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FUNNEL CONTROL FOR THE MONODOMAIN EQUATIONS

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Monodomain equations

- (simple) model for the electric activity of the human heart to describe defibrillation processes
- “reentry waves” can be interpreted as fibrillation processes that should be terminated by external control

$$\partial_t \mathbf{v} = \operatorname{div}(D \nabla \mathbf{v}) + I_{ion}(\mathbf{v}, \mathbf{w}) + I_{s,j} + B I_{s,e}, \quad (\nu^\top \cdot D \nabla \mathbf{v} + \alpha \mathbf{v})|_{\partial \Omega} = \mathbf{0},$$

$$\partial_t \mathbf{w} = c_5 \mathbf{v} - c_4 \mathbf{w}$$

- D – coercive diffusion matrix
- B – input operator
- \mathbf{v} – transmembrane electric potential
- \mathbf{w} – cellular state variable
- I_{ion} – ionic current
- $I_{s,j}$ – intracellular stimulation current
- $I_{s,e}$ – extracellular stimulation current
- α – nonnegative weight function

FitzHugh-Nagumo model

$$p_3(v) = -c_1v + c_2v^2 - c_3v^3,$$

output: $y = B'v,$

$$I_{ion}(v, w) = p_3(v) - w$$

input: $I_{s,e}(t)$

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$$\text{input: } I_{s,e}(t)$$

$$\alpha : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \rightarrow \mathbb{R}, (z_1, z_2) \mapsto \langle \nabla z_1, D\nabla z_2 \rangle + \langle z_1, \alpha z_2 \rangle_{L^2(\partial\Omega)}$$

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$\implies \exists!$ operator $A : \mathcal{D}(A) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ with

$$\begin{aligned}
 \mathcal{D}(A) &= \left\{ z_2 \in W^{1,2}(\Omega) \mid \exists y \in L^2(\Omega) \forall z_1 \in W^{1,2}(\Omega) : \alpha(z_1, z_2) = -\langle z_1, y \rangle \right\} \\
 &\quad \forall z_1 \in W^{1,2}(\Omega) \forall z_2 \in \mathcal{D}(A) : \alpha(z_1, z_2) = -\langle z_1, Az_2 \rangle
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 \end{aligned}$$

A is called *Robin elliptic operator*; it is *closed* and *self-adjoint*
if $\partial\Omega$, D and α are sufficiently smooth:

$$A v = \operatorname{div}(D \nabla v), \quad \mathcal{D}(A) = \left\{ v \in W^{2,2}(\Omega) \mid (\nu^\top \cdot D \nabla v + \alpha v)|_{\partial\Omega} = 0 \right\}$$

FitzHugh-Nagumo model

FHN model:

$$\begin{aligned}\partial_t v(t) &= Av(t) + p_3(v)(t) - w(t) + I_{s,i}(t) + BI_{s,e}(t) \\ \partial_t w(t) &= c_5 v(t) - c_4 w(t) \\ y(t) &= B'v(t)\end{aligned}$$

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solutions via weak formulation:

$$v, w \in C([0, T]; L^2(\Omega)), v \in L^2(0, T; W^{1,2}(\Omega)) \text{ s.t.}$$

$$\begin{aligned}\frac{d}{dt} \langle v(t), \theta \rangle &= -a(v(t), \theta) + \langle p_3(v)(t) - w(t) + I_{s,i}(t), \theta \rangle + \langle I_{s,e}(t), B'\theta \rangle_{\mathbb{R}^m} \\ \frac{d}{dt} \langle w(t), \psi \rangle &= \langle c_5 v(t) - c_4 w(t), \psi \rangle \\ y(t) &= B'v(t)\end{aligned}$$

Input-output configurations

- $q \in L^2(\Omega)$ and distributed input of the form $Bz = z \cdot q$, i.e. $B \in \mathcal{L}(\mathbb{R}, L^2(\Omega))$

$$\implies y(t) = B'v(t) = \int_{\Omega} q(\xi) \cdot (v(t))(\xi) d\xi$$

e.g. $q = \mathbb{1}_{\omega}$ for $\omega \subseteq \Omega$

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- $q \in W^{1/2,2}(\partial\Omega)'$ and input operator $Bw \in W^{1,2}(\Omega)'$, $w \in \mathbb{R}$, with

$$(Bw)(z) = \langle q \cdot w, \text{tr}(z) \rangle_{W^{1/2,2}(\partial\Omega)', W^{1/2,2}(\partial\Omega)}$$

