effort.

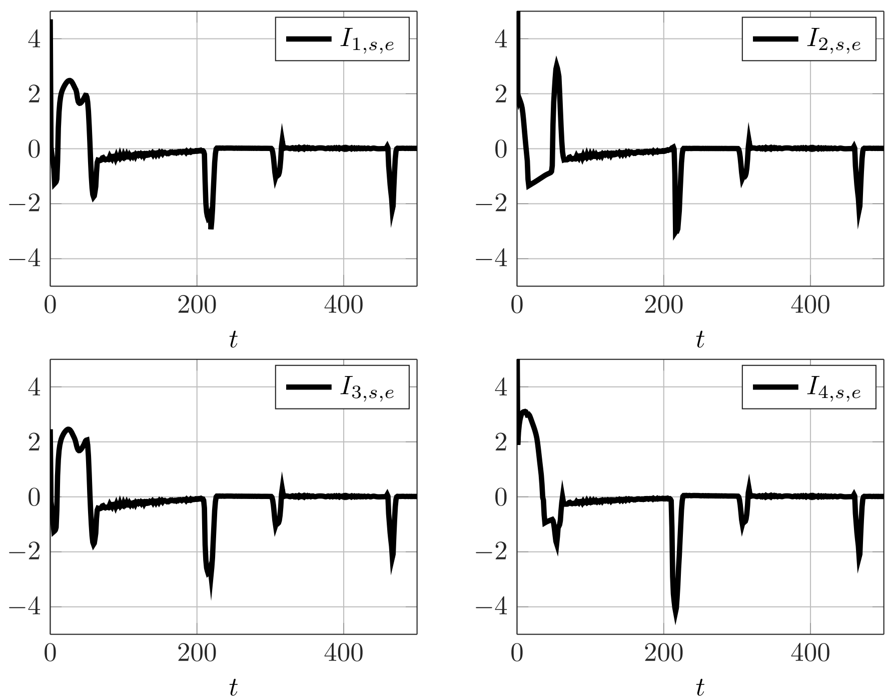

Figure 5: Funnel control laws.

6. Conclusions

In this work we have proved existence and uniqueness of global bounded solutions of a reaction diffusion system under funnel control. The considered monodomain equations with the FitzHugh-Nagumo model (3) are a relevant system arising in mathematical biology. The considered input-output configurations allow for both distributed and boundary control and observation. The proposed funnel control feedback law (9) renders the closed-loop system (10) a nonlinear and non-autonomous PDE with the requirement that the tracking error associated to any solution evolves in the performance funnel $\mathcal{F}_\varphi$. In the main result Theorem 3.3 we have put special emphasis on the regularity properties of the solutions of (10).

The present work is the basis for extensions in several directions. Apart from more general reaction diffusion systems with more complex nonlinearities and other nonlinear parabolic equations from mathematical biology (such as the Keller-Segel system modelling chemotactic behavior [52]) an important topic for future research is the investigation of the bidomain model of the human heart [23]. Since this model is closer to reality, the authors expect that successfully applying funnel control methods will potentially lead to real-world applications such as in implantable cardioverter defibrillators.

Appendix

A. Interpolation spaces

We collect some results on interpolation spaces, which are necessary for the proof of Theorem 3.3. For a (more) general interpolation theory, we refer to [44].

Definition A.1. *Let X, Y be Hilbert spaces and let $\alpha \in [0,1]$. Consider the function*

$$K : (0,\infty) \times (X+Y) \to \mathbb{R},\ (t,x) \mapsto \inf_{\substack{a\in X,\, b\in Y,\\ x=a+b}} \|a\|_X + t\|b\|_Y.$$

The interpolation space $(X,Y)_\alpha$ *is defined by*

$$(X,Y)_\alpha := \left\{ \ x \in X+Y \ \middle|\ \Big(t \mapsto t^{-\alpha} K(t,x)\Big) \in L^2(0,\infty) \ \right\},$$

and it is a Hilbert space with the norm

$$\|x\|_{(X,Y)_\alpha} = \|t \mapsto t^{-\alpha}K(t,x)\|_{L^2}.$$

Note that interpolation can be performed in a more general fashion for Banach spaces X, Y. More precise, we may utilize the L^p-norm of the map $t \mapsto t^{-\alpha}K(t,x)$ for some $p \in [1,\infty)$ instead of the L^2-norm in the above definition. However, this does not lead to Hilbert spaces $(X,Y)_\alpha$, not even when X and Y are Hilbert spaces.

For a self-adjoint operator $A : \mathcal{D}(A) \subset X \to X$, X a Hilbert space and $n \in \mathbb{N}$, we may define the space $X_n := \mathcal{D}(A^n)$ by $X_0 = X$ and $X_{n+1} := \{\ x \in X_n \ \mid\ Ax \in X_n\ \}$. This is a Hilbert space with norm $\|z\|_{X_{n+1}} = \|-\lambda z + Az\|_{X_n}$, where $\lambda \in \mathbb{C}$ is in the resolvent set of A. Likewise, we introduce X_{-n} as the completion of X with respect to the norm $\|z\|_{X_{-n}} = \|(-\lambda I + A)^{-n} z\|$. Note that X_{-n} is the dual of X_n with respect to the pivot space X, cf. [38, Sec. 2.10]. Using interpolation theory, we may further introduce the spaces X_α for any $\alpha \in \mathbb{R}$ as follows.

