

Prescribed performance control of uncertain higher-order nonlinear systems in the presence of delays

Thomas Berger, Lampros N. Bikas, Jan Hachmeister, and George A. Rovithakis

Abstract—We propose a novel feedback controller for a class of uncertain higher-order nonlinear systems, subject to delays in both state measurement and control input signals. Building on the prescribed performance control framework, a delay-dependent performance correction mechanism is introduced to ensure the boundedness of all signals in the closed-loop and to keep the output tracking error strictly within a dynamically adjusted performance envelope. This mechanism adapts in response to large delays that may cause performance degradation. In the absence of delays, the correction term vanishes, and the controller recovers the nominal (user-defined) performance envelope. The effectiveness of the proposed approach is validated through simulation studies.

Index Terms—Delays, prescribed performance control, uncertain nonlinear systems.

I. INTRODUCTION

In recent years, significant research efforts have been devoted to the design of controllers for linear and nonlinear systems. Early works often assumed ideal communication between the controller and system, including measurement and control input signals. However, from both theoretical and practical perspectives, delays are inherent in most control systems and can significantly affect performance and stability [1]. Therefore, designing robust controllers that account for communication delays is of paramount importance. It is especially crucial to consider both measurement and control input delays, as these are common in practical applications, and recent progress has been reported in this direction [2]–[11].

A common focus of these studies has been on establishing stability conditions, often without explicit consideration of performance objectives. In trajectory tracking problems for uncertain nonlinear systems, beyond stability, it is essential to ensure strict bounds on the output tracking error, encompassing transient and steady-state behaviors. Ideally, these bounds should be prescribed and user-defined.

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Thomas Berger and Jan Hachmeister are with the Universität Paderborn, Institut für Mathematik, Warburger Str. 100, 33098 Paderborn, Germany (e-mail: thomas.berger@math.upb.de, janha@mail.uni-paderborn.de).

Lampros N. Bikas and George A. Rovithakis are with the Department of Electrical and Computer Engineering, Aristotle University of Thessaloniki, 54124 Thessaloniki, Greece (e-mail: lnmpikas@ece.auth.gr, rovithak@ece.auth.gr).

Two prominent methodologies addressing this are funnel control (FC) and prescribed performance control (PPC). FC was introduced in [12], with a recent survey provided in [13]. PPC was initially presented in [14] with further developments in [15]. When assuming delay-free communication, both methods can guarantee prescribed performance characteristics—such as maximum overshoot, convergence rate, and steady-state error.

In the presence of measurement and control input delays, approaches such as the bang-bang controller based on FC were proposed in [16] where constant delays were considered. However, these methods often involve increased complexity and require high-order derivatives of reference signals. Under the PPC framework, an initial attempt to address communication delays was made in [17], where the control design aimed to satisfy prescribed performance objectives. However, the boundedness of all closed-loop signals is only assured for first-order systems under the proposed control scheme.

Extending prescribed performance guarantees under delays to uncertain nonlinear systems of arbitrary order remains a challenging open problem. Motivated by this gap, in this paper we propose a modified control scheme based on [17], enhanced with a delay-dependent performance correction term. This controller enforces prescribed performance for a class of uncertain higher-order nonlinear systems under constant measurement delays and time-varying control input delays. Our approach guarantees the boundedness of all signals in the closed-loop and ensures that the output tracking error evolves strictly within a dynamically adjusted performance envelope, thereby overcoming the limitations of [17]. The correction term explicitly depends on the delay magnitude and vanishes in the delay-free case, thereby recovering the nominal prescribed performance. The proposed controller is of low-complexity, maintaining simplicity in implementation.

The remainder of this paper is organized as follows: Section II states the problem, Section III details the proposed controller, and Section IV presents the main results. Simulation studies validating the approach are provided in Section V, and conclusions are drawn in Section VI. The proof of the main theorem is included in the Appendix.

A. Notation

In the following let \mathbb{N} denote the natural numbers, and $\mathbb{R}_{\geq \tau} = [\tau, \infty)$ for $\tau \in \mathbb{R}$. By $\|x\|$ we denote the Euclidean

norm of $x \in \mathbb{R}^n$. For some interval $I \subseteq \mathbb{R}$, some $V \subseteq \mathbb{R}^m$ and $k \in \mathbb{N}$, $\mathcal{L}^\infty(I, \mathbb{R}^n)$ ($\mathcal{L}_{\text{loc}}^\infty(I, \mathbb{R}^n)$) is the Lebesgue space of measurable, (locally) essentially bounded functions $f: I \rightarrow \mathbb{R}^n$, $\mathcal{W}^{k,\infty}(I, \mathbb{R}^n)$ is the Sobolev space of all functions $f: I \rightarrow \mathbb{R}^n$ with k -th order weak derivative $f^{(k)}$ and $f, f^{(1)}, \dots, f^{(k)} \in L^\infty(I, \mathbb{R}^n)$, and $\mathcal{C}^k(V, \mathbb{R}^n)$ is the set of k -times continuously differentiable functions $f: V \rightarrow \mathbb{R}^n$, with $\mathcal{C}(V, \mathbb{R}^n) := \mathcal{C}^0(V, \mathbb{R}^n)$. Furthermore, \mathcal{K}_∞ denotes the set of continuous, strictly increasing and unbounded functions, and \mathcal{KL} is the set of continuous functions, strictly decreasing with limit zero in the first argument and strictly increasing in the second argument.

II. PROBLEM STATEMENT

A. System class

We consider nonlinear multi-input, multi-output systems of n -th order of the form

$$\begin{aligned} \dot{x}_{i,j}(t) &= f_{i,j}(t, \bar{x}_j(t)) + x_{i,j+1}(t), \\ \dot{x}_{i,n}(t) &= f_{i,n}(t, \bar{x}_n(t), \eta(t)) \\ &\quad + \sum_{k=1}^m g_{i,k}(t, \bar{x}_n(t), \eta(t)) u_k(t - \tau_u(t)), \quad (1) \\ \dot{\eta}(t) &= h(t, \bar{x}_n(t), \eta(t)), \\ j &= 1, \dots, n-1, \quad i = 1, \dots, m, \end{aligned}$$

where $\bar{x}_j = (x_{1,1}, \dots, x_{1,j}, \dots, x_{m,1}, \dots, x_{m,j})^\top$, $j = 1, \dots, n$, is the part of the state available for measurement, $\eta: \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^q$ is the state of the internal dynamics, $u_i: [-\tau_s - \bar{\tau}_u, \infty) \rightarrow \mathbb{R}$, $i = 1, \dots, m$, are the control inputs and $y_i := x_{i,1}$, $i = 1, \dots, m$, are the system outputs. The functions $h, f_{i,j}, g_{i,k}$, $i, k = 1, \dots, m$, are assumed to be piecewise continuous and bounded in t and locally Lipschitz in (\bar{x}_n, η) for $j = n$, or locally Lipschitz in \bar{x}_j for $j = 1, \dots, n-1$, respectively.

We consider the problem of output tracking control with the objective of achieving a prescribed performance of the tracking error, under the effect of time-varying input delays (described by $\tau_u(t) \geq 0$) and constant state measurement delays (described by $\tau_s \geq 0$), see Fig. 1. The latter means that only the delayed information $x_{i,j}(t - \tau_s)$ of the state measurement is available for controller design. Because of this delay, in order for the problem to be well-posed, an initial history of the state is required on the interval $[-\tau_s - \bar{\tau}_u, 0]$, where $\bar{\tau}_u \geq \tau_u(t)$ for all $t \geq 0$, i.e.,

$$(\bar{x}_n, \eta)|_{[-\tau_s - \bar{\tau}_u, 0]} = \varphi = (\bar{x}_n^\varphi, \eta^\varphi) \in \mathcal{C}([- \tau_s - \bar{\tau}_u, 0], \mathbb{R}^{nm+q}). \quad (2)$$

For $u_i \in \mathcal{L}_{\text{loc}}^\infty([- \tau_s - \bar{\tau}_u, \infty), \mathbb{R})$, $i = 1, \dots, m$, we call $(\bar{x}_n, \eta): [- \tau_s - \bar{\tau}_u, \omega) \rightarrow \mathbb{R}^{nm+q}$, $\omega \in (0, \infty]$, a solution of (1), (2), if it is locally absolutely continuous and satisfies (1) for almost all $t \in [0, \omega)$. A solution is called maximal, if it has no right extension that is also a solution; it is global, if $\omega = \infty$.

We need the following additional assumption on the nonlinearities $f_{i,n}$ and h .

Assumption 1: For each $i = 1, \dots, m$ there exists $d_i \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ such that

$$\begin{aligned} \forall (t, x, \eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{nm+q}: \\ |f_{i,n}(t, x, \eta)| \leq |d_i(t)|(\|\bar{x}_n\| + \|\eta\| + 1). \end{aligned}$$

Furthermore, the last equation in (1) is practically input-to-state stable in the sense that there exists a constant $c \geq 0$, a \mathcal{KL} -function κ and a \mathcal{K}_∞ -function γ such that, for any $\xi \in \mathcal{L}_{\text{loc}}^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^{nm})$ and any $\eta^0 \in \mathbb{R}^q$ the solution of $\dot{\eta}(t) = h(t, \xi(t), \eta(t))$, $\eta(0) = \eta^0$, is global and satisfies

$$\forall t \geq 0: \|\eta(t)\| \leq \kappa(t, \|\eta^0\|) + \gamma(\sup_{s \in [0, t]} \|\xi(s)\|) + c.$$

Additionally, the function γ is linearly bounded, i.e., $\gamma(x) \leq \bar{\gamma}_1 x + \bar{\gamma}_2$ for all $x \geq 0$ and some $\bar{\gamma}_1, \bar{\gamma}_2 > 0$.

Remark 1: We note that the Lipschitz assumption on the nonlinearities $f_{i,n}$ cannot be waived in general. For instance, consider the system

$$\dot{x}(t) = x(t)^2 + u(t - \tau), \quad x|_{[-\tau, 0]} \equiv x^0 > 0.$$

Since the only reasonable choice for the input for $t \in [-\tau, 0]$ is $u(t) = 0$, as the system is not “active” yet, this leads to the initial-value problem

$$\dot{x}(t) = x(t)^2, \quad x(0) = x^0, \quad t \in [0, \tau],$$

the solution of which is given by

$$x(t) = \left(\frac{1}{x^0} - t \right)^{-1}, \quad t \in [0, \min\{\tau, 1/x^0\}).$$

If $\frac{1}{x^0} < \tau$, then this leads to a blow-up of the solutions, hence existence of global solutions of the closed-loop system cannot be guaranteed by any control algorithm.