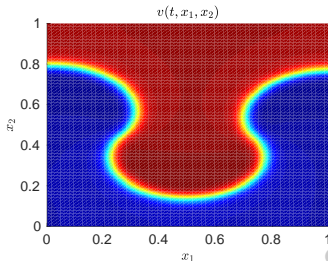
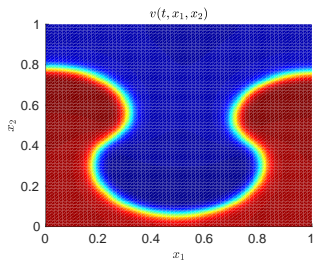
corresponds to a Robin boundary control

$$(\nu^{\top} \cdot D\nabla v(t) + \alpha v(t))|_{\partial\Omega} = q \cdot I_{s,e}(t)$$

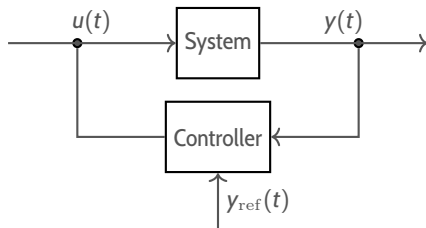
$$\implies y(t) = B'v(t) = \int_{\partial\Omega} q(\xi) \cdot (v(t))(\xi) \, d\sigma \quad (\text{Dirichlet boundary obs.})$$

Control objective

- under certain initial conditions, reentry phenomena and spiral waves may occur [Kunisch, Nagaiah, Wagner '11]
- reentry phenomena resemble a dysfunctional heart rhythm which impedes the intracellular stimulation current $I_{S,j}$
- such situations can be interpreted as fibrillation processes of the heart that should be terminated by an external control



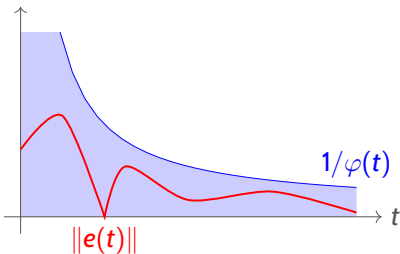
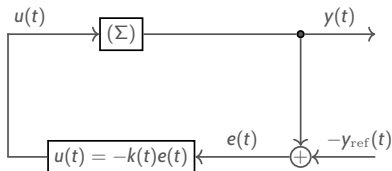
Control objective



$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), & x(t) &\in X \\ y(t) &= h(x(t)) \end{aligned}$$

- **Goal:** simple controller, so that “ $y(t)$ tracks $y_{\text{ref}}(t)$ ”
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters

Funnel control



$$(\Sigma) : \dot{y}(t) = F(T(y)(t)) + G(T(y)(t)) u(t)$$

$$k(t) = \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}$$

[Ilchmann, Ryan, Sangwin '02]:
Feasible, if T is

- causal and loc. Lipschitz
- BIBO stable

Funnel control for ∞ -dimensional systems

- systems which have a relative degree: [Ilchmann, Selig, Trunk '16], [B., Puche, Schwenninger '20]
- boundary controlled heat equation [Reis, Selig '15]

$$\partial_t x(t) = \Delta x(t), \quad u(t) = (\nu^\top \cdot \nabla x(t))|_{\partial\Omega},$$

$$y(t) = \int_{\partial\Omega} (x(t))(\zeta) d\zeta$$

- general class of boundary control systems based on m -dissipative operators [Puche, Reis, Schwenninger '21], [Puche '19]

$$\dot{x}(t) = \mathfrak{A}x(t), \quad x(0) = x_0 \in \mathcal{D}(\mathfrak{A}) \subseteq X,$$

$$u(t) = \mathfrak{B}x(t), \quad y(t) = \mathfrak{C}x(t)$$

e.g. lossy transmission line, wave equation

Funnel controller for the monodomain equations

Goal: controller independent of $\Omega, B, D, \alpha, c_i > 0$ and $I_{s,i}$

$$\varphi \in \Phi_\gamma := \left\{ \varphi \in W^{1,\infty}([0, \infty); \mathbb{R}) \mid \begin{array}{l} \varphi|_{[0,\gamma]} = 0, \\ \forall \delta > 0 : \inf_{t \geq \gamma + \delta} \varphi(t) > 0 \end{array} \right\}$$

→ φ is a *design parameter!*

Controller:
$$I_{s,e}(t) = -\frac{k_0}{1 - \varphi(t)^2 \|y(t) - y_{\text{ref}}(t)\|_{\mathbb{R}^m}^2} (y(t) - y_{\text{ref}}(t)),$$

$$y(t) = B'v(t),$$

$$\varphi \in \Phi_\gamma, y_{\text{ref}} \in W^{1,\infty}([0, \infty); \mathbb{R}^m)$$

$$\forall t \in [0, \gamma] : I_{s,e}(t) = -k_0(y(t) - y_{\text{ref}}(t))$$

Closed-loop system

$$\begin{aligned}\partial_t v(t) &= Av(t) + p_3(v)(t) - w(t) + I_{s,i}(t) \\ &\quad - \frac{k_0 B(B'v(t) - y_{\text{ref}}(t))}{1 - \varphi(t)^2 \|B'v(t) - y_{\text{ref}}(t)\|_{\mathbb{R}^m}^2}, \\ \partial_t w(t) &= c_5 v(t) - c_4 w(t)\end{aligned}$$

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nonlinear time-varying PDE!