Definition A.2. *Let $\alpha \in \mathbb{R}$, X a Hilbert space and $A : \mathcal{D}(A) \subset X \to X$ be self-adjoint. Further, let $n \in \mathbb{Z}$ be such that $\alpha \in [n, n+1)$. The space X_α is defined as the interpolation space*

$$X_\alpha = (X_n, X_{n+1})_{\alpha-n}.$$

The reiteration theorem, see [44, Cor. 1.24], together with [44, Prop. 3.8] yields that for all $\alpha \in [0,1]$ and $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 \leq \alpha_2$ we have that

$$(X_{\alpha_1}, X_{\alpha_2})_\alpha = X_{\alpha_1+\alpha(\alpha_2-\alpha_1)}. \tag{53}$$

Next we characterize interpolation spaces associated with the Robin elliptic operator $\mathcal{A}$ as in (2).

Remark A.3. *Let Assumption 2.1 hold and $\mathcal{A}$ be the Robin elliptic operator as in* (2)*. Further let X_α, $\alpha \in \mathbb{R}$, be the corresponding interpolation spaces with, in particular, $X = X_0 = L^2(\Omega)$. Then the equation $X_{1/2} = W^{1,2}(\Omega)$ is an immediate consequence of a combination of* (5) *with Kato's second representation theorem [29, Sec. VI.2, Thm. 2.23]. Further,* (53) *implies that*

$$X_{r/2} = (X_0, X_{1/2})_r \quad \text{for all } r \in [0,1].$$

On the other hand, [32, Thm. 1.35] gives $(L^2(\Omega), W^{1,2}(\Omega))_r = W^{r,2}(\Omega)$, whence we obtain

$$X_{r/2} = W^{r,2}(\Omega) \quad \text{for all } r \in [0,1]. \tag{54}$$

In terms of the spectral decomposition (6)*, the interpolation space has the representation*

$$X_\alpha = \left\{ \sum_{j=0}^{\infty} \lambda_j \theta_j \;\middle|\; (\lambda_j)_{j\in\mathbb{N}_0} \text{ with } \sum_{j=0}^{\infty} \alpha_j^{2\alpha} |\lambda_j|^2 < \infty \right\}, \tag{55}$$

which follows from a combination of [44, Thm. 4.33] with [44, Thm. 4.36].

B. Abstract Cauchy problems and regularity

We consider mild solutions of certain abstract Cauchy problems and the concept of admissible control operators. This notion is well-known in infinite-dimensional linear systems theory with unbounded control and observation operators and we refer to [38] for further details.

Let X be a real Hilbert space and recall that a semigroup $(\mathbb{T}_t)_{t\geq 0}$ on X is a $\mathcal{L}(X,X)$-valued map satisfying $\mathbb{T}_0 = I_X$ and $\mathbb{T}_{t+s} = \mathbb{T}_t\mathbb{T}_s$, $s,t \geq 0$, where I_X denotes the identity operator, and $t \mapsto \mathbb{T}_t x$ is continuous for every $x \in X$. Semigroups are characterized by their generator A, which is a, not necessarily bounded, operator on X. If $A : \mathcal{D}(A) \subset X \to X$ is self-adjoint with $\langle x, Ax\rangle \leq 0$ for all $x \in \mathcal{D}(A)$, then it generates a contractive, analytic semigroup $(\mathbb{T}_t)_{t\geq 0}$ on X, cf. [45, Thm. 4.2]. Furthermore, if additionally there exists $\omega_0 > 0$ such that $\langle x, Ax\rangle \leq -\omega_0\|x\|^2$ for all $x \in \mathcal{D}(A)$, then the semigroup $(\mathbb{T}_t)_{t\geq 0}$ generated by A satisfies $\|\mathbb{T}_t\| \leq \mathrm{e}^{-\omega_0 t}$ for all $t \geq 0$; the smallest number ω_0 for which this is true is called *growth bound* of $(\mathbb{T}_t)_{t\geq 0}$. We can further conclude from [46, Thm. 6.13 (b)] that, for all $\alpha \in \mathbb{R}$, $(\mathbb{T}_t)_{t\geq 0}$ restricts (resp. extends) to an analytic semigroup $((\mathbb{T}|_\alpha)_t)_{t\geq 0}$ on X_α with same growth bound as $(\mathbb{T}_t)_{t\geq 0}$. Furthermore, we have $\operatorname{im} \mathbb{T}_t \subset X_r$ for all $t > 0$ and $r \in \mathbb{R}$, see [46, Thm. 6.13(a)]. In the following we present an estimate for the corresponding operator norm.

Lemma B.1. *Assume that $A : \mathcal{D}(A) \subset X \to X$, X a Hilbert space, is self-adjoint and there exists $\omega_0 > 0$ with $\langle x, Ax \rangle \leq -\omega_0 \|x\|^2$ for all $x \in \mathcal{D}(A)$. Then there exist $M, \omega > 0$ such that the semigroup $(\mathbb{T}_t)_{t \geq 0}$ generated by A satisfies*

$$\forall\, \alpha \in [0,2]\ \forall\, t > 0 :\ \|\mathbb{T}_t\|_{\mathcal{L}(X, X_\alpha)} \leq M(1 + t^{-\alpha})\mathrm{e}^{-\omega t}.$$

Thus, for each $\alpha \in [0,2]$ there exists $K > 0$ such that

$$\sup_{t \in [0,\infty)} t^\alpha \|\mathbb{T}_t\|_{\mathcal{L}(X, X_\alpha)} < K.$$

Proof. Since A with the above properties generates an exponentially stable analytic semigroup $(\mathbb{T}_t)_{t \geq 0}$, the cases $\alpha \in [0,1]$ and $\alpha = 2$ follow from [47, Cor. 3.10.8 & Lem. 3.10.9]. The result for $\alpha \in [1,2]$ is a consequence of [47, Lem 3.9.8] and interpolation between X_1 and X_2, cf. Appendix A. □

Next we consider the abstract Cauchy problem with source term.

