Fault tolerant funnel control

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A new approach to adaptive fault tolerant tracking control for uncertain linear systems is presented. Based on recent results in funnel control and the time-varying Byrnes-Isidori form, we introduce a low-complexity model-free controller which achieves prescribed performance of the tracking error for any given sufficiently smooth reference signal. Within the considered system class, we allow for more inputs than outputs as long as a certain redundancy of the actuators is satisfied. An important role in the controller design is played by the controller weight matrix, which is a rectangular input transformation chosen such that in the resulting system the zero dynamics, which are assumed to be uniformly exponentially stable, are independent of the new input.

1 System class and control objective

We consider linear systems with time-varying and nonlinear uncertainties and possible actuator faults of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + BL(t)u(t) + f\big(t, x(t), u(t)\big), \\ y(t) &= Cx(t) \end{aligned} \tag{1.1}$$

with initial condition $x(0) = x^0 \in \mathbb{R}^n$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ with $m \geq p$, $f \in \mathcal{C}(\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n)$ is bounded and the following properties are satisfied:

(P1) $L \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times m})$ such that $L, \dot{L}, \ldots, L^{(n)}$ are bounded and $\operatorname{rk} BL(t) = q \geq p$ for all $t \in \mathbb{R}$, $q \in \mathbb{N}$;

(P2) the system has (strict) relative degree $r \in \mathbb{N}$, i.e.,
- $CA^kBL(\cdot) = 0$ and $CA^kf(\cdot) = 0$ for all $k = 0, \ldots, r-2$ and
- the "high-frequency gain matrix" $\Gamma := CA^{r-1}B \in \mathbb{R}^{p\times m}$ and L satisfy $\operatorname{rk} \Gamma L(t) = p$ for all $t \in \mathbb{R}$.

The objective is fault tolerant tracking of a reference trajectory $y_{\rm ref} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^p)$ with prescribed performance, i.e., we seek an output error derivative feedback such that in the closed-loop system the tracking error $e(t) = y(t) - y_{\rm ref}(t)$ evolves within a prescribed performance funnel, i.e., $\varphi(t)\|e(t)\| < 1$ for all $t \geq 0$, where φ belongs to

$$\Phi_r := \left\{ \varphi \in \mathcal{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \ldots, \varphi^{(r)} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}.$$

Furthermore, the state x and the input u in (1.1) should remain bounded. We follow the framework of *Funnel Control* which was developed in [1], see also the survey [2] and the references therein. The funnel controller is an adaptive controller of high-gain type and thus inherently robust, which makes it a suitable choice for fault tolerant control tasks. The funnel controller has been successfully applied e.g. in control of industrial servo-systems [3] and voltage and current control of electrical circuits [4].

A certain redundancy of the actuators is necessary in (1.1), thus m is usually much larger than p. A typical situation is that $\operatorname{rk} B = p$, i.e., the number of linearly independent actuators equals the number of outputs of the system; one may think of p groups of actuators, where actuators in the same group perform the same control task.

The (unknown) time-varying matrix function L from (P1) describes the *reliability* of the actuators. We assume that the system parameters A, B, C, $L(\cdot)$, $f(\cdot)$, x^0 are unknown. We only require knowledge of the relative degree $r \in \mathbb{N}$. Furthermore, we will derive a class of rectangular input transformations of the form $u(t) = K(t)v(t)$, where $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times p})$, such that in the resulting system the zero dynamics (cf. [5–8]) are independent of the new input v. As a structural assumption, we will require that the zero dynamics of the time-varying linear system $(A, BL(\cdot)K(\cdot), C)$ are uniformly exponentially stable for one (and hence any) K in this class; it is hence independent of the choice of K. Some additional knowledge of system parameters, such as the high-frequency gain matrix $\Gamma = CA^{r-1}B$ from (P2), may be helpful for the construction of K.