Assumption 2: The delays $\tau_u \in \mathcal{C}^1(\mathbb{R}_{\geq 0}, \mathbb{R})$ and $\tau_s \geq 0$ are assumed to be known exactly and bounded such that $\bar{\tau}_u \geq \tau_u(t)$ for all $t \geq 0$. Furthermore, we assume that there exists $\dot{\tau}_u < 1$ such that $\dot{\tau}_u(t) \leq \dot{\tau}_u$ for all $t \geq 0$.

The strict boundedness requirement on $\dot{\tau}_u(t)$ is necessary to guarantee the satisfaction of the first-in first-out principle [17]. The required knowledge of the delays $\tau_s(\cdot)$ and $\tau_u(\cdot)$ might seem to be a strong assumption, but in many applications they can indeed be estimated very well, or, by intentionally delaying some measurements, a prescribed quantity can be achieved.

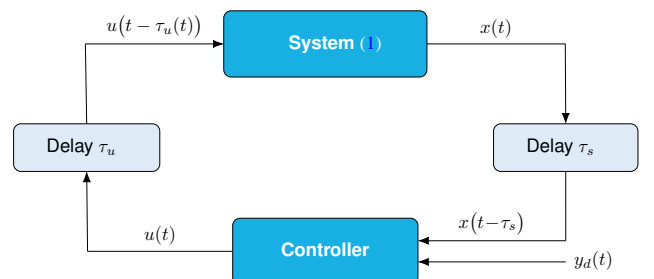


Fig. 1. Structure of the closed-loop system.

B. Control objective

The objective is, for given functions $\psi_{i,1} \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq -\tau_s}, \mathbb{R})$ which are bounded and satisfy $\inf_{t \geq -\tau_s} \psi_{i,1}(t) > 0$, and reference signals $y_{d,i} \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq -\tau_s}, \mathbb{R})$, $i = 1, \dots, m$, to design a controller which achieves that

$$\forall i = 1, \dots, m \quad \forall t \geq 0 : |y_i(t) - y_{d,i}(t)| < \psi_{i,1}(t).$$

Furthermore, all closed-loop signals should remain bounded. The controller should not require any knowledge of the system parameters (initial history φ , nonlinearities $f_{i,j}, g_{i,k}, h$) and should be of low complexity (no approximation or adaptive structures are used to obtain that knowledge, no hard calculations are performed to create the control signal). A controller which satisfies these requirements is inherently robust with respect to uncertainties or disturbances (with the exception of measurement noise).

Remark 2: The performance functions $\psi_{i,1}$, $i = 1, \dots, m$, are user-defined and can be appropriately constructed to introduce performance bounds on the output tracking error with respect to transient and steady-state behavior. A candidate selection is the exponentially decreasing $\psi_{i,1}(t) = (\lambda_{i,1}^0 - \lambda_{i,1}^\infty)e^{-c_{i,1}t} + \lambda_{i,1}^\infty$ with $\lambda_{i,1}^0 > |y_i(0) - y_{d,i}(0)|$. The constants $\lambda_{i,1}^\infty > 0$ and $c_{i,1} > 0$ are selected to prescribe the maximum output tracking error at steady-state and the minimum convergence rate, respectively.

Remark 3: The control objective, as formulated above, cannot be achieved by any control algorithm, when the control input is “switched on” at a certain time (usually $t = 0$), and zero before (that is, $u(t) = 0$ for $t \leq 0$). Nevertheless, the new design that we propose in Section III achieves the prescribed performance objective with an additional correction term, the magnitude of which depends on the magnitude of the delays (and vanishes for zero delay). Similar control objectives were considered in [17], however, the proposed control scheme cannot guarantee the boundedness of all closed-loop signals for the general case of systems of arbitrary order. The present work overcomes this limitation by introducing a modified control design.

III. CONTROLLER DESIGN

In this Section we propose a control scheme to achieve the control objectives stated in Section II. The control design philosophy is based on the line of analysis of PPC structure and incorporates a delay-dependent transformation of the output tracking error, resulting in a closed-loop system that includes at least one delay-free control input.

Let $\psi_{i,1} \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq -\tau_s}, \mathbb{R})$, $i = 1, \dots, m$, be given and select the auxiliary functions $\psi_{i,j} \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq -\tau_s}, \mathbb{R})$ such that $\inf_{t \geq 0} \psi_{i,j}(t) > -\tau_s$, for $i = 1, \dots, m$ and $j = 1, \dots, n$. Further, choose the controller design parameters:

- a matrix $S = (s_{i,j}) \in \mathbb{R}^{m \times m}$, which is sign definite, i.e., there exist $s^* > 0$ and $\sigma \in \{-1, +1\}$ such that $\sigma v^\top S v \geq s^* \|v\|^2$ for all $v \in \mathbb{R}^m$,
- some freely selected control gains $\alpha > 0$ and $k_{i,j}, k_n > 0$, $i = 1, \dots, m$, $j = 1, \dots, n - 1$.

The following expressions define the controller we propose for $i = 1, \dots, m$, and $t \geq 0$:

$$\begin{aligned} z_{i,1}(t) &= \frac{x_{i,1}(t - \tau_s) - y_{d,i}(t - \tau_s) + I_{i,1}(t)}{\psi_{i,1}(t - \tau_s)}, \\ z_{i,j}(t) &= \frac{x_{i,j}(t - \tau_s) - a_{i,j-1}(t) + \sum_{k=1}^j \binom{j-1}{j-k} (-\alpha)^{j-k} I_{i,k}(t)}{\psi_{i,j}(t - \tau_s)}, \\ &\quad j = 2, \dots, n, \\ a_{i,j}(t) &= -k_{i,j} \frac{z_{i,j}(t)}{1 - z_{i,j}(t)^2}, \quad j = 1, \dots, n - 1 \\ z_n(t) &= (z_{1,n}(t), \dots, z_{m,n}(t))^\top, \\ u_i(t) &= -\sigma k_n \chi(\|z_n(t)\|) \frac{z_{i,n}(t)}{1 - \|z_n(t)\|^2}, \end{aligned} \quad (3)$$

with

$$\begin{aligned} \dot{I}_{i,j}(t) &= I_{i,j+1}(t) - \alpha I_{i,j}(t), \quad I_{i,j}(0) = 0, \\ &\quad j = 1, \dots, n - 1, \\ \dot{I}_{i,n}(t) &= -\alpha I_{i,n}(t) \\ &\quad + \sum_{k=1}^m s_{i,k} (u_k(t) - u_k(t - \tau_s - \tau_u(t - \tau_s))), \\ I_{i,n}(0) &= 0, \end{aligned} \quad (4)$$

and $\chi : [0, 1] \rightarrow [0, 1 - \delta]$, $\delta \in [0, 1)$, an activation function of the form:

$$\chi(s) = \begin{cases} 0, & s \leq \delta, \\ s - \delta, & s > \delta. \end{cases} \quad (5)$$

Furthermore, we assume that $u_i(t) = 0$ for $t \in [-\tau_s - \bar{\tau}_u, 0]$ and, for simplicity,

$$\forall t \geq -\tau_s : \psi_{1,n}(t) = \dots = \psi_{m,n}(t) =: \psi_n(t). \quad (6)$$

We like to note that the terms $I_{i,j}$ serve as *correction terms* of the performance functions $\psi_{i,j}$, which are prescribed for $x_{i,1} - y_{d,i}$ and $x_{i,j} - a_{i,j-1}$, $j = 2, \dots, n$, respectively. Owing to the input and state measurement delays, the desired performance $|x_{i,1}(t) - y_{d,i}(t)| < \psi_{i,1}(t)$ and $|x_{i,j}(t) - a_{i,j-1}(t + \tau_s)| < \psi_{i,j}(t)$ cannot be achieved, but utilizing the correction terms we are able to achieve $|z_{i,j}(t)| < 1$ for $i = 1, \dots, m$, $j = 1, \dots, n$. If no delays are present, i.e., $\tau_s \equiv \tau_u \equiv 0$, then by construction we have $I_{i,j}(t) = 0$ for all $i = 1, \dots, m$, $j = 1, \dots, n$ and $t \geq 0$, thus no correction occurs. Compared to [17], the correction terms are the crucial modification of the controller design, which facilitates its feasibility.

Remark 4: By incorporating the activation function χ to the control input, we enforce u_i to be zero when z_n is small (i.e., $\|z_n\| < \delta$). In such case the $I_{i,j}$ -terms will converge to zero, thus recovering the original shape of the performance envelope. This modification allows the performance envelope to be adjusted only when the error evolves sufficiently close to the boundaries; a beneficial property in practice as one cannot exclude the case where despite the presence of a large delay the error evolves close to zero. In the context of funnel control and funnel MPC, activation functions have been employed in [18], [19].

IV. MAIN RESULTS

Before we state the main result of this article we require an additional assumption for feasibility of the control. Essentially, the assumption states that the delays τ_s and $\tau_u(t)$ need to be sufficiently small and, at the same time, S needs to be a sufficiently good estimate of the control input matrix $G(t, x, \eta) = (g_{i,j}(t, x, \eta)) \in \mathbb{R}^{m \times m}$ for $(t, x, \eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{nm+q}$.

