Funnel control for linear non-minimum phase systems

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We consider tracking control for linear non-minimum phase systems with known relative degree. For a given reference signal we design a low-complexity controller which achieves that the tracking error evolves within a prescribed performance funnel. We present a novel approach where a new output is constructed, with respect to which the system has a higher relative degree, but the unstable part of the internal dynamics is eliminated. Using recent results in funnel control, we then design a controller for this new output, which also incorporates a new reference signal. The original output stays within a prescribed performance funnel around the original reference trajectory and all signals in the closed-loop system are bounded.

1 System class and control objective

We consider linear systems given by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \tag{1.1}$$

where $x(0) = x^0 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ and $B, C^\top \in \mathbb{R}^{n\times m}$. We assume that (1.1) has strict relative degree $r \in \mathbb{N}$, that is

$$CB = CAB = \ldots = CA^{r-2}B = 0, \ \ CA^{r-1}B \in \mathbf{Gl}_n(\mathbb{R}).$$

We do not assume that (1.1) is minimum phase or, equivalently, its zero dynamics (cf. [1–4]) are asymptotically stable, which would mean that $\operatorname{rk}\left[\begin{smallmatrix} A-\lambda I_n & B \\ C & 0\end{smallmatrix}\right] = n+m$ for all $\lambda \in \mathbb{C}_-$. As an important tool we recall the Byrnes-Isidori form for linear systems (1.1). As shown in [5, Lem. 3.5], if (1.1) has strict relative degree r, then there exists $U \in \mathbf{Gl}_n(\mathbb{R})$ such that $Ux(t) = \big(y(t)^\top, \dot{y}(t)^\top, \ldots, y^{(r-1)}(t)^\top, \eta(t)^\top\big)^\top$, where $\eta : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n-rm}$, transforms (1.1) into

$$\begin{aligned} y^{(r)}(t) &= \sum\nolimits_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \Gamma u(t), \\ \dot{\eta}(t) &= Py(t) + Q\eta(t), \end{aligned} \tag{1.2}$$

where $R_i \in \mathbb{R}^{m\times m}$ for $i = 1, \ldots, r$, $S, P^\top \in \mathbb{R}^{m\times(n-rm)}$, $Q \in \mathbb{R}^{(n-rm)\times(n-rm)}$ and $\Gamma := CA^{r-1}B$. Furthermore, (1.1) is minimum phase if, and only if, $\sigma(Q) \subseteq \mathbb{C}_-$. The second equation in (1.2) represents the internal dynamics of the linear system (1.1).

Systems (1.1) which are non-minimum phase are a main obstacle for feedback controllers, since the unstable parts of the internal dynamics may impose fundamental limitations on the transient tracking performance as shown in [6]. Some approaches to resolve these limitations are discussed in the literature, see e.g. [7–10], however none of the available approaches is able to achieve tracking with prescribe performance for non-minimum phase systems.

To treat non-minimum phase systems (1.1) we need to assume that the system parameters A, B, C are known and the state x can be measured at all times and is available to the controller. However, we stress that knowledge of the initial value x^0 is not required for the presented controller design. Therefore, the objective is to design a dynamic state feedback such that tracking of a sufficiently smooth reference signal $y_{\rm ref} : \mathbb{R}_{\ge 0} \to \mathbb{R}^m$ is achieved so that in the closed-loop system the tracking error $e(t) = y(t) - y_{\rm ref}(t)$ evolves within a prescribed performance funnel, i.e., $\varphi(t)\|e(t)\| < 1$ for all $t \ge 0$, where φ belongs to

$$\Phi_r := \left\{ \varphi \in \mathcal{C}^r(\mathbb{R}_{\ge 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \ldots, \varphi^{(r)} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}.$$

Furthermore, all signals should remain bounded, even though (1.1) is non-minimum phase. We follow the framework of *Funnel Control* which was developed in [11], see also the survey [12] and the references therein. The funnel controller is an adaptive controller of high-gain type and has been successfully applied e.g. in control of industrial servo-systems [13] and underactuated multibody systems [14], voltage and current control of electrical circuits [15], temperature control of chemical reactor models [16] and adaptive cruise control [17].