Definition B.2. *Let X be a Hilbert space, $A : \mathcal{D}(A) \subset X \to X$ be self-adjoint with $\langle x, Ax \rangle \leq 0$ for all $x \in \mathcal{D}(A)$, $T \in (0, \infty]$, and $\alpha \in [0,1]$. Let $(\mathbb{T}_t)_{t \geq 0}$ be the semigroup on X generated by A, and let $B \in \mathcal{L}(\mathbb{R}^m, X_{-\alpha})$. For $x_0 \in X$, $p \in [1, \infty]$, $f \in L^p_{\mathrm{loc}}(0,T;X)$ and $u \in L^p_{\mathrm{loc}}(0,T;\mathbb{R}^m)$, we call $x : [0,T) \to X$ a* mild solution *of*

$$\dot{x}(t) = Ax(t) + f(t) + Bu(t), \quad x(0) = x_0 \tag{56}$$

on $[0,T)$, if it satisfies

$$\forall\, t \in [0,T) :\ x(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s\, + \int_0^t (\mathbb{T}|_{-\alpha})_{t-s} Bu(s)\,\mathrm{d}s\,. \tag{57}$$

We further call $x : [0,T) \to X$ a strong solution *of* (56) *on $[0,T)$, if x in* (57) *satisfies $x \in C([0,T);X) \cap W^{1,p}_{\mathrm{loc}}(0,T;X_{-1})$.*

Definition B.2 requires that the integral $\int_0^t (\mathbb{T}|_{-\alpha})_{t-s} Bu(s)\,\mathrm{d}s$ is in X, whilst the integrand is not necessarily in X. This motivates the definition of admissibility, which is now introduced for self-adjoint A. Note that admissibility can also be defined for arbitrary generators of semigroups, see [38].

Definition B.3. *Let X be a Hilbert space, $A : \mathcal{D}(A) \subset X \to X$ be self-adjoint with $\langle x, Ax \rangle \leq 0$ for all $x \in \mathcal{D}(A)$, $T \in (0, \infty]$, $\alpha \in [0,1]$ and $p \in [1, \infty]$. Let $(\mathbb{T}_t)_{t \geq 0}$ be the semigroup on X generated by A, and let $B \in \mathcal{L}(\mathbb{R}^m, X_{-\alpha})$. Then B is called an* L^p-admissible (control operator) for $(\mathbb{T}_t)_{t \geq 0}$, *if for some (and hence any) $t > 0$ we have*

$$\forall\, u \in L^p(0,t;\mathbb{R}^m) :\ \Phi_t u := \int_0^t (\mathbb{T}|_{-\alpha})_{t-s} Bu(s)\,\mathrm{d}s\, \in X.$$

By a closed graph theorem argument this implies that $\Phi_t \in \mathcal{L}(L^p(0,t;\mathbb{R}^m), X)$ for all $t > 0$. We call B an infinite-time L^p-admissible (control operator) for $(\mathbb{T}_t)_{t \geq 0}$, *if*

$$\sup_{t > 0} \|\Phi_t\| < \infty.$$

In the following we show that for $p \geq 2$ and $\alpha \leq 1/2$ any B is admissible and the mild solution of the abstract Cauchy problem is indeed a strong solution.

Lemma B.4. *Let X be a Hilbert space, $A : \mathcal{D}(A) \subset X \to X$ be self-adjoint with $\langle x, Ax\rangle \leq 0$ for all $x \in \mathcal{D}(A)$, $B \in \mathcal{L}(\mathbb{R}^m, X_{-\alpha})$ for some $\alpha \in [0, 1/2]$, and $(\mathbb{T}_t)_{t\geq 0}$ be the analytic semigroup generated by A. Then for all $p \in [2, \infty]$ we have that B is L^p-admissible for $(\mathbb{T}_t)_{t\geq 0}$.*

Furthermore, for all $x_0 \in X$, $T \in (0, \infty]$, $f \in L^p_{\mathrm{loc}}(0, T; X)$ and $u \in L^p_{\mathrm{loc}}(0, T; \mathbb{R}^m)$, the function x in (57) is a strong solution of (56) on $[0, T)$.