Assumption 3: Assume that

$$\sup_{(t,x,\eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{nm+q}} \|G(t, x, \eta) - S\| + C < s^* \frac{\inf_{t \geq 0} \psi_n(t)}{\sup_{t \geq 0} \psi_n(t)},$$

where $C \geq 0$ is defined as

$$C := \frac{\tilde{c} \|S\| M}{\mu} \left(\frac{\dot{\tau}_u}{1 - \dot{\tau}_u} + \frac{2 - \dot{\tau}_u}{1 - \dot{\tau}_u} \|A\| (\tau_s + \bar{\tau}_u) e^{\|A\|(\tau_s + \bar{\tau}_u)} \right)$$

with $A = \text{diag}(\tilde{A}, \dots, \tilde{A}) \in \mathbb{R}^{nm \times nm}$,

$$\tilde{A} = \begin{bmatrix} -\alpha & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -\alpha \end{bmatrix} \in \mathbb{R}^{n \times n},$$

$M, \mu > 0$ such that (note that $\mu < \alpha$ for $n > 1$)

$$\forall t \geq 0 : \|e^{At}\| \leq M e^{-\mu t} \quad (7)$$

and

$$\tilde{c} := \max\{\alpha^{n-1}, 1\} \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right) n \cdot \max_{i=1, \dots, m} \|d_i\|_\infty + \max\{\alpha^n, 1\} \left(\frac{n}{\lfloor \frac{n}{2} \rfloor}\right).$$

Remark 5: We like to note that Assumption 3 is indeed quite restrictive and requires a significant amount of knowledge of the control input matrix $G(\cdot)$, only up to slight uncertainties. Furthermore, since S is a constant matrix, $G(\cdot)$ is restricted to be globally bounded. In future research, we aim to relax these assumptions.

The following result shows feasibility of the application of the controller (3), (4) to the system (1) in the presence of input and state measurement delays.

Theorem 1: Consider the system (1) satisfying Assumptions 1 and 2. Further consider the controller (3), (4) with reference signals $y_{d,i} \in \mathcal{W}^{1,\infty}(\mathbb{R}_{\geq -\tau_s}, \mathbb{R})$, $i = 1, \dots, m$, and design parameters satisfying Assumption 3. Let φ as in (2) be an initial history such that all signals in (3), (4) are well-defined for $t \in [-\bar{\tau}_u, \tau_s]$ (with $I_{i,j}(t) = 0$ for $t \leq 0$) and, in particular, $|z_{i,j}(t)| < 1$ for $i = 1, \dots, m$, $j = 1, \dots, n-1$ and $\|z_n(t)\| < 1$. Then the application of the controller (3), (4) to the system (1) leads to a closed-loop system, which has a solution and every solution can be extended to a maximal solution $(\bar{x}_n, \eta, I_{1,1}, \dots, I_{m,n}) : [-\tau_s - \bar{\tau}_u, \omega) \rightarrow \mathbb{R}^{2mn+q}$, $\omega \in (0, \infty]$, with the properties:

- (i) global existence: $\omega = \infty$;
- (ii) all closed-loop signals $x_{i,j}, I_{i,j}, \eta$, $i = 1, \dots, m$, $j = 1, \dots, n$, and the control input u are bounded;
- (iii) the system output exhibits a prescribed performance in the sense that, for all $t \geq 0$ and $i = 1, \dots, m$,

$$|y_i(t - \tau_s) - y_{d,i}(t - \tau_s) + I_{i,1}(t)| < \psi_{i,1}(t - \tau_s)$$

if $n > 1$, and if $n = 1$, then we have that

$$\sum_{i=1}^m (y_i(t - \tau_s) - y_{d,i}(t - \tau_s) + I_{i,1}(t))^2 < \psi_1(t - \tau_s)^2.$$

The proof of Theorem 1 is relegated to the Appendix.

V. SIMULATIONS

To illustrate the applicability of our controller we consider a mass-spring system mounted on a moving car, see Fig. 2. This system was originally presented in [20] and has been used to demonstrate the application of a funnel controller in a delay-free setting in [21].

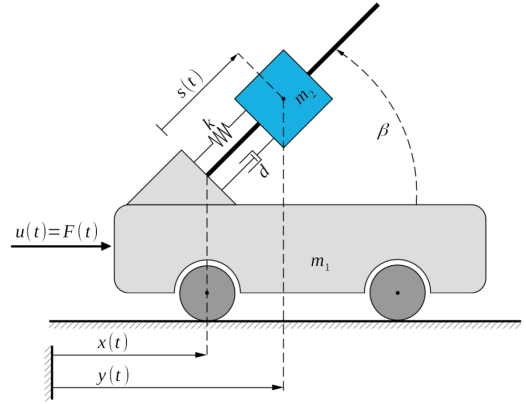


Fig. 2. Mass on car system

The car with mass m_1 (in kg) moves horizontally and is actuated by input force $u(t) = F(t)$ (in N). On the car, the mass m_2 (in kg) is mounted via a spring-damper combination and moves along an axis inclined by the angle β (in rad). The equations of motion are

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos \beta \\ m_2 \cos \beta & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} + \begin{pmatrix} 0 \\ ks + d\dot{s} \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix} \quad (8)$$

where $x(t)$ is the horizontal position of the car and $s(t)$ is the relative position of mass m_1 on the car along the inclined axis. The parameters c (in N/m) and d (in Ns/m) denote the spring and damper coefficients, respectively. The system's output is the horizontal position of mass m_1 ,

$$y(t) = x(t) + s(t) \cos \beta.$$

As we can only control the force acting on the car and not m_1 itself, the system possesses internal dynamics. For the simulations we choose the same parameters as in [21], i.e., $m_1 = 4$, $m_2 = 1$, $k = 2$ and $d = 1$. The initial history of the system for $t \in [-\tau_s - \bar{\tau}_u, 0)$ is given by

$$x(t) = s(t) \equiv 0 \quad \text{and} \quad \dot{x}(t) = \dot{s}(t) \equiv 0.$$

The goal is to control the input $u(t)$ so that the output $y(t)$ tracks the reference trajectory

$$y_d(t) = \cos t.$$

Both the control input and the state measurement are subject to a constant delay $\tau_s = \tau_u = 0.05$ [s]. All simulations are

MATLAB generated (solver: dde23, rel. tol.: 10^{-8} , abs. tol.: 10^{-8}) and over the time interval $[0, 10]$.

Case 1 ($\beta = \frac{\pi}{4}$): In this case, the system is of the form (1) with $n = q = 2$ and $m = 1$ as well as $G(t, x, \eta) = \frac{1}{9}$ as shown in [13]. We choose the controller design parameters $S = \frac{1}{9}$, $k_{1,1} = k_2 = 1$ and $\alpha = 1$. The activation function is chosen as $\chi(s) = s$ with $\delta = 0$. For the performance functions we choose

$$\psi_{1,1}(t) = 5e^{-2t} + 0.1 \quad \text{and} \quad \psi_{1,2}(t) = 10e^{-2t} + 0.5.$$

The application of the controller (3), (4) to (8) is depicted in Fig. 3, showing the tracking error and funnel boundary in the top image, the output and reference signal in the middle and the input function in the bottom image. We see that apart from the time frame $t \in [1.8, 2.1]$ the error $y(t - \tau_s) - y_d(t - \tau_s)$ remains in the original funnel $\pm\psi_{1,1}(t - \tau_s)$ even without the correction term $I_{1,1}(t)$.

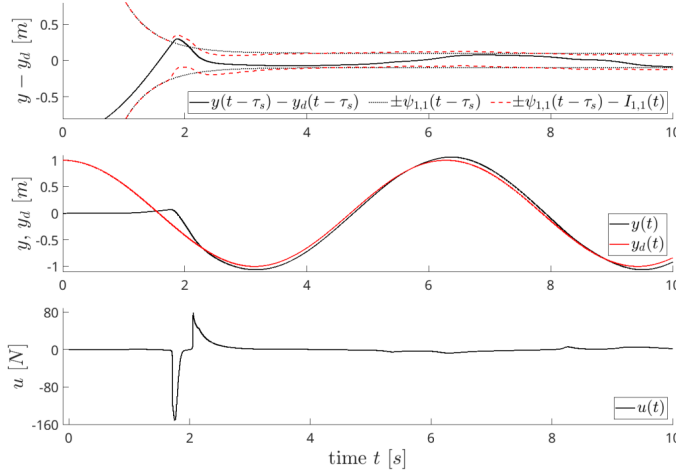


Fig. 3. Simulation of system (8) with delays $\tau_s = \tau_u = 0.05$ under controller (3), (4) for $\beta = \frac{\pi}{4}$.

As mentioned before, the same system has already been considered in a delay-free setting in [21]. However, when using the controller therein and applying a state measurement delay τ_s as little as a 0.005 [s] and no control input delay τ_u , the closed-loop system becomes unstable and the simulation fails at $t \approx 2.5$, as the controller is not able to keep the error between the funnel boundaries.

Case 2 ($\beta = 0$): For $\beta = 0$, the system is of the form (1) with $n = 3$ and $q = m = 1$ as well as $G(t, x, \eta) = \frac{1}{4}$ as shown in [13]. We choose parameters $\frac{1}{4}$, $k_{1,1} = k_{1,2} = 10$, $k_3 = 3000$, $\alpha = 1$ and performance functions

$$\psi_{1,1}(t) = 5e^{-2t} + 0.1$$

and

$$\psi_{1,2}(t) = \psi_{1,3}(t) = 10e^{-2t} + 0.5.$$

Again, the activation function is chosen as $\chi(s) = s$ with $\delta = 0$. The simulations we performed can be found in Fig. 4. We see that in this case the output stays between the original funnel boundaries for all $t \geq 0$. The higher system order and the larger gain constants result in a larger control effort.

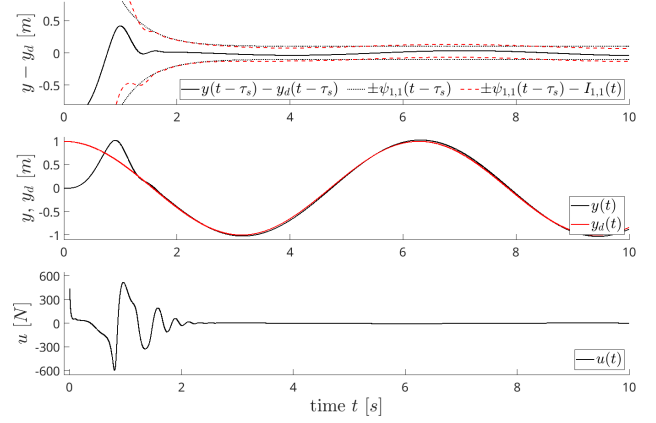


Fig. 4. Simulation of system (8) with delays $\tau_s = \tau_u = 0.05$ under controller (3), (4) for $\beta = 0$.