Theorem [B., Breiten, Puche, Reis '21]

- $\Omega \subset \mathbb{R}^d$, $d \leq 3$, bounded domain with Lipschitz boundary $\partial\Omega$
- $D \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ coercive; $\mathbf{a} \in L^\infty(\partial\Omega)$
- $B \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega)')$, $\ker B = \{\mathbf{0}\}$
- $l_{s,j} \in L^\infty(\mathbf{0}, \infty; L^2(\Omega))$, $v_0, w_0 \in L^2(\Omega)$
- $\mathbf{y}_{\text{ref}} \in W^{1,\infty}(\mathbf{0}, \infty; \mathbb{R}^m)$, $\gamma > \mathbf{0}$ and $\varphi \in \Phi_\gamma$

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$\implies \exists!$ solution s.t. $v, w, \dot{w} \in BC([0, \infty); L^2(\Omega))$ and

$$v \in BUC([\delta, \infty); W^{1,2}(\Omega)) \cap C^{0,1/2}([\delta, \infty); L^2(\Omega)),$$

$$y, l_{s,e} \in BUC([\delta, \infty); \mathbb{R}^m),$$

$$\forall t \geq \gamma : \varphi(t)^2 \|B'v(t) - \mathbf{y}_{\text{ref}}(t)\|_{\mathbb{R}^m}^2 \leq 1 - \varepsilon$$

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$$\forall t \geq \gamma : \varphi(t)^2 \|B'\mathbf{v}(t) - \mathbf{y}_{\text{ref}}(t)\|_{\mathbb{R}^m}^2 \leq 1 - \varepsilon$$

$$\implies \mathbf{v}(\gamma) \in W^{1,2}(\Omega)$$

Funnel control for the monodomain equations

a) $B \in \mathcal{L}(\mathbb{R}^m, W^{r,2}(\Omega)'), r \in (0, 1)$

$$\implies v \in C^{0,1-r/2}([\delta, \infty); L^2(\Omega)), y, l_{s,e} \in C^{0,1-r}([\delta, \infty); \mathbb{R}^m)$$

b) $B \in \mathcal{L}(\mathbb{R}^m, L^2(\Omega))$

$$\implies \forall \lambda \in (0, 1) : v \in C^{0,\lambda}([\delta, \infty); L^2(\Omega)), y, l_{s,e} \in C^{0,\lambda}([\delta, \infty); \mathbb{R}^m)$$

c) $B \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega))$

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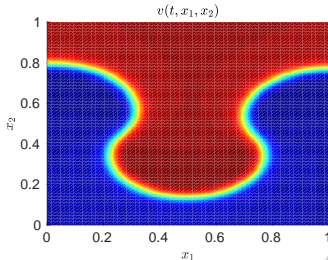
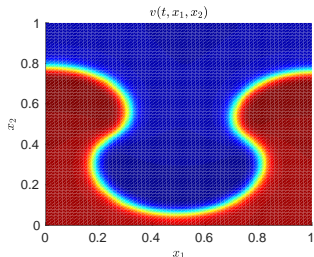
based on results by [Lunardi '95] on optimal regularity in parabolic problems

Monodomain equations – Simulation

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Example: standard test example for termination of reentry waves

$\Omega = (0, 1)^2$; $(v(0), w(0)) = (v_0^*, w_0^*)$ – snapshot of a reentry wave, resembling a dysfunctional heart rhythm



Monodomain equations – Simulation

Task: design stimulation current $I_{s,e}$, which restores a natural heart rhythm

Reference $y_{\text{ref}} = B'v_{\text{ref}}$ corresponds to $(v_{\text{ref}}, w_{\text{ref}})$ with
 $(v_{\text{ref}}(\mathbf{0}), w_{\text{ref}}(\mathbf{0})) = (\mathbf{0}, \mathbf{0}), I_{s,e} = \mathbf{0}$ and

$$I_{s,i}(t) = 101 \cdot q(\xi) (\mathbb{1}_{[49,51]}(t) + \mathbb{1}_{[299,301]}(t)),$$
$$q(\xi) = \begin{cases} 1, & \text{falls } (\xi_1 - \frac{1}{2})^2 + (\xi_2 - \frac{1}{2})^2 \leq 0.0225, \\ 0, & \text{otherwise} \end{cases}$$

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Control: input operator B corresponds to **boundary control**

$$I_{s,e}(t) = -\frac{0.75}{1 - \varphi(t)^2 \|B'v(t) - y_{\text{ref}}(t)\|_{\mathbb{R}^2}^2} (B'v(t) - y_{\text{ref}}(t))$$

Monodomain equations – Simulation