2 Normal form and control

We introduce the following for all $t \in \mathbb{R}$:

$$\begin{aligned} \mathcal{B}(t) &:= \Big[BL(t), \big(\tfrac{\rm d}{{\rm d}t} - A\big)\big(BL(t)\big), \\ &\qquad \ldots, \big(\tfrac{\rm d}{{\rm d}t} - A\big)^{r-1}\big(BL(t)\big)\Big] \in \mathbb{R}^{n\times rm}, \\ \mathcal{C} &:= \Big[C^\top, A^\top C^\top, \ldots, (A^{r-1})^\top C^\top\Big]^\top \in \mathbb{R}^{rp\times n}. \end{aligned}$$

Let $\rho := \operatorname{rk} \mathcal{C}$, choose $V \in \mathbb{R}^{n\times(n-\rho)}$ such that $\operatorname{im} V = \ker \mathcal{C}$ and define

$$\mathcal{N}(t) := V^\dagger \Big[I_n - \mathcal{B}(t)\big(\mathcal{C}\mathcal{B}(t)\big)^\dagger \mathcal{C}\Big] \in \mathbb{R}^{(n-\rho)\times n}, \quad t \in \mathbb{R},$$

where $M^\dagger$ denotes the *Moore-Penrose pseudoinverse* of a matrix M. Set

$$U(t) := \begin{bmatrix} \mathcal{C} \\ \mathcal{N}(t) \end{bmatrix} \in \mathbb{R}^{(n-\rho+pr)\times n}, \quad t \in \mathbb{R}. \tag{2.1}$$

As shown in [9] we have that $\rho = \operatorname{rk}\mathcal{C} = \operatorname{rk}\mathcal{C}\mathcal{B}(t) = pr$ for all $t \in \mathbb{R}$, and hence $U(\cdot)$ as in (2.1) is invertible.

We call a matrix function $M \in \mathcal{C}^1(\mathbb{R} \to \mathbf{Gl}_n(\mathbb{R}))$ a *Lyapunov transformation*, if M, M^{-1} and $\dot{M}$ are bounded.

Theorem 2.1 *Consider a system* (1.1) *with (P1) and (P2) such that U as in* (2.1) *is a Lyapunov transformation. Then*

$$\begin{aligned} (\hat{A}, \hat{B}, \hat{C}) &:= \Big((UA + \dot{U})U^{-1}, UBL, CU^{-1}\Big), \\ \hat{f}(t, z, u) &:= U(t)f\big(t, U(t)^{-1}z, u\big), \;\; (t, z, u) \in \mathbb{R}^{1+n+m} \end{aligned}$$

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satisfy

$$\hat{A}(t) = \begin{bmatrix} 0 & I_p & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_p & & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \cdots & 0 & I_p & 0 \\ R_1(t) & R_2(t) & \cdots & R_{r-1}(t) & R_r(t) & S(t) \\ P_1(t) & P_2(t) & \cdots & P_{r-1}(t) & P_r(t) & Q(t) \end{bmatrix},$$

$$\hat{B}(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Gamma L(t) \\ N(t) \end{bmatrix}, \qquad \hat{f}(t,z,u) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f_r(t,z,u) \\ f_\eta(t,z,u) \end{pmatrix}, \tag{2.2}$$

$$\hat{C} = \begin{bmatrix} I_p & 0 & \cdots & 0 \end{bmatrix},$$

where $Q \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{(n-pr)\times(n-pr)})$, R_i, P_i, S *and* N *are smooth, bounded, and of appropriate sizes and* f_r, f_η *are continuous, bounded, and of appropriate sizes. If, additionally,* $q = p$ *for* q *in (P1), then we have*

$$P_2 = P_3 = \ldots = P_r = 0 \quad \text{and} \quad N = 0.$$

The proof of the theorem can be found in [9].

The funnel control design that we introduce here extends the recently developed funnel controller for systems with arbitrary relative degree [10]. The controller is of the form

$$\boxed{\begin{aligned} e_0(t) &= e(t) = y(t) - y_{\text{ref}}(t), \\ e_1(t) &= \dot{e}_0(t) + k_0(t)\, e_0(t), \\ e_2(t) &= \dot{e}_1(t) + k_1(t)\, e_1(t), \\ &\;\;\vdots \\ e_{r-1}(t) &= \dot{e}_{r-2}(t) + k_{r-2}(t)\, e_{r-2}(t), \\ k_i(t) &= \tfrac{1}{1-\varphi_i(t)^2\|e_i(t)\|^2}, \quad i = 0,\ldots,r-1, \\ u(t) &= -k_{r-1}(t)\, K(t)\, e_{r-1}(t) \end{aligned}} \tag{2.3}$$

where the reference signal and funnel functions have the following properties:

$$\boxed{\begin{aligned} &y_{\text{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^p), \\ &\varphi_0 \in \Phi_r,\;\; \varphi_1 \in \Phi_{r-1},\; \ldots,\;\; \varphi_{r-1} \in \Phi_1. \end{aligned}} \tag{2.4}$$