VI. CONCLUSION

A novel control approach is presented for a class of uncertain nonlinear systems of arbitrary order, addressing the challenges of state and control input delays. Extending the PPC framework, we introduce a delay-dependent correction term that actively compensates for communication delays. This results in guaranteed, dynamically adjusted output tracking performance. A key advantage of our controller is its adaptability: it automatically recovers nominal performance when delays are minimal, eliminating the need for re-tuning. The effectiveness of this approach is validated through simulations.

APPENDIX

PROOF OF THEOREM 1

The proof of the theorem consists of four phases. In Phase 1 we derive the differential equations of the transformed closed-loop system and in Phase 2 we guarantee the existence of solution in a maximal time interval. Further, in Phase 3 we prove that the solution evolves strictly within the prescribed performance envelope during this interval. This enables us to show that the solution is global and all signals in the closed-loop system are bounded in Phase 4.

Phase 1: First, let $\delta_I > 0$ be a constant, chosen large enough with lower bounds to be specified later in the proof. Define the set $\Omega_I := (-\delta_I, \delta_I)$. We derive differential equations for $z_{i,j}(t)$, $i = 1, \dots, m$, $j = 1, \dots, n$ for $t \geq \tau_s$. To compute the derivative of $z_{i,1}$ we utilize the equation

$$\begin{aligned} \dot{x}_{i,1}(t - \tau_s) &= f_{i,1}(t, \bar{x}_1(t - \tau_s)) + x_{i,2}(t - \tau_s) \\ &= f_{i,1}(t, \bar{x}_1(t - \tau_s)) + z_{i,2}(t)\psi_{i,2}(t - \tau_s) + a_{i,1}(t) \\ &\quad + \alpha I_{i,1}(t) - I_{i,2}(t), \end{aligned}$$

thus obtaining

$$\begin{aligned}\dot{z}_{i,1}(t) &= \frac{1}{\psi_{i,1}(t-\tau_s)} \left(f_{i,1}(t, \bar{x}_1(t-\tau_s)) + z_{i,2}(t) \psi_{i,2}(t-\tau_s) \right. \\ &\quad \left. + a_{i,1}(t) + \alpha I_{i,1}(t) - I_{i,2}(t) - \dot{y}_{d,i}(t-\tau_s) \right. \\ &\quad \left. - \alpha I_{i,1}(t) + I_{i,2}(t) - \dot{\psi}_{i,1}(t-\tau_s) z_{i,1}(t) \right) \\ &= \frac{1}{\psi_{i,1}(t-\tau_s)} \left(f_{i,1}(t, \bar{x}_1(t-\tau_s)) + z_{i,2}(t) \psi_{i,2}(t-\tau_s) \right. \\ &\quad \left. - \frac{k_{i,1} z_{i,1}(t)}{1 - z_{i,1}(t)^2} - \dot{y}_{d,i}(t-\tau_s) - \dot{\psi}_{i,1}(t-\tau_s) z_{i,1}(t) \right) \\ &=: h_{i,1}(t, z_{1,1}(t), \dots, z_{m,1}(t), z_{i,2}(t), I_{i,1}(t), \dots, I_{m,1}(t)),\end{aligned}$$

where we used that we can express $\bar{x}_1(t-\tau_s)$ (and hence the dependency of $f_{i,1}$ on it) in terms of

- $\psi_{1,1}(t-\tau_s) z_{1,1}(t), \dots, \psi_{m,1}(t-\tau_s) z_{m,1}(t),$
- $y_{d,1}(t-\tau_s), \dots, y_{d,m}(t-\tau_s)$ and
- $I_{1,1}(t), \dots, I_{m,1}(t),$

hence $h_{i,1} : \mathbb{R}_{\geq \tau_s} \times (-1, 1)^{m+1} \times \Omega_I^m \rightarrow \mathbb{R}$ is well-defined and continuous. For $j = 2, \dots, n-1$ we show in a similar way that

$$\begin{aligned}\dot{z}_{i,j}(t) &= \frac{1}{\psi_{i,j}(t-\tau_s)} \left(f_{i,j}(t, \bar{x}_j(t-\tau_s)) + x_{i,j+1}(t-\tau_s) \right. \\ &\quad \left. - \dot{a}_{i,j-1}(t) - \dot{\psi}_{i,j}(t-\tau_s) z_{i,j}(t) \right. \\ &\quad \left. + \sum_{k=1}^j \binom{j-1}{j-k} (-\alpha)^{j-k} (-\alpha I_{i,k}(t) + I_{i,k+1}(t)) \right) \\ &= \frac{1}{\psi_{i,j}(t-\tau_s)} \left(z_{i,j+1}(t) \psi_{i,j+1}(t-\tau_s) - \frac{k_{i,j} z_{i,j}(t)}{1 - z_{i,j}(t)^2} \right. \\ &\quad \left. - \dot{\psi}_{i,j}(t-\tau_s) z_{i,j}(t) - \sum_{k=1}^{j+1} \binom{j}{j+1-k} (-\alpha)^{j+1-k} I_{i,k}(t) \right. \\ &\quad \left. + \sum_{k=1}^j \binom{j-1}{j-k} ((-\alpha)^{j+1-k} I_{i,k}(t) + (-\alpha)^{j-k} I_{i,k+1}(t)) \right. \\ &\quad \left. + f_{i,j}(t, \bar{x}_j(t-\tau_s)) - \dot{a}_{i,j-1}(t) \right) \\ &= \frac{1}{\psi_{i,j}(t-\tau_s)} \left(z_{i,j+1}(t) \psi_{i,j+1}(t-\tau_s) - \frac{k_{i,j} z_{i,j}(t)}{1 - z_{i,j}(t)^2} \right. \\ &\quad \left. + f_{i,j}(t, \bar{x}_j(t-\tau_s)) - \dot{\psi}_{i,j}(t-\tau_s) z_{i,j}(t) - \dot{a}_{i,j-1}(t) \right) \\ &=: h_{i,j}(t, z_{1,1}(t), \dots, z_{m,1}(t), \dots, z_{1,j}(t), \dots, z_{m,j}(t), \\ &\quad z_{i,j+1}(t), I_{1,1}(t), \dots, I_{m,1}(t), \dots, I_{1,j}(t), \dots, I_{m,j}(t))\end{aligned}$$

where we used that

$$\begin{aligned}&\sum_{k=1}^j \binom{j-1}{j-k} ((-\alpha)^{j+1-k} I_{i,k}(t) + (-\alpha)^{j-k} I_{i,k+1}(t)) \\ &= \sum_{k=1}^j \binom{j-1}{j-k} (-\alpha)^{j+1-k} I_{i,k}(t) + \sum_{k=2}^{j+1} \binom{j-1}{j+1-k} (-\alpha)^{j+1-k} I_{i,k}(t) \\ &= \sum_{k=1}^{j+1} \binom{j}{j+1-k} (-\alpha)^{j+1-k} I_{i,k}(t).\end{aligned}$$

Furthermore, we can express $\bar{x}_j(t-\tau_s)$ (and hence the dependency of $f_{i,j}$ on it) in terms of

- $\psi_{1,1}(t-\tau_s) z_{1,1}(t), \dots, \psi_{m,j}(t-\tau_s) z_{m,j}(t),$
- $y_{d,1}(t-\tau_s), \dots, y_{d,m}(t-\tau_s),$
- $a_{1,1}(t), \dots, a_{m,j-1}(t)$ and
- $I_{1,1}(t), \dots, I_{m,j}(t),$

thus $h_{i,j} : \mathbb{R}_{\geq \tau_s} \times (-1, 1)^{mj+1} \times \Omega_I^{mj} \rightarrow \mathbb{R}$ is well-defined and continuous. For $j = n$ we obtain

$$\begin{aligned}\dot{z}_{i,n}(t) &= \frac{1}{\psi_{i,n}(t-\tau_s)} \left(f_{i,n}(t-\tau_s, \bar{x}_n(t-\tau_s), \eta(t-\tau_s)) \right. \\ &\quad \left. + \sum_{k=1}^m g_{i,k}(t-\tau_s, \bar{x}_n(t-\tau_s), \eta(t-\tau_s)) u_k(t-\tau_s - \tau_u(t-\tau_s)) \right. \\ &\quad \left. - \dot{a}_{i,n-1}(t) - \dot{\psi}_{i,n}(t-\tau_s) z_{i,n}(t) \right. \\ &\quad \left. - \sum_{k=1}^n \binom{n}{n+1-k} (-\alpha)^{n+1-k} I_{i,k}(t) \right. \\ &\quad \left. + \sum_{k=1}^m s_{i,k} (u_k(t) - u_k(t-\tau_s - \tau_u(t-\tau_s))) \right) \\ &= \frac{1}{\psi_{i,n}(t-\tau_s)} \left(f_{i,n}(t-\tau_s, \bar{x}_n(t-\tau_s), \eta(t-\tau_s)) - \dot{a}_{i,n-1}(t) \right. \\ &\quad \left. + \sum_{k=1}^m (g_{i,k}(t-\tau_s, \bar{x}_n(t-\tau_s), \eta(t-\tau_s)) \right. \\ &\quad \left. - s_{i,k}) u_k(t-\tau_s - \tau_u(t-\tau_s)) \right. \\ &\quad \left. - \sum_{k=1}^n \binom{n}{n+1-k} (-\alpha)^{n+1-k} I_{i,k}(t) \right. \\ &\quad \left. + \sum_{k=1}^m s_{i,k} u_k(t) - \dot{\psi}_{i,n}(t-\tau_s) z_{i,n}(t) \right) \\ &=: h_{i,n}(t, z_{1,1}(t), \dots, z_{m,1}(t), \dots, z_{1,n-1}(t), \dots, z_{m,n-1}(t), \\ &\quad z_n(t), z_n(t-\tau_s - \tau_u(t-\tau_s)), I_{1,1}(t), \dots, I_{m,n}(t), \eta(t-\tau_s))\end{aligned}$$

where we can express $\bar{x}_n(t-\tau_s)$ (and hence the dependency of $f_{i,n}$ and $g_{i,k}$ on it) in terms of

- $\psi_{1,1}(t-\tau_s) z_{1,1}(t), \dots, \psi_{m,n}(t-\tau_s) z_{m,n}(t),$
- $y_{d,1}(t-\tau_s), \dots, y_{d,m}(t-\tau_s),$
- $a_{1,1}(t), \dots, a_{m,n-1}(t)$ and
- $I_{1,1}(t), \dots, I_{m,n}(t),$

i.e., there exists a continuous function

$$F_n : \mathbb{R}_{\geq \tau_s} \times (-1, 1)^{m(n-1)} \times \Omega \times \Omega_I^{mn} \rightarrow \mathbb{R}^{mn}$$

such that, for all $t \geq \tau_s$,

$$\begin{aligned}\bar{x}_n(t-\tau_s) &= F_n(t, z_{1,1}(t), \dots, z_{m,1}(t), \dots, z_{1,n-1}(t), \\ &\quad \dots, z_{m,n-1}(t), z_n(t), I_{1,1}(t), \dots, I_{m,n}(t)).\end{aligned}$$

The right-hand side of the differential equation for $z_{i,n}$ is then a continuous function

$$h_{i,n} : \mathbb{R}_{\geq \tau_s} \times (-1, 1)^{m(n-1)} \times \Omega^2 \times \Omega_I^{mn} \times \mathbb{R}^q \rightarrow \mathbb{R},$$

where $\Omega := B(0, 1) \subset \mathbb{R}^m$ is the open unit ball in \mathbb{R}^m .

