Funnel-Regelung nichtlinearer DAEs mit gemischtem