In the approach that we present here we define a new output for the system such that the unstable part of the internal dynamics is completely removed, but the relative degree is increased eventually. A suitable redefinition of the reference signal then guarantees that the funnel controller developed in the recent work [18] may be applied. By an appropriate choice of the design parameters it can be achieved that the original output stays within a prescribed performance funnel around the original reference trajectory.

We stress that a main feature of funnel control is that it is model-free and hence inherently robust. Moreover, it was recently shown that even for higher relative degree systems funnel control is feasible using output error feedback only, and no derivatives of the output are required, see [19, 20]. These features are lost when dealing with non-minimum phase systems, where knowledge of the system parameters (apart from the initial value) and measurement of the complete state is required in general.

2 Trackability and controller

It is revealed in [9] that for tracking non-minimum systems certain *trackability assumptions* are necessary. Here we make the following assumptions:

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(A1) There exists $T \in \mathbf{Gl}_{n-rm}(\mathbb{R})$ and $\ell \in \mathbb{N}$ such that

$$TQT^{-1} = \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ 0 & \tilde{Q} \end{bmatrix}, \quad TP = \begin{bmatrix} \hat{P} \\ \tilde{P} \end{bmatrix},$$

where $\hat{Q}_1 \in \mathbb{R}^{k\times k}$, $\hat{Q}_2 \in \mathbb{R}^{k\times \ell m}$, $\tilde{Q} \in \mathbb{R}^{\ell m\times \ell m}$, $\hat{P} \in \mathbb{R}^{k\times m}$, $\tilde{P} \in \mathbb{R}^{\ell m\times m}$, $k = n - rm - \ell m \geq 0$ with $\sigma(\hat{Q}_1) \subseteq \mathbb{C}_-$ and $[\tilde{P}, \tilde{Q}\tilde{P}, \ldots, \tilde{Q}^{\ell-1}\tilde{P}] \in \mathbf{Gl}_{\ell m}(\mathbb{R})$.

(A2) Let $y_{\rm ref} \in \mathcal{W}^{r-1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ be a given reference signal and $W \in \mathbf{Gl}_{\ell m}(\mathbb{R})$ be such that

$$W\tilde{Q}W^{-1} = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix},$$

where $Q_j \in \mathbb{R}^{k_j\times k_j}$, $j = 1, 2, 3$, and $\sigma(Q_1) \subseteq \mathbb{C}_-$, $\sigma(Q_2) \subseteq \mathbb{C}_+$ and $\sigma(Q_3) \subseteq \mathrm{i}\mathbb{R}$. Then the equation

$$\dot{\eta}_3(t) = Q_3\eta_3(t) + P_3 y_{\rm ref}(t), \quad \eta_3(0) = 0$$

has a bounded solution $\eta_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}^{k_3}$.

Now, let $T\eta = (\eta_1^\top, \eta_2^\top)^\top$ and

$$K := [0, \ldots, 0, \Gamma^{-1}][\tilde{P}, \tilde{Q}\tilde{P}, \ldots, \tilde{Q}^{\ell-1}\tilde{P}]^{-1} \in \mathbb{R}^{m\times \ell m},$$

then we define the new output by $y_{\rm new}(t) := K\eta_2(t)$. Then we find that

$$y_{\rm new}^{(r+\ell)}(t) = \sum_{i=1}^{r+\ell} \hat{R}_i\, y_{\rm new}^{(i-1)}(t) + S_1\eta_1(t) + u(t),$$
$$\dot{\eta}_1(t) = \sum_{i=1}^{\ell+1} \hat{P}_i\, y_{\rm new}^{(i-1)}(t) + \hat{Q}_1\eta_1(t),$$

where the matrices involved are of appropriate sizes. The new reference signal $\hat{y}_{\rm ref}$ is generated by

$$\begin{aligned} \dot{\eta}_{2,\rm ref}(t) &= \tilde{Q}\eta_{2,\rm ref}(t) + \tilde{P}y_{\rm ref}(t), \;\; \eta_{2,\rm ref}(0) = \eta_{2,\rm ref}^0, \\ \hat{y}_{\rm ref}(t) &= K\eta_{2,\rm ref}(t). \end{aligned} \tag{2.1}$$