Phase 2: We show existence of a local solution. Considering (4) we have

$$\begin{pmatrix} \dot{I}_{1,1}(t) \\ \vdots \\ \dot{I}_{1,n}(t) \\ \vdots \\ \dot{I}_{m,1}(t) \\ \vdots \\ \dot{I}_{m,n}(t) \end{pmatrix} = \underbrace{\begin{bmatrix} \tilde{A} & & \\ & \ddots & \\ & & \tilde{A} \end{bmatrix}}_{=A} \underbrace{\begin{pmatrix} I_{1,1}(t) \\ \vdots \\ I_{1,n}(t) \\ \vdots \\ I_{m,1}(t) \\ \vdots \\ I_{m,n}(t) \end{pmatrix}}_{=: \bar{I}(t) \in \mathbb{R}^{mn}} + \underbrace{\begin{pmatrix} b_1(t) \\ \vdots \\ b_m(t) \end{pmatrix}}_{=: b(t) \in \mathbb{R}^{mn}} \quad (9)$$

for $t \geq 0$, where for $i = 1, \dots, m$ we have $b_i(t) \in \mathbb{R}^n$ defined by

$$b_i(t) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{k=1}^m s_{i,k} (u_k(t) - u_k(t - \tau_s - \tau_u(t - \tau_s))) \end{pmatrix}.$$

Defining $\zeta(t) := \eta(t - \tau_s)$ the last of equations (1) becomes

$$\dot{\zeta}(t) = h(t - \tau_s, \bar{x}_n(t - \tau_s), \zeta(t)).$$

Then, by (1), (3), (4) and Step 1 there exists a function

$$H : \mathbb{R}_{\geq \tau_s} \times (-1, 1)^{m(n-1)} \times \Omega^2 \times \Omega_I^{mn} \times \mathbb{R}^q \rightarrow \mathbb{R}^{2mn+q}$$

satisfying

$$\begin{pmatrix} \dot{\bar{z}}(t) \\ \dot{\bar{I}}(t) \\ \dot{\zeta}(t) \end{pmatrix} = H(t, \bar{z}(t), z_n(t - \tau_s - \tau_u(t - \tau_s)), \bar{I}(t), \zeta(t)), \quad (10)$$

where

$$\bar{z}(t) = \begin{pmatrix} z_{1,1}(t) \\ \vdots \\ z_{m,1}(t) \\ \vdots \\ z_{1,n}(t) \\ \vdots \\ z_{m,n}(t) \end{pmatrix}.$$

Note that H is piecewise continuous and bounded in t and locally Lipschitz in all other variables. We define the operator

$$\begin{aligned} \mathcal{S} : \mathcal{C}([- \bar{\tau}_u, \infty), \mathbb{R}^m) &\rightarrow \mathcal{L}_{\text{loc}}^\infty(\mathbb{R}_{\geq \tau_s}, \mathbb{R}^n), \\ \xi &\mapsto (t \mapsto \xi(t - \tau_s - \tau_u(t - \tau_s))). \end{aligned}$$

We can now rewrite (10) as

$$\begin{pmatrix} \dot{\bar{z}}(t) \\ \dot{\bar{I}}(t) \\ \dot{\zeta}(t) \end{pmatrix} = \tilde{H}(t, \bar{z}(t), \bar{I}(t), \zeta(t), \tilde{\mathcal{S}}(\bar{z})(t)) \quad (11)$$

for some

$$\tilde{H} : \mathbb{R}_{\geq \tau_s} \times (-1, 1)^{m(n-1)} \times \Omega \times \Omega_I^{mn} \times \mathbb{R}^{q+n} \rightarrow \mathbb{R}^{mn+q},$$

and where $\tilde{\mathcal{S}}$ is defined such that $\tilde{\mathcal{S}}(\bar{z}) = \mathcal{S}(z_n)$. Furthermore, by assumption of the theorem, there exists a well-defined

initial history function $z^\varphi \in \mathcal{C}([- \bar{\tau}_u, \tau_s], \mathbb{R}^{mn})$ for \bar{z} with $z^\varphi(t) \in (-1, 1)^{m(n-1)} \times \Omega$ for all $t \in [- \bar{\tau}_u, \tau_s]$. Therefore, it follows from (3) and the fact that $u_i(t) = 0$ for $t \in [- \tau_s - \bar{\tau}_u, 0]$ and $i = 1, \dots, m$ that

$$u_i(t) = \begin{cases} \frac{-\sigma k_n \chi(\|z_n^\varphi(t)\|) z_{i,n}^\varphi(t)}{1 - \|z_n^\varphi(t)\|^2}, & t \in [0, \tau_s], \\ 0, & t \in [- \bar{\tau}_u, 0]. \end{cases}$$

Hence, there exists a well-defined initial history function $I^\varphi \in \mathcal{C}([0, \tau_s], \mathbb{R}^{mn})$ for \bar{I} , given by

$$I^\varphi(t) = \begin{cases} \int_0^t e^{A(t-s)} b^\varphi(s) ds, & t \in [0, \tau_s], \\ 0, & t \in [- \bar{\tau}_u, 0] \end{cases}$$

with $b^\varphi(s) = (b_1^\varphi(s)^\top, \dots, b_m^\varphi(s)^\top)^\top$ and

$$b_i^\varphi(s) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{k=1}^m s_{i,k} u_k(s) \end{pmatrix}.$$

Now, as a first lower bound on δ_I , we assume that $\delta_I > |I_{i,j}^\varphi(t)|$, for all $t \in [0, \tau_s]$, all $i = 1, \dots, m$ and all $j = 1, \dots, n$; this bound can be determined a priori.

The initial history for ζ is, according to (2), simply given by $\zeta^\varphi(\cdot) = \eta^\varphi(\cdot - \tau_s) \in \mathcal{C}([- \bar{\tau}_u, \tau_s], \mathbb{R}^q)$. Overall, (11) is equipped with the initial history

$$(\bar{z}, \bar{I}, \zeta)|_{[- \bar{\tau}_u, \tau_s]} = (z^\varphi, I^\varphi, \zeta^\varphi). \quad (12)$$

The existence of a maximal solution $(\bar{z}, \bar{I}, \zeta) : [- \bar{\tau}_u, \omega) \rightarrow \mathbb{R}^{2mn+q}$, $\omega \in (\tau_s, \infty]$, of (11), (12) such that the closure of

$$\{ (t, \bar{z}(t), \bar{I}(t), \zeta(t)) \mid t \in [\tau_s, \omega) \}$$

is not a compact subset of $\mathbb{R}_{\geq \tau_s} \times ((-1, 1)^{m(n-1)} \times \Omega) \times \Omega_I^{mn} \times \mathbb{R}^q$ then follows from [22, Thm. B.1].

Phase 3: We show that each $z_{i,j}(t)$ evolves strictly within $(-1, 1)$ and $z_n(t)$ evolve strictly within Ω , $i = 1, \dots, m$, $j = 1, \dots, n-1$, for all $t \in [\tau_s, \omega)$. The proof follows a recursive procedure.

Step 1 ($j = 1$, $t \in [\tau_s, \omega)$): We begin by considering the positive functions

$$V_{i,1}(t) := \frac{1}{2} z_{i,1}(t)^2$$

for $t \in [\tau_s, \omega)$ and $i = 1, \dots, m$. By Step 1 we have

$$\begin{aligned} \dot{V}_{i,1}(t) &= \frac{z_{i,1}(t)}{\psi_{i,1}(t - \tau_s)} \left(z_{i,2}(t) \psi_{i,2}(t - \tau_s) - \frac{k_{i,1} z_{i,1}(t)}{1 - z_{i,1}(t)^2} \right. \\ &\quad \left. - \dot{y}_{d,i}(t - \tau_s) - \dot{\psi}_{i,1}(t - \tau_s) z_{i,1}(t) + f_{i,1}(t, \bar{x}_1(t - \tau_s)) \right). \end{aligned} \quad (13)$$

Recall that \bar{x}_1 can be rewritten in terms of $z_{i,1}$, $\psi_{i,1}$, $y_{d,i}$ and $I_{i,1}$, as mentioned in Step 1, i.e., there exists a continuous function $F_1 : \mathbb{R}^{4m} \rightarrow \mathbb{R}^{mn}$ such that, for all $t \geq \tau_s$,

$$\begin{aligned} \bar{x}_1(t - \tau_s) &= F_1(z_{1,1}(t), \dots, z_{m,1}(t), I_{1,1}(t), \dots, I_{m,1}(t), \\ &\quad \psi_{1,1}(t), \dots, \psi_{m,1}(t), y_{d,1}(t), \dots, y_{d,m}(t)). \end{aligned}$$