Proof. For the case $p = 2$, there exists a unique strong solution in X_{-1} (that is, we replace X by X_{-1} and X_{-1} by X_{-2} in the definition) given by (57) and at most one strong solution in X, see for instance [47, Thm. 3.8.2 (i) & (ii)], so we only need to check that all the elements are in the correct spaces. Since A is self-adjoint, the semigroup generated by A is self-adjoint as well. Further, by combining [38, Prop. 5.1.3] with [38, Thm. 4.4.3], we find that B is an L^2-admissible control operator for $(\mathbb{T}_t)_{t\geq 0}$. Moreover, by [38, Prop. 4.2.5] we have that

$$\left(t \mapsto \mathbb{T}_t x_0 + \int_0^t (\mathbb{T}|_{-\alpha})_{t-s} Bu(s)\,\mathrm{d}s\right) \in C([0, T); X) \cap W^{1,2}_{\mathrm{loc}}(0, T; X_{-1})$$

and from [47, Thm. 3.8.2 (iv)],

$$\left(t \mapsto \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s\right) \in C([0, T); X) \cap W^{1,2}_{\mathrm{loc}}(0, T; X_{-1}),$$

whence $x \in C([0, T); X) \cap W^{1,2}_{\mathrm{loc}}(0, T; X_{-1})$, which proves that x is a strong solution of (56) on $[0, T)$.

Since B is L^2-admissible, it follows from the nesting property of L^p on finite intervals that B is an L^p-admissible control operator for $(\mathbb{T}_t)_{t\geq 0}$ for all $p \in [2, \infty]$. Furthermore, for $p > 2$, set $\tilde{f} \coloneqq f + Bu$ and apply [47, Thm. 3.10.10] with $\tilde{f} \in L^\infty_{\mathrm{loc}}(0, T; X_{-\alpha})$ to conclude that x is a strong solution. □

Next we show the regularity properties of the solution of (56), if $A = \mathcal{A}$ and $B = \mathcal{B}$ are as in the model (3). Note that this result also holds when considering some $t_0 \geq 0$, $T \in (t_0, \infty]$, and the initial condition $x(t_0) = x_0$ (instead of $x(0) = x^0$) by some straightforward modifications, cf. [47, Sec. 3.8].

Proposition B.5. *Let Assumption 2.1 hold, $\mathcal{A}$ be the Robin elliptic operator as in (2), $T \in (0, \infty]$, and $c > 0$. Define $\mathcal{A}_0 \coloneqq \mathcal{A} - cI$ with $\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A})$ and consider $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, W^{r,2}(\Omega)')$ for $r \in [0, 1]$, $u \in L^2_{\mathrm{loc}}(0, T; \mathbb{R}^m) \cap L^\infty(\delta, T; \mathbb{R}^m)$ and $f \in L^2_{\mathrm{loc}}(0, T; L^2(\Omega)) \cap L^\infty(\delta, T; L^2(\Omega))$ for all $\delta > 0$. Then for all $x_0 \in L^2(\Omega)$ and all $\delta > 0$, the mild solution of (56) (with $A = \mathcal{A}_0$ and $B = \mathcal{B}$) on $[0, T)$, given by x as in (57), satisfies*

(i) *if $r = 0$, then*

$$\forall\, \lambda \in (0, 1): \;\; x \in BC([0, T); L^2(\Omega)) \cap C^{0,\lambda}([\delta, T); L^2(\Omega));$$

(ii) *if* $r \in (0,1)$*, then*

$$x \in BC([0,T);L^2(\Omega)) \cap C^{0,1-r/2}([\delta,T);L^2(\Omega)) \cap C^{0,1-r}([\delta,T);W^{r/2,2}(\Omega));$$

(iii) *if* $r = 1$*, then*

$$x \in BC([0,T);L^2(\Omega)) \cap C^{0,1/2}([\delta,T);L^2(\Omega)) \cap BUC([\delta,T);W^{1,2}(\Omega)).$$

Proof. For brevity we set $X = X_0 = L^2(\Omega)$, and let X_α, $\alpha \in \mathbb{R}$, be the interpolation spaces corresponding to $\mathcal{A}$ according to Definition A.2. Observe that the Robin elliptic operator satisfies the assumptions of Lemma B.4 with $p = 2$, hence x as in (57) is a strong solution of (56) on $[0,T)$ in the sense of Definition B.2. In the following we restrict ourselves to the case $T = \infty$, and the assertions for $T < \infty$ follow from these arguments by considering the restrictions to $[0,T)$. Define, for $t \geq 0$, the functions

$$x_h(t) := \mathbb{T}_t x_0, \quad x_f(t) := \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s\,, \quad x_u(t) := \int_0^t (\mathbb{T}|_{-\alpha})_{t-s}\mathcal{B}u(s)\,\mathrm{d}s\,, \tag{58}$$

so that $x = x_h + x_f + x_u$.

Step 1: We show that $x \in BC([0,\infty);X)$. We obtain from Remark 2.2 c) that for all $z \in \mathcal{D}(\mathcal{A})$ we have $\langle z, \mathcal{A}_0 z\rangle \leq -c\|z\|^2$. The self-adjointness of $\mathcal{A}$ moreover implies that $\mathcal{A}_0$ is self-adjoint, whence [45, Thm. 4.2] gives that $\mathcal{A}_0$ generates an analytic, contractive semigroup $(\mathbb{T}_t)_{t\geq 0}$ on X, which satisfies

$$\forall\, t \geq 0\ \forall\, x \in X: \ \|\mathbb{T}_t x\| \leq \mathrm{e}^{-ct}\|x\|. \tag{59}$$

Since, by Lemma B.4, x is a strong solution, we have $x \in C([0,\infty);X) \cap W^{1,2}_{\mathrm{loc}}(0,\infty;X_{-1})$. Further observe that $\mathcal{B}$ is L^∞-admissible by Lemma B.4. Then it follows from (59) and [48, Lem. 2.9 (i)] that $\mathcal{B}$ is infinite-time L^∞-admissible, which implies that for x_u as in (58) we have

$$\|x_u\|_\infty \leq \left(\sup_{t>0}\|\Phi_t\|\right)\|u\|_\infty < \infty,$$

thus $x_u \in BC([0,\infty);X)$. A direct calculation using (59) further shows that $x_h, x_f \in BC([0,\infty);X)$, whence $x \in BC([0,\infty);X)$.