We may show the following result: If $y_{\rm ref} \in \mathcal{W}^{r-1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, assumption (A2) holds and

$$\eta_{2,\rm ref}^0 = W^{-1}\begin{bmatrix} 0_{k_1\times k_2} \\ -I_{k_2} \\ 0_{k_3\times k_2} \end{bmatrix} \int_0^\infty e^{-Q_2 s} P_2 y_{\rm ref}(s)\,\mathrm{d}s\,, \tag{2.2}$$

then the initial value problem (2.1) has a unique global solution such that $\hat{y}_{\rm ref} \in \mathcal{W}^{r+\ell,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$.

The generator (2.1) of the new reference signal is incorporated as a dynamic part into the controller design and the funnel controller from [18] is applied to system (1.1) with new output $y_{\rm new}$. The final controller design is given by:

$$\begin{aligned} \dot{\eta}_{2,\rm ref}(t) &= \tilde{Q}\eta_{2,\rm ref}(t) + \tilde{P}y_{\rm ref}(t), \;\; \eta_{2,\rm ref}(0) = \eta_{2,\rm ref}^0, \\ \hat{y}_{\rm ref}(t) &= K\eta_{2,\rm ref}(t), \\ e_0(t) &= y_{\rm new}(t) - \hat{y}_{\rm ref}(t), \\ e_1(t) &= \dot{e}_0(t) + k_0(t)\,e_0(t), \\ e_2(t) &= \dot{e}_1(t) + k_1(t)\,e_1(t), \\ &\;\;\vdots \\ e_{r+\ell-1}(t) &= \dot{e}_{r+\ell-2}(t) + k_{r+\ell-2}(t)\,e_{r+\ell-2}(t), \\ k_i(t) &= \tfrac{1}{1-\varphi_i(t)^2\|e_i(t)\|^2}, \quad i = 0, \ldots, r+\ell-1, \\ u(t) &= -k_{r+\ell-1}(t)\,e_{r+\ell-1}(t), \end{aligned} \tag{2.3}$$

where the initial value $\eta_{2,\rm ref}^0$ is as in (2.2) and the reference signal and funnel functions have the following properties:

$$\begin{aligned} & y_{\rm ref} \in \mathcal{W}^{r-1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \\ & \varphi_0 \in \Phi_{r+\ell}, \;\; \varphi_1 \in \Phi_{r+\ell-1}, \;\ldots,\; \varphi_{r+\ell-1} \in \Phi_1. \end{aligned} \tag{2.4}$$

The novel funnel controller design (2.3) is feasible for every system (1.1) which satisfies the assumptions (A1) and (A2).

Theorem 2.1 *Consider a linear system* (1.1) *with strict relative degree $r \in \mathbb{N}$, which satisfies assumptions (A1) and (A2). Let $\ell \in \mathbb{N}$ be the smallest number such that (A1) is satisfied. Further let $y_{\rm ref}, \varphi_0, \ldots, \varphi_{r+\ell-1}$ be as in* (2.4) *and $x^0 \in \mathbb{R}^n$ be an initial value such that $e_0, \ldots, e_{r+\ell-1}$ as defined in* (2.3) *satisfy $\varphi_i(0)\|e_i(0)\| < 1$ for all $i = 0, \ldots, r+\ell-1$.*

Then the controller (2.3) *applied to* (1.1) *yields a closed-loop system which has a unique global solution $(x, \eta_{2,\rm ref}) : [0,\infty) \to \mathbb{R}^{n+\ell m}$ such that all involved signals $x(\cdot)$, $\eta_{2,\rm ref}(\cdot)$, $u(\cdot)$, $k_0(\cdot), \ldots, k_{r+\ell-1}(\cdot)$ are bounded and the errors evolve uniformly within the respective performance funnels in the sense*

$$\forall\, i = 0, \ldots, r+\ell-1\; \exists\, \varepsilon_i > 0\; \forall\, t \geq 0: \; \|e_i(t)\| \leq \varphi_i(t)^{-1} - \varepsilon_i.$$

The proof of the theorem as well as further details on obtaining a prescribed performance for the original tracking error can be found in [21].

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