Since $z_{i,1}(t), z_{i,2}(t) \in (-1, 1)$, $I_{i,1}(t) \in \Omega_I$, $y_{d,i}$, $\psi_{i,1}$ are bounded and $f_{i,1}$ are locally Lipschitz in $\bar{x}_{i,1}$ and bounded in t , we deduce that

$$|f_{i,1}(t, \bar{x}_1(t - \tau_s))| \leq \sup_{\xi \in \Omega_{\xi,1}} |f_{i,1}(t, F_1(\xi))| := \bar{f}_{i,1},$$

where

$$\Omega_{\xi,1} := (-1, 1)^m \times \Omega_I^m \times [-\bar{\psi}_1, \bar{\psi}_1]^m \times [-\bar{y}_d, \bar{y}_d]^m$$

and $\bar{\psi}_1 = \max_{i=1,\dots,m} \sup_{t \geq 0} |\psi_{i,1}(t)|$, $\bar{y}_d = \max_{i=1,\dots,m} \sup_{t \geq 0} |y_{d,i}(t)|$. Moreover, $\bar{\psi}_{i,1}$ and $\bar{\psi}_{i,2}$ are bounded by construction and $\dot{y}_{d,i}$ is bounded by assumption, we can utilize (13) to obtain

$$\dot{V}_{i,1}(t) \leq \frac{|z_{i,1}(t)|}{\psi_{i,1}(t - \tau_s)} \left(C_{i,1} - \frac{k_{i,1}|z_{i,1}(t)|}{1 - z_{i,1}(t)^2} \right) \quad (14)$$

for some $C_{i,1} > 0$. Choose $\varepsilon_{i,1} \in (0, 1)$ such that

$$\varepsilon_{i,1} > \max \left\{ -\frac{k_{i,1}}{2C_{i,1}} + \sqrt{\left(\frac{k_{i,1}}{2C_{i,1}}\right)^2 + 1}, \sup_{t \in [-\bar{\tau}_u, \tau_s]} |z_{i,1}(t)| \right\}.$$

Assume that there exists $t_1 \in (\tau_s, \omega)$ with $|z_{i,1}(t_1)| > \varepsilon_{i,1}$. Since $|z_{i,1}(t)| \leq \varepsilon_{i,1}$ for all $t \in [-\bar{\tau}_u, \tau_s]$,

$$t_0 := \max \{t \in [\tau_s, t_1] \mid |z_{i,1}(t)| = \varepsilon_{i,1}\}$$

is well-defined. Then we have

$$|z_{i,1}(t)| > \varepsilon_{i,1} \quad \text{and} \quad \frac{k_{i,1}|z_{i,1}(t)|}{1 - z_{i,1}(t)^2} > \frac{k_{i,1}\varepsilon_{i,1}}{1 - \varepsilon_{i,1}^2}$$

for all $t \in (t_0, t_1]$. Utilizing that for $a \in [0, 1)$ we have

$$\begin{aligned} C_{i,1} - \frac{k_{i,1}a}{1-a^2} &< 0 \\ \Leftrightarrow -C_{i,1}a^2 - k_{i,1}a + C_{i,1} &< 0 \\ \Leftrightarrow a^2 + \frac{k_{i,1}}{C_{i,1}}a - 1 &> 0 \\ \Leftrightarrow -\frac{k_{i,1}}{2C_{i,1}} + \sqrt{\left(\frac{k_{i,1}}{2C_{i,1}}\right)^2 + 1} &< a, \end{aligned}$$

we find that, by construction of $\varepsilon_{i,1}$,

$$\forall t \in (t_0, t_1] : C_{i,1} - \frac{k_{i,1}|z_{i,1}(t)|}{1 - z_{i,1}(t)^2} < 0.$$

Then it follows from (14) that

$$\forall t \in (t_0, t_1] : \dot{V}_{i,1}(t) < 0,$$

thus we arrive at the contradiction

$$\varepsilon_{i,1} = |z_{i,1}(t_0)| > |z_{i,1}(t_1)| > \varepsilon_{i,1}.$$

Therefore, we have shown that

$$\forall t \in [-\bar{\tau}_u, \omega) : |z_{i,1}(t)| \leq \varepsilon_{i,1}. \quad (15)$$

We once again recall Step 1 to infer that also $\dot{z}_{i,1}$ is bounded on $[-\bar{\tau}_u, \omega)$ for all $i = 1, \dots, m$. Then it follows from (3) that $\dot{a}_{i,1}$ is bounded on $[-\bar{\tau}_u, \omega)$, $i = 1, \dots, m$.

Step j ($j = 2, \dots, n-1$, $t \in [\tau_s, \omega)$): Again consider the positive functions

$$V_{i,j}(t) := \frac{1}{2} z_{i,j}(t)^2$$

for $t \in [\tau_s, \omega)$ and $i = 1, \dots, m$. By Step 1, the only difference between the differential equations for $V_{i,1}$ and $V_{i,j}$, except

from the changed indices, is the term $\dot{y}_{d,i}(t - \tau_s)$ appearing in $\dot{V}_{i,1}(t)$, which is replaced by $\dot{a}_{i,j-1}(t)$ in $\dot{V}_{i,j}(t)$. Furthermore, $\bar{x}_j(t - \tau_s)$ depends (continuously) on more of the previous variables (cf. Step 1), again all of which are bounded. Since $\dot{a}_{i,1}$ is bounded by the proof of the Case $j = 1$, we may inductively show that there exists $\varepsilon_{i,j} \in (0, 1)$ such that

$$\forall t \in [-\bar{\tau}_u, \omega) : |z_{i,j}(t)| \leq \varepsilon_{i,j}. \quad (16)$$

and $\dot{z}_{i,j}$ as well as $\dot{a}_{i,j}$ are bounded on $[-\bar{\tau}_u, \omega)$ for all $i = 1, \dots, m$.

Step n ($j = n$, $t \in [\tau_s, \omega)$): We consider the positive function

$$V_n(t) := \frac{1}{2} \|z_n(t)\|^2 = \frac{1}{2} \sum_{i=1}^m z_{i,n}(t)^2.$$

By (3), (4) and Step 1 we calculate the derivative of $V_n(t)$ as

$$\begin{aligned} \dot{V}_n(t) &= z_n(t)^\top \begin{pmatrix} \dot{z}_{1,n}(t) \\ \vdots \\ \dot{z}_{m,n}(t) \end{pmatrix} \\ &= \frac{z_n(t)^\top}{\psi_n(t - \tau_s)} \left(\begin{pmatrix} f_1(t - \tau_s, \bar{x}_n(t - \tau_s), \eta(t - \tau_s)) \\ \vdots \\ f_m(t - \tau_s, \bar{x}_n(t - \tau_s), \eta(t - \tau_s)) \end{pmatrix} \right. \\ &\quad \left. - \dot{\psi}_n(t - \tau_s) \begin{pmatrix} z_{1,n}(t) \\ \vdots \\ z_{m,n}(t) \end{pmatrix} - \begin{pmatrix} \dot{a}_{1,n-1}(t) \\ \vdots \\ \dot{a}_{m,n-1}(t) \end{pmatrix} - \tilde{I}(t) + S\bar{u}(t) \right. \\ &\quad \left. + (G(t - \tau_s, \bar{x}_n(t - \tau_s), \eta(t - \tau_s)) - S)\bar{u}(t - \tau_s - \tau_u(t - \tau_s)) \right), \end{aligned} \quad (17)$$

where

$$\begin{aligned} \tilde{I}(t) &:= \begin{pmatrix} \sum_{k=1}^n \binom{n}{n+1-k} (-\alpha)^{n+1-k} I_{1,k}(t) \\ \vdots \\ \sum_{k=1}^n \binom{n}{n+1-k} (-\alpha)^{n+1-k} I_{m,k}(t) \end{pmatrix} \\ \text{and } \bar{u}(t) &:= \begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{pmatrix}. \end{aligned}$$

By recalling Assumption 1 and (3), (4) we can derive the

estimate

$$\begin{aligned}
& |f_{i,n}(t-\tau_s, \bar{x}_n(t-\tau_s), \eta(t-\tau_s))| \\
& \leq |d_i(t-\tau_s)| (\|\bar{x}_n(t-\tau_s)\| + \|\eta(t-\tau_s)\| + 1) \\
& \leq |d_i(t-\tau_s)| \left(1 + \left\| \begin{pmatrix} \psi_{1,1}(t-\tau_s) z_{1,1}(t) \\ \vdots \\ \psi_{m,n}(t-\tau_s) z_{m,n}(t) \end{pmatrix} \right\| \right. \\
& \quad \left. + \kappa(t-\tau_s, \|\eta(0)\|) + \bar{\gamma}_1 \sup_{s \in [0, t-\tau_s]} \|\bar{x}_n(s)\| + \bar{\gamma}_2 + c \right. \\
& \quad \left. + \left\| \begin{pmatrix} y_{d,1}(t-\tau_s) \\ \vdots \\ y_{d,m}(t-\tau_s) \\ a_{1,1}(t) \\ \vdots \\ a_{m,n-1}(t) \end{pmatrix} \right\| + \left\| \begin{pmatrix} I_{1,1}(t) \\ \vdots \\ \sum_{k=1}^j \binom{j-1}{j-k} (-\alpha)^{j-k} I_{i,k}(t) \\ \vdots \\ \sum_{k=1}^n \binom{n-1}{n-k} (-\alpha)^{n-k} I_{m,k}(t) \end{pmatrix} \right\| \right) \\
& \leq |d_i(t-\tau_s)| \left(\tilde{C}_1 + \max \{ \alpha^{n-1}, 1 \} \binom{n}{\lfloor \frac{n}{2} \rfloor} n (\|\bar{I}(t)\| \right. \\
& \quad \left. + \bar{\gamma}_1 \sup_{s \in [\tau_s, t]} \|\bar{I}(s)\|) \right) \\
& \leq C_1 (1 + \sup_{s \in [\tau_s, t]} \|\bar{I}(s)\|)
\end{aligned} \tag{18}$$

where $\tilde{C}_1, C_1 > 0$ exist due to the boundedness of $\kappa(\cdot, \|\eta(0)\|)$, $\psi_{i,j}$, $z_{i,j}$, $y_{d,i}$, d_i , $i = 1, \dots, m$, $j = 1, \dots, n$ and $a_{i,j}$, $i = 1, \dots, m$, $j = 1, \dots, n-1$.

We like to emphasize at this point, that the above estimate is quite conservative and one might find a better one depending on the given system. As this estimate will essentially dictate the upper bound for $\tau_s + \tau_u(t-\tau_s)$ later on, a relaxation would allow for higher state measurement and control input delays.