Step 2: We show (i). Let $\delta > 0$ and set $\tilde{f} := f + Bu \in L^2_{\mathrm{loc}}(0,\infty;X) \cap L^\infty(\delta,\infty;X)$, then we may infer from [27, Props. 4.2.3 & 4.4.1 (i)] that

$$\forall\, \lambda \in (0,1): \ x \in C^{0,\lambda}([\delta,\infty);X).$$

From this together with Step 1 we may infer (i).

Step 3: We show (ii). Let $\delta > 0$, then it follows from [27, Props. 4.2.3 & 4.4.1 (i)] together with $x_0 \in X$ and $f \in L^\infty(\delta,\infty;X)$, that

$$\begin{aligned} x_h + x_f &\in C^{0,1-r/2}([\delta,\infty);X_{r/2}) \cap C^1([\delta,\infty);X) \\ &= C^{0,1-r}([\delta,\infty);X_{r/2}) \cap C^{0,1-r/2}([\delta,\infty);X). \end{aligned}$$

Since we have shown in Step 1 that $x \in BC([0,\infty),X)$, it remains to show that $x_u \in C^{0,1-r}([\delta,\infty);X_{r/2}) \cap C^{0,1-r/2}([\delta,\infty);X)$.
To this end, consider the space $Y := X_{-r/2}$. Then $(\mathbb{T}_t)_{t\geq 0}$ extends to a semigroup $\big((\mathbb{T}|_{-r/2})_t\big)_{t\geq 0}$ on Y with generator $\mathcal{A}_{0,r/2} : \mathcal{D}(\mathcal{A}_{0,r/2}) = X_{-r/2+1} \subset X_{-r/2} = Y$, cf. [27, pp. 50]. Now, for $\alpha \in \mathbb{R}$, consider the interpolation spaces Y_α as in Definition A.2 by means of the operator $\mathcal{A}_{0,r/2}$. Then it is straightforward to show that $Y_n = \mathcal{D}(\mathcal{A}^n_{0,r/2}) = X_{n-r/2}$ for all $n \in \mathbb{N}$ using the representation (55). Similarly, we may show that $Y_n = X_{n-r/2}$ for all $n \in \mathbb{Z}$. Then the reiteration theorem, see [44, Cor. 1.24] and also (53), gives

$$\forall\, \alpha \in \mathbb{R} : \quad Y_\alpha = X_{\alpha - r/2}.$$

Since $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, Y)$, [27, Props. 4.2.3 & 4.4.1 (i)] now imply

$$\begin{aligned} x_u &\in C^{0,1-r}([\delta,\infty);Y_r) \cap C^{0,1-r/2}([\delta,\infty);Y_{r/2}) \\ &= C^{0,1-r}([\delta,\infty);X_{r/2}) \cap C^{0,1-r/2}([\delta,\infty);X), \end{aligned}$$

which leads to $x \in C^{0,1-r}([\delta,\infty);X_{r/2}) \cap C^{0,1-r/2}([\delta,\infty);X)$, and by further using (54), we may conclude statement (ii).

Step 4: We show (iii). The proof of $x \in C^{0,1/2}([\delta,\infty);X)$ is analogous to that of $x \in C^{0,1-r/2}([\delta,\infty);X)$ in Step 3. Boundedness and continuity of x on $[0,\infty)$ was proved in Step 1. Since we have $X_{1/2} = W^{1,2}(\Omega)$ by (54), it remains to show that x is uniformly continuous as a mapping to $X_{1/2}$: Again consider the additive decomposition of x into x_h, x_f and x_u as in (58). Similar to Step 3 it can be shown that $x_h, x_f \in C^{0,1/2}([\delta,\infty);X_{1/2})$, whence $x_h, x_f \in BUC([\delta,\infty);X_{1/2})$. It remains to show that $x_u \in BUC([\delta,\infty);X_{1/2})$.

Note that Lemma B.4 gives that $x_\delta := x(\delta) \in X$. Then x_u solves $\dot z(t) = \mathcal{A}_0 z(t) + \mathcal{B}u(t)$ with $z(\delta) = x_u(\delta)$ and hence, for all $t \geq \delta$ we have

$$x_u(t) = \mathbb{T}_{t-\delta} x_u(\delta) + \underbrace{\int_\delta^t (\mathbb{T}|_{-\alpha})_{t-s} \mathcal{B}u(s)\,\mathrm{d}s}_{=:x_u^\delta(t)} \tag{60}$$