We choose the bounded controller weight matrix function $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times p})$, if possible, such that

$$\boxed{\begin{aligned} &\exists\, \alpha > 0 :\; \Gamma L(t)K(t) + \big(\Gamma L(t)K(t)\big)^\top \geq \alpha I_p \\ &\text{and} \quad N(t)K(t) = 0, \end{aligned}} \tag{2.5}$$

where we use the notation from Theorem 2.1.

We show feasibility of the controller (2.3) for every system (1.1) which satisfies the assumptions (P1), (P2) and

(P3) U as in (2.1) is a Lyapunov transformation,

(P4) for $Q(\cdot)$ in (2.2), the system $\dot{x}(t) = Q(t)x(t)$ is uniformly exponentially stable.

We stress that assumptions (P1)–(P4) are only of structural nature and hold for a large class of systems.

Theorem 2.2 *Consider a system* (1.1) *which satisfies assumptions (P1)–(P4). Let* $y_{\text{ref}}, \varphi_0, \ldots, \varphi_{r-1}$ *be as in* (2.4) *and* $x^0 \in \mathbb{R}^n$ *be an initial value such that* $e_0, \ldots, e_{r-1}$ *as defined in* (2.3) *satisfy*

$$\varphi_i(0)\|e_i(0)\| < 1 \quad \text{for } i = 0,\ldots,r-1. \tag{2.6}$$

Assume that there exists a bounded $K \in \mathcal{C}^\infty(\mathbb{R} \to \mathbb{R}^{m\times p})$ *such that* (2.5) *is satisfied. Then the funnel controller* (2.3) *applied to* (1.1) *yields an initial-value problem which has a solution, and every solution can be extended to a maximal solution* $x : [0,\omega) \to \mathbb{R}^n$, $\omega \in (0,\infty]$, *which satisfies:*

(i) *The solution is global (i.e.,* $\omega = \infty$*).*

(ii) *The input* $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$*, the gain functions* $k_0, \ldots, k_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ *and* $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ *are bounded.*

(iii) *The functions* $e_0, \ldots, e_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$ *evolve in their respective performance funnels and are uniformly bounded away from the funnel boundaries in the sense:*

$$\forall i = 0,\ldots,r-1\, \exists\, \varepsilon_i > 0\, \forall t > 0 :\; \|e_i(t)\| \leq \varphi_i(t)^{-1} - \varepsilon_i.$$

The proof of the theorem can be found in [9].

We comment on the choice of K: If $\operatorname{rk} BL(t) = q > p$, then the system has an unnecessary high redundancy. Typically, there are q groups of actuators and p outputs. When it can still be guaranteed that at least one actuator without total fault remains in each group, then whole groups of actuators may be switched off so that $q = p$ is achieved. In this case, under (P1)–(P3) it follows from Theorem 2.1 that we have $N = 0$, so the second condition in (2.5) is satisfied for any choice of K. In order to satisfy the first condition in (2.5), a natural choice is $K(t) = \Gamma^\top$ and the requirement that

$$\exists\, \alpha > 0\, \forall\, t \in \mathbb{R} :\; \Gamma\big(L(t) + L(t)^\top\big)\Gamma^\top \geq \alpha I_p.$$

This condition means that we have at least p linearly independent actuators, the reliability of which does not converge to zero. For this choice of K we have to assume that the high frequency gain matrix Γ of (1.1) is known; apart from that, no knowledge of the system parameters is required.

We stress that the derivatives of the output must be available for the controller. However, this is not satisfied in several applications. A first approach to treat this problem using a "funnel pre-compensator" has been developed in [11, 12] for systems with relative degree $r = 2$ or $r = 3$.

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