To utilize (18) in (17) we need to find an estimate for $\|\bar{I}(t)\|$. We begin by defining

$$\tilde{u}_i(t) := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \sum_{k=1}^m s_{i,k} u_k(t) \end{pmatrix} \in \mathbb{R}^n, \quad i = 1, \dots, m$$

and

$$\tilde{u}(t) := \begin{pmatrix} \tilde{u}_1(t) \\ \vdots \\ \tilde{u}_m(t) \end{pmatrix} \in \mathbb{R}^{mn}$$

for $t \in [\tau_s, \omega)$. Since $t \mapsto t - \tau_s - \tau_u(t - \tau_s)$ is strictly monotonically increasing on $\mathbb{R}_{\geq \tau_s}$ by Assumption 2, there exists a strictly monotonically increasing function $\Gamma : [-\tau_u(0), \infty) \rightarrow \mathbb{R}$ such that

$$\forall t \geq \tau_s : \Gamma(t - \tau_s - \tau_u(t - \tau_s)) = t.$$

By (9) and the fact that $u_i(t) = 0$ for $t \in [-\tau_s - \bar{\tau}_u, 0]$ and

$i = 1, \dots, m$, variation of constants leads to

$$\begin{aligned}
\bar{I}(t) &= \int_0^t e^{A(t-s)} \tilde{u}(s) ds - \int_{\Gamma(0)}^t e^{A(t-s)} \tilde{u}(s - \tau_s - \tau_u(s - \tau_s)) ds \\
&= \int_0^t e^{A(t-s)} \tilde{u}(s) ds \\
&\quad - \int_0^{t-\tau_s-\tau_u(t-\tau_s)} e^{A(t-\Gamma(r))} \tilde{u}(r) \frac{dr}{1 - \dot{\tau}_u(\Gamma(r))} \\
&= \int_{t-\tau_s-\tau_u(t-\tau_s)}^t e^{A(t-s)} \tilde{u}(s) ds \\
&\quad - \int_0^{t-\tau_s-\tau_u(t-\tau_s)} \left(I - \frac{e^{A(r-\Gamma(r))}}{1 - \dot{\tau}_u(\Gamma(r))} \right) e^{A(t-r)} \tilde{u}(r) dr.
\end{aligned}$$

We can now estimate $\|\bar{I}(t)\|$ by employing Assumption 3 and the estimates (7), $\mu \leq \|A\|$, $|r - \Gamma(r)| \leq \tau_s + \bar{\tau}_u$ for $r \geq 0$ and

$$\forall B \in \mathbb{R}^{n \times n} : \|I - e^B\| \leq \|B\| e^{\|B\|},$$

so that

$$\begin{aligned}
\|\bar{I}(t)\| &\leq \sup_{s \in [0, t]} \|\tilde{u}(s)\| \left(\int_{t-\tau_s-\tau_u(t-\tau_s)}^t \|e^{A(t-s)}\| ds \right. \\
&\quad \left. + \int_0^{t-\tau_s-\tau_u(t-\tau_s)} \left\| I - \frac{e^{A(r-\Gamma(r))}}{1 - \dot{\tau}_u(\Gamma(r))} \right\| \cdot \|e^{A(t-r)}\| dr \right) \\
&\leq \|S\| \sup_{s \in [0, t]} \|\tilde{u}(s)\| \left(\int_{t-\tau_s-\tau_u(t-\tau_s)}^t M e^{-\mu(t-s)} ds \right. \\
&\quad \left. + \sup_{r \geq 0} \left\| I - \frac{e^{A(r-\Gamma(r))}}{1 - \dot{\tau}_u(\Gamma(r))} \right\| \int_0^{t-\tau_s-\tau_u(t-\tau_s)} M e^{-\mu(t-s)} ds \right) \\
&\leq \|S\| \sup_{s \in [0, t]} \|\tilde{u}(s)\| \frac{M}{\mu} \left((1 - e^{-\mu(\tau_s + \tau_u(t-\tau_s))}) \right. \\
&\quad \left. + \sup_{r \geq 0} \frac{1}{|1 - \dot{\tau}_u(\Gamma(r))|} \left(\|I - e^{A(r-\Gamma(r))}\| + \dot{\tau}_u \right) \cdot (e^{-\mu(\tau_s + \tau_u(t-\tau_s))} - e^{-\mu t}) \right) \\
&\leq \|S\| \sup_{s \in [0, t]} \|\tilde{u}(s)\| \frac{M}{\mu} \left(\mu(\tau_s + \tau_u(t-\tau_s)) e^{\mu(\tau_s + \tau_u(t-\tau_s))} \right. \\
&\quad \left. + \sup_{r \geq 0} \frac{1}{1 - \dot{\tau}_u} \left(\|A\| \cdot |r - \Gamma(r)| e^{\|A\| \cdot |r - \Gamma(r)|} + \dot{\tau}_u \right) \right) \\
&\leq \|S\| \sup_{s \in [0, t]} \|\tilde{u}(s)\| \frac{M}{\mu} \left(\mu(\tau_s + \bar{\tau}_u) e^{\mu(\tau_s + \bar{\tau}_u)} \right. \\
&\quad \left. + \frac{1}{1 - \dot{\tau}_u} \left(\|A\| (\tau_s + \bar{\tau}_u) e^{\|A\|(\tau_s + \bar{\tau}_u)} + \dot{\tau}_u \right) \right) \\
&\leq \|S\| \sup_{s \in [0, t]} \|\tilde{u}(s)\| \frac{M}{\mu} \left(\frac{\dot{\tau}_u}{1 - \dot{\tau}_u} \right. \\
&\quad \left. + \frac{2 - \dot{\tau}_u}{1 - \dot{\tau}_u} \|A\| (\tau_s + \bar{\tau}_u) e^{\|A\|(\tau_s + \bar{\tau}_u)} \right),
\end{aligned} \tag{19}$$

where $\dot{\tau}_u$ is defined in Assumption 2. Now observe that

$$\bar{u}(t) = -\sigma k_n \chi(\|z_n(t)\|) \frac{z_n(t)}{1 - \|z_n(t)\|^2}.$$

Using this and (18) in combination with the boundedness of $\dot{\psi}_n$, $z_{i,n}$ and $\dot{a}_{i,n-1}$, $i = 1, \dots, m$ as well as

$$\lambda := \frac{1}{\inf_{t \geq 0} \psi_n(t)},$$

$$\Theta := \sup_{(t,x,\eta) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{nm+q}} \|G(t, x, \eta) - S\| < \infty,$$

which exists by Assumption 3, in (17) we arrive at the inequality

$$\begin{aligned} \dot{V}_n(t) &\leq \|z_n(t)\| \lambda C_1 (1 + \sup_{s \in [\tau_s, t]} \|\bar{I}(s)\|) \\ &\quad + \Theta \lambda \|z_n(t)\| \|\bar{u}(t - \tau_s - \tau_u(t - \tau_s))\| \\ &\quad + \lambda \|z_n(t)\| \max\{\alpha^n, 1\} \left(\lfloor \frac{n}{2} \rfloor\right) \|\bar{I}(t)\| \\ &\quad + C_2 \lambda \|z_n(t)\| + \frac{1}{\psi_n(t - \tau_s)} z_n(t)^\top S \bar{u}(t) \\ &\leq \|z_n(t)\| \left(C_3 + \lambda C_4 \sup_{s \in [\tau_s, t]} \|\bar{I}(s)\| \right. \\ &\quad \left. + \Theta \lambda \|\bar{u}(t - \tau_s - \tau_u(t - \tau_s))\| - \frac{k_n \chi(\|z_n(t)\|)}{\|\psi_n\|_\infty} \frac{s^* \|z_n(t)\|}{1 - \|z_n(t)\|^2} \right) \end{aligned}$$

for some $C_2, C_3, C_4 > 0$, where s^* is defined in Section III and

$$C_4 := (1 + \bar{\gamma}_1) \max\{\alpha^{n-1}, 1\} \left(\lfloor \frac{n}{2} \rfloor\right) n \cdot \max_{i=1, \dots, m} \|d_i\|_\infty + \max\{\alpha^n, 1\} \left(\lfloor \frac{n}{2} \rfloor\right).$$

Next we employ (19) to further estimate

$$\begin{aligned} \dot{V}_n(t) &\leq \|z_n(t)\| \left(C_3 + C_5 \lambda \sup_{s \in [0, t]} \|\bar{u}(s)\| \right. \\ &\quad \left. + \Theta \lambda \|\bar{u}(t - \tau_s - \tau_u(t - \tau_s))\| - \frac{k_n \chi(\|z_n(t)\|)}{\|\psi_n\|_\infty} \frac{s^* \|z_n(t)\|}{1 - \|z_n(t)\|^2} \right) \\ &\leq \|z_n(t)\| \left(C_3 + C_5 \lambda \sup_{s \in [0, t]} \frac{\|z_n(s)\|}{1 - \|z_n(s)\|^2} \right. \\ &\quad \left. + \Theta \lambda \|\bar{u}(t - \tau_s - \tau_u(t - \tau_s))\| - \frac{k_n \chi(\|z_n(t)\|)}{\|\psi_n\|_\infty} \frac{s^* \|z_n(t)\|}{1 - \|z_n(t)\|^2} \right) \end{aligned} \quad (20)$$

for

$$C_5 = C_5(\tau_s, \bar{\tau}_u) := C_4 \frac{\|S\|M}{\mu} \left(\frac{\dot{\tau}_u}{1 - \dot{\tau}_u} + \frac{2 - \dot{\tau}_u}{1 - \dot{\tau}_u} \|A\|(\tau_s + \bar{\tau}_u) e^{\|A\|(\tau_s + \bar{\tau}_u)} \right)$$

according to (19). We note that

$$\begin{aligned} &\|\bar{u}(t - \tau_s - \tau_u(t - \tau_s))\| \\ &= \begin{cases} \frac{k_n \chi(\|z_n(t - \tau_s - \tau_u(t - \tau_s))\|) \|z_n(t - \tau_s - \tau_u(t - \tau_s))\|}{1 - \|z_n(t - \tau_s - \tau_u(t - \tau_s))\|^2}, & t \geq \tau_s \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Choose $\varepsilon_n \in (0, 1)$ with