Since $x_u(\delta) \in X$ by Lemma B.4, it remains to show that $x_u^\delta \in BUC([\delta,\infty);X_{1/2})$. We obtain from Remark 2.2 c) that $\mathcal{A}_0$ has an eigendecomposition of type (6) with eigenvalues $(-\beta_j)_{j\in\mathbb{N}_0}$, $\beta_j := \alpha_j + c$, and eigenfunctions $(\theta_j)_{j\in\mathbb{N}_0}$. Moreover, there exist $b_i \in X_{-1/2}$ for $i = 1,\dots,m$ such that $\mathcal{B}\xi = \sum_{i=1}^m b_i \cdot \xi_i$ for all $\xi \in \mathbb{R}^m$. Therefore,

$$\begin{aligned} x_u^\delta(t) &= \int_\delta^t \sum_{j=0}^\infty \mathrm{e}^{-\beta_j(t-\tau)} \theta_j \sum_{i=1}^m \langle b_i \cdot u_i(\tau), \theta_j\rangle \,\mathrm{d}\tau \\ &= \int_\delta^t \sum_{j=0}^\infty \mathrm{e}^{-\beta_j(t-\tau)} \theta_j \sum_{i=1}^m u_i(\tau) \langle b_i, \theta_j\rangle \,\mathrm{d}\tau\,, \end{aligned}$$

where the last equality holds since $u_i(\tau) \in \mathbb{R}$ and can be treated as a constant in X. By considering each of the factors in the sum over $i = 1, \ldots, m$, we can assume without loss of generality that $m = 1$ and $b := b_1$, so that

$$x_u^\delta(t) = \int_\delta^t \sum_{j=0}^\infty \mathrm{e}^{-\beta_j(t-\tau)} u(\tau) \langle b, \theta_j \rangle \theta_j \, \mathrm{d}\tau \,.$$

Define $b^j := \langle b, \theta_j \rangle$ for $j \in \mathbb{N}_0$. Since $b \in X_{-1/2}$ we have that $\sum_{j=0}^\infty b_j^2/\beta_j$ converges, which implies

$$S := \sum_{j=0}^\infty \frac{(b^j)^2}{\beta_j} < \infty. \tag{61}$$

Recall that the spaces X_α, $\alpha \in \mathbb{R}$, are defined by using $\lambda \in \mathbb{C}$ belonging to the resolvent set of $\mathcal{A}$, and they are independent of the choice of λ. Since $c > 0$ in the statement of the proposition is in the resolvent set of $\mathcal{A}$, the spaces X_α coincide for $\mathcal{A}$ and $\mathcal{A}_0 = \mathcal{A} - cI$.
Using the diagonal representation from Remark A.3 and [38, Prop. 3.4.8], we may infer that $x_u^\delta(t) \in X_{1/2}$ for a.a. $t \geq \delta$, namely,

$$\begin{aligned}
\|x_u^\delta(t)\|_{X_{1/2}}^2 &\leq \sum_{j=0}^\infty \beta_j (b^j)^2 \|u\|_{L^\infty(\delta,\infty)}^2 \left(\int_\delta^t \mathrm{e}^{-\beta_j(t-s)} \, \mathrm{d}s \right)^2 \\
&= \|u\|_{L^\infty(\delta,\infty)}^2 \sum_{j=0}^\infty \frac{(b^j)^2}{\beta_j} \left(1 - \mathrm{e}^{-\beta_j(t-\delta)}\right)^2 \\
&\leq \|u\|_{L^\infty(\delta,\infty)}^2 \sum_{j=0}^\infty \frac{(b^j)^2}{\beta_j} < \infty.
\end{aligned}$$

Hence,

$$\|x_u^\delta(t)\|_{X_{1/2}} \leq \|u\|_{L^\infty(\delta,\infty)} \sqrt{S}. \tag{62}$$

Now let $t > s > \delta$ and $\sigma > 0$ such that $t - s < \sigma$. By dominated convergence [40, Thm. II.2.3], summation and integration can be interchanged, so that

$$\begin{aligned}
&\|x_u^\delta(t) - x_u^\delta(s)\|_{X_{1/2}}^2 \\
&\leq \|u\|_{L^\infty(\delta,\infty)}^2 \sum_{j=0}^\infty \beta_j (b^j)^2 \left(\int_\delta^s \mathrm{e}^{-\beta_j(s-\tau)} - \mathrm{e}^{-\beta_j(t-\tau)} \, \mathrm{d}\tau + \int_s^t \mathrm{e}^{-\beta_j(t-\tau)} \, \mathrm{d}\tau \right)^2 \\
&\leq 4\|u\|_{L^\infty(\delta,\infty)}^2 \sum_{j=0}^\infty \frac{(b^j)^2}{\beta_j} \left(1 - \mathrm{e}^{-\beta_j(t-s)}\right)^2 \\
&\leq 4\|u\|_{L^\infty(\delta,\infty)}^2 \sum_{j=0}^\infty \frac{(b^j)^2}{\beta_j} \left(1 - \mathrm{e}^{-\beta_j \sigma}\right)^2 .
\end{aligned}$$

We can conclude from (61) that the series $F : (0,\infty) \to (0, S)$ with

$$F(\sigma) := \sum_{j=0}^{\infty} \frac{(b^j)^2}{\beta_j}(1 - \mathrm{e}^{-\beta_j \sigma})^2$$

converges uniformly to a strictly monotone, continuous and surjective function. Therefore, F has an inverse. The function x_u^δ is thus uniformly continuous on $[\delta,\infty)$ and by (59) we obtain boundedness, i.e., $x_u^\delta \in BUC([\delta,\infty); X_{1/2})$. $\square$

Finally we present a consequence of the Banach-Alaoglu theorem, see e.g. [49, Thm. 3.15].