$$\varepsilon_n > \max \left\{ -\frac{C_6}{2C_3} + \sqrt{\left(\frac{C_6}{2C_3}\right)^2 + 1}, \sup_{t \in [-\bar{\tau}_u, \tau_s]} \|z_n(t)\|, \frac{1 + \delta}{2} \right\},$$

where $\delta \in [0, 1)$ is as in (5) and

$$C_6 := \frac{k_n(1 - \delta)s^*}{2\|\psi_n\|_\infty} - \lambda C_5 - \lambda \Theta > 0$$

by Assumption 3. Assume that there exist $\zeta \in (\varepsilon_n, 1)$ and $t \in (\tau_s, \omega)$ with $\|z_n(t)\| > \zeta$. Define

$$t_0 := \inf \{ t \in [\tau_s, \omega) \mid \|z_n(t)\| > \zeta \},$$

which is well-defined since $\|z_n(t)\| \leq \varepsilon_n$ for all $t \in [-\bar{\tau}_u, \tau_s]$. By continuity there exists $t_1 \in (t_0, \omega)$ such that $\|z_n(t_1)\| > \zeta$ and $\|z_n(t)\| \geq \varepsilon_n > \delta$ for all $t \in [t_0, t_1]$. Therefore, we find that

$$\begin{aligned} \forall t \in [t_0, t_1] : \chi(\|z_n(t)\|) &\geq \|z_n(t)\| - \delta \\ &\geq \varepsilon_n - \delta > \frac{1 + \delta}{2} - \delta = \frac{1 - \delta}{2}. \end{aligned} \quad (21)$$

Furthermore, we observe that

$$\forall t \in [-\bar{\tau}_u, t_0] : \|z_n(t)\| \leq \zeta = \|z_n(t_0)\|. \quad (22)$$

Next we distinguish two cases.

Case I: There exists $t_2 \in (t_0, t_1)$ such that $\dot{V}_n(t) \leq 0$ for all $t \in [t_0, t_2]$. Clearly, t_2 can be chosen such that $\|z_n(t_2)\| > \zeta$, otherwise this contradicts the definition of t_0 . In this case, we directly obtain the contradiction

$$\zeta = \|z_n(t_0)\| \geq \|z_n(t_2)\| > \zeta.$$

Case II: Assuming the opposite of Case I leads to existence of a sequence $(t_k) \subset (t_0, t_1)$ with $t_k \rightarrow t_0$ for $k \rightarrow \infty$ such that $\dot{V}_n(t_k) > 0$ for all $k \in \mathbb{N}$. For $k \in \mathbb{N}$, define

$$\tau_k := \max \left\{ t \in [t_0, t_k] \mid \dot{V}_n(t) = 0 \right\},$$

then we have $\dot{V}_n(t) \geq 0$ for all $t \in [\tau_k, t_k]$ and all $k \in \mathbb{N}$. Choose $\rho > 0$ small enough so that

$$\varepsilon_n > -\frac{C_6}{2(C_3 + \rho C_5)} + \sqrt{\left(\frac{C_6}{2(C_3 + \rho C_5)}\right)^2 + 1}, \quad (23)$$

which is always possible. Then, by continuity, there exists $k \in \mathbb{N}$ sufficiently large such that

$$\sup_{s \in [t_0, \tau_k]} \frac{\|z_n(s)\|}{1 - \|z_n(s)\|^2} \leq \frac{\|z_n(\tau_k)\|}{1 - \|z_n(\tau_k)\|^2} + \rho.$$

Together with (22) and monotonicity of V_n on $[\tau_k, t_k]$ this implies that

$$\forall t \in [\tau_k, t_k] : \sup_{s \in [0, t]} \frac{\|z_n(s)\|}{1 - \|z_n(s)\|^2} \leq \frac{\|z_n(t)\|}{1 - \|z_n(t)\|^2} + \rho. \quad (24)$$

Incorporating (21) and (24) in (20) leads to

$$\begin{aligned}\dot{V}_n(t) &\leq \|z_n(t)\| \left(C_3 + \rho C_5 - C_6 \frac{\|z_n(t)\|}{1 - \|z_n(t)\|^2} \right) \\ &< \varepsilon_n \left(C_3 + \rho C_5 - C_6 \frac{\varepsilon_n}{1 - \varepsilon_n^2} \right) \\ &< 0,\end{aligned}$$

where for the last step we used (23) and the fact that for any $a \in [0, 1)$

$$\begin{aligned}\Leftrightarrow & \quad (C_3 + \rho C_5) - C_6 \frac{a}{1-a^2} < 0 \\ \Leftrightarrow & \quad -(C_3 + \rho C_5)a^2 - C_6a + (C_3 + \rho C_5) < 0 \\ \Leftrightarrow & \quad a^2 + \frac{C_6}{(C_3 + \rho C_5)}a - 1 < 0 \\ \Leftrightarrow & \quad \frac{-C_6}{2(C_3 + \rho C_5)} + \sqrt{\left(\frac{C_6}{2(C_3 + \rho C_5)}\right)^2 + 1} < a.\end{aligned}$$

Now $\dot{V}_n(t) < 0$ for all $t \in [\tau_k, t_k]$ directly contradicts the choice of the interval $[\tau_k, t_k]$.

Overall, we have shown that

$$\forall t \in [-\bar{\tau}_u, \omega) : \|z_n(t)\| \leq \varepsilon_n. \quad (25)$$

Phase 4: Employing (25), we further conclude the existence of constants $\bar{u}_i > 0$, $i = 1, \dots, n$, which are independent of δ_I (since C_3 and C_6 which define ε_n are independent of δ_I), such that $|u_i(t)| \leq \bar{u}_i$, $i = 1, \dots, m$, for all $t \in [\tau_s, \omega)$. Therefore, invoking the definition of $I_{i,j}$, we conclude that we may choose δ_I sufficiently large such that for all $i = 1, \dots, m$, $j = 1, \dots, n$, and $t \in [\tau_s, \omega)$,

$$|I_{i,j}(t)| \leq \sum_{\ell=1}^m \frac{2s_{i,\ell}\bar{u}_i}{\alpha^{n-j}} + \sum_{w=j+1}^n \frac{|I_{i,j}^\varphi(\tau_s)|}{\alpha^{w-j}} + |I_{i,j}^\varphi(\tau_s)| < \delta_I, \quad (26)$$

guaranteeing that $I_{i,j}$ evolves strictly within a compact subset of Ω_I for all $t \in [\tau_s, \omega)$. Finally, it follows from (15), (16), and (25) that \bar{z} evolves in a compact subset of $(-1, 1)^{m(n-1)} \times \Omega$. Therefore, Step 2 implies that $\omega = \infty$. Furthermore, we have shown that all closed-loop signals remain bounded. This finishes the proof of the theorem. \square

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Thomas Berger was born in Germany in 1986. He received his B.Sc. (2008), M.Sc. (2010), and Ph.D. (2013), all in Mathematics and from Technische Universität Ilmenau, Germany. From 2013 to 2018 Dr. Berger was a postdoctoral researcher at the Department of Mathematics, Universität Hamburg, Germany. Since January 2019 he is a Juniorprofessor at the Institute for Mathematics, Universität Paderborn, Germany. His research interest encompasses adaptive control, optimization-based control, differential–algebraic systems and multibody dynamics. In 2024, Dr. Berger received a Heisenberg grant from the German Research Foundation (DFG). For his exceptional scientific achievements in the field of Applied Mathematics and Mechanics, he received the "Richard-von-Mises Prize 2021" of the International Association of Applied Mathematics and Mechanics (GAMM). Dr. Berger further received several awards for his dissertation, including the "2015 European Ph.D. Award on Control for Complex and Heterogeneous Systems" from the European Embedded Control Institute and the "Dr.-Körper-Preis 2015" from the GAMM. He serves as an Associate Editor for Mathematics of Control, Signals, and Systems, the IMA Journal of Mathematical Control and Information and the DAE Panel.



Lampros N. Bikas is currently a postdoctoral researcher in the Automation and Robotics Lab of Aristotle University of Thessaloniki (AUTH). He received the diploma and the M.S. degrees in Electrical and Computer Engineering from the Democritus University of Thrace (DUTH) in 2012 and 2015, respectively, as well as the Ph.D. degree in Automatic Control from AUTH in 2023. He has received funding from the National Scholarships Foundation (IKY) and from the Research Committee of AUTH to conduct his Ph.D.

and postdoctoral research, respectively. His research interests include nonlinear control, prescribed performance control and networked control systems.



Jan Hachmeister was born in Germany in 1998. He obtained a B.Sc. in Mathematics in 2021 and an M.Sc. in Mathematics in 2023, both from Universität Paderborn. Additionally, he has completed his B.Sc. in Mechanical Engineering there in 2025. Since 2024, he has been working as a research associate, first with Dr. Berger and later within the Department of Mechanical Engineering. His involvement with the topic of this paper began during his master's thesis, which subsequently led to the present collaboration.



George A. Rovithakis (Senior Member, IEEE), received the diploma in Electrical Engineering from the Aristotle University of Thessaloniki, Greece in 1990 and the M.S. and Ph.D. degrees in Electronic and Computer Engineering both from the Technical University of Crete, Greece in 1994 and 1995, respectively. After holding a visiting Assistant Professor position with the Department of Electronic and Computer Engineering, Technical University of Crete from 1995 to 2002, he joined the Aristotle University of

Thessaloniki where he is currently a Professor at the Department of Electrical and Computer Engineering. His research interests include nonlinear control, robust adaptive control, prescribed performance control, robot control and control-identification of uncertain systems using neural networks, where he has authored or co-authored 3 books and over 190 papers in scientific journals, conference proceedings and book chapters. Dr. Rovithakis is currently serving as an Associate Editor of the IEEE Transactions on Automatic Control and served as Associate Editor of the IEEE Transactions on Neural Networks, of the IEEE Transactions on Control Systems Technology, as a member of the IEEE Control Systems Society Conference Editorial Board and of the European Control Association (EUCA) Conference Editorial Board. He is a member of the Technical Chamber of Greece, an elected member of EUCA, and former elected chair of the IEEE Greece Section Control Systems Chapter.