Lemma B.6. *Let $T > 0$ and Z be a reflexive and separable Banach space. Then*

(i) *every bounded sequence $(w_n)_{n\in\mathbb{N}}$ in $L^\infty(0,T;Z)$ has a weak* convergent subsequence in $L^\infty(0,T;Z)$;*

(ii) *every bounded sequence $(w_n)_{n\in\mathbb{N}}$ in $L^p(0,T;Z)$ with $p \in (1,\infty)$ has a weakly convergent subsequence in $L^p(0,T;Z)$.*

Proof. Let $p \in [1,\infty)$. Then $W := L^p(0,T;Z')$ is a separable Banach space, see [40, Sec. IV.1]. Since Z is reflexive, by [40, Cor. III.4] it has the Radon-Nikodým property. Then it follows from [40, Thm. IV.1] that $W' = L^q(0,T;Z)$ is the dual of W, where $q \in (1,\infty]$ such that $p^{-1} + q^{-1} = 1$. Assertion (i) now follows from [49, Thm. 3.17] with $p = 1$ and $q = \infty$. On the other hand, statement (ii) follows from [51, Thm. V.2.1] by further using that W is reflexive for $p \in (1,\infty)$. $\square$

Acknowledgments

The authors would like to thank Felix L. Schwenninger (U Twente) and Mark R. Opmeer (U Bath) for helpful comments on maximal regularity.

References

[1] M. Agranovich, Sobolev Spaces, Their Generalizations, and Elliptic Problems in Smooth and Lipschitz Domains, Monographs in Mathematics, Springer-Verlag, Berlin Heidelberg, Germany, 2015.

[2] A. Ilchmann, E. P. Ryan, C. J. Sangwin, Tracking with prescribed transient behaviour, ESAIM: Control, Optimisation and Calculus of Variations 7 (2002) 471–493.

[3] A. Ilchmann, E. P. Ryan, High-gain control without identification: a survey, GAMM Mitt. 31 (1) (2008) 115–125.

[4] A. Ilchmann, S. Trenn, Input constrained funnel control with applications to chemical reactor models, Syst. Control Lett. 53 (5) (2004) 361–375.

[5] C. M. Hackl, Non-identifier Based Adaptive Control in Mechatronics–Theory and Application, Vol. 466 of Lecture Notes in Control and Information Sciences, Springer-Verlag, Cham, Switzerland, 2017.

[6] T. Berger, S. Otto, T. Reis, R. Seifried, Combined open-loop and funnel control for underactuated multibody systems, Nonlinear Dynamics 95 (2019) 1977–1998.

[7] C. M. Hackl, Funnel control for wind turbine systems, in: Proc. 2014 IEEE Int. Conf. Contr. Appl., Antibes, France, 2014, pp. 1377–1382.

[8] C. M. Hackl, Speed funnel control with disturbance observer for wind turbine systems with elastic shaft, in: Proc. 54th IEEE Conf. Decis. Control, Osaka, Japan, 2015, pp. 12005–2012.

[9] C. M. Hackl, Current PI-funnel control with anti-windup for synchronous machines, in: Proc. 54th IEEE Conf. Decis. Control, Osaka, Japan, 2015, pp. 1997–2004.

[10] A. Senfelds, A. Paugurs, Electrical drive DC link power flow control with adaptive approach, in: Proc. 55th Int. Sci. Conf. Power Electr. Engg. Riga Techn. Univ., Riga, Latvia, 2014, pp. 30–33.

[11] T. Berger, T. Reis, Zero dynamics and funnel control for linear electrical circuits, J. Franklin Inst. 351 (11) (2014) 5099–5132.

[12] A. Pomprapa, S. R. Alfocea, C. Göbel, B. J. Misgeld, S. Leonhardt, Funnel control for oxygenation during artificial ventilation therapy, in: Proceedings of the 19th IFAC World Congress, Cape Town, South Africa, 2014, pp. 6575–6580.

[13] A. Pomprapa, S. Weyer, S. Leonhardt, M. Walter, B. Misgeld, Periodic funnel-based control for peak inspiratory pressure, in: Proc. 54th IEEE Conf. Decis. Control, Osaka, Japan, 2015, pp. 5617–5622.

[14] T. Berger, A.-L. Rauert, Funnel cruise control, Automatica 119 (2020), Article 10906.

[15] A. Isidori, Nonlinear Control Systems, 3rd Edition, Communications and Control Engineering Series, Springer-Verlag, Berlin, 1995.

[16] T. Berger, H. H. Lê, T. Reis, Funnel control for nonlinear systems with known strict relative degree, Automatica 87 (2018) 345–357.

[17] T. Berger, M. Puche, F. L. Schwenninger, Funnel control in the presence of infinite-dimensional internal dynamics, Syst. Control Lett. 139 (2020), Article 104678.

[18] T. Berger, M. Puche, F. Schwenninger, Funnel control for a moving water tank, submitted for publication. Available at arXiv: `https://arxiv.org/abs/1902.00586` (2019).

[19] T. Reis, T. Selig, Funnel control for the boundary controlled heat equation, SIAM J. Control Optim. 53 (1) (2015) 547–574.

[20] M. Puche, T. Reis, F. L. Schwenninger, Funnel control for boundary control systems, Evol. Eq. Control Th., to appear (2020), doi: 10.3934/eect.2020079.

[21] T. Berger, Funnel control of the Fokker-Planck equation for a multi-dimensional Ornstein-Uhlenbeck process, submitted for publication. Available at arXiv: `https://arxiv.org/abs/2005.13377v2` (2020).

[22] L. Tung, A bi-domain model for describing ischemic myocardial DC potentials, Ph.D. thesis, Dept. of Electrical Engineering and Computer Science (1978).

[23] J. Sundnes, G. T. Lines, X. Cai, B. F. Nielsen, K.-A. Mardal, A. Tveito, Computing the electrical activity in the heart, Vol. 1 of Monographs in Computational Science and Engineering, Springer-Verlag, Berlin Heidelberg, Germany, 2007.

[24] K. Kunisch, C. Nagaiah, M. Wagner, A parallel Newton-Krylov method for optimal control of the monodomain model in cardiac electrophysiology, Computing and Visualization in Science 14 (2011) 257–269.

[25] C. Nagaiah, K. Kunisch, G. Plank, Optimal control approach to termination of re-entry waves in cardiac electrophysiology, Journal of Mathematical Biology 67 (2013) 359–388.

[26] R. A. Adams, Sobolev Spaces, no. 65 in Pure and Applied Mathematics, Academic Press, New York, London, 1975.

[27] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Birkhäuser, Basel, Switzerland, 1995.

[28] L. Evans, Partial Differential Equations, 2nd Edition, Vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2010.

[29] T. Kato, Perturbation Theory for Linear Operators, 2nd Edition, Springer-Verlag, Berlin Heidelberg, Germany, 1980.

[30] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, Biophysical journal 1 (6) (1961) 445–466.

[31] K. Kunisch, D. A. Souza, On the one-dimensional nonlinear monodomain equations with moving controls, Journal de Mathématiques Pures et Appliquées 117 (2018) 94–122.

[32] A. Yagi, Abstract Parabolic Evolution Equations and their Applications, Springer Monographs in Mathematics, Springer-Verlag, Berlin Heidelberg, Germany, 2010.

[33] A. Ilchmann, Decentralized tracking of interconnected systems, in: K. Hüper, J. Trumpf (Eds.), Mathematical System Theory - Festschrift in Honor of Uwe Helmke on the Occasion of his Sixtieth Birthday, CreateSpace, 2013, pp. 229–245.

[34] H. L. Trentelman, A. A. Stoorvogel, M. L. J. Hautus, Control Theory for Linear Systems, Communications and Control Engineering, Springer-Verlag, London, 2001.

[35] D. E. Jackson, Existence and regularity for the FitzHugh-Nagumo equations with inhomogeneous boundary conditions, Nonlin. Anal. Th. Meth. Appl. 14 (3) (1990) 201–216.

[36] J. L. Lions, Quelques methodes de resolution des problemes aux limits non lineaires, Dunod Gauthier-Villars, France, 1969.

[37] W. Walter, Ordinary Differential Equations, Springer-Verlag, New York, 1998.

[38] M. Tucsnak, G. Weiss, Observation and Control for Operator Semigroups, Birkhäuser Advanced Texts Basler Lehrbücher, Birkhäuser, Basel, Switzerland, 2009.

[39] M. Hinze, R. Pinnau, M. Ulbrich, S. Ulbrich, Optimization with PDE Constraints, Vol. 23 of Mathematical Modelling: Theory and Applications, Springer-Verlag, The Netherlands, 2009.

[40] J. Diestel, J. Uhl, Vector Measures, Vol. 15 of Mathematical surveys and monographs, American Mathematical Society, Providence, RI, 1977.

[41] G. F. Simmons, Introduction to topology and modern analysis, McGraw-Hill, New York, 1963.

[42] T. Breiten, K. Kunisch, Compensator design for the monodomain equations with the FitzHugh-Nagumo model, ESAIM: Control, Optimisation and Calculus of Variations 23 (2017) 241–262.

[43] R. Nittka, Regularity of solutions of linear second order elliptic and parabolic boundary value problems on Lipschitz domains, J. Diff. Eqns. 251 (4-5) (2011) 860—880.

[44] A. Lunardi, Interpolation Theory, no. 16 in Lecture Notes (Scuola Normale Superiore), Edizioni della Normale, Pisa, Italy, 2018.

[45] W. Arendt, A. ert Elst, From forms to semigroups, in: W. Arendt, J. A. Ball, J. Behrndt, K.-H. Förster, V. Mehrmann, C. Trunk (Eds.), Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations, Vol. 221 of Operator Theory: Advances and Applications, Birkhäuser, Basel, Switzerland, 2012, pp. 47–69.

[46] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.

[47] O. Staffans, Well-Posed Linear Systems, Vol. 103 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2005.

[48] B. Jacob, R. Nabiullin, J. R. Partington, F. L. Schwenninger, Infinite-dimensional input-to-state stability and Orlicz spaces, SIAM J. Control Optim. 56 (2) (2018) 868–889.

[49] W. Rudin, Functional Analysis, 2nd Edition, McGraw-Hill, New York, 1991.

[50] L. Tartar, An Introduction to Sobolev Spaces and Interpolation Spaces, Springer-Verlag, Berlin, Heidelberg, New York, 2007.

[51] K. Yosida, Functional Analysis, 6th Edition, Springer-Verlag, Berlin, Germany, 1980.

[52] E. F. Keller, L. A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (3) (1970) 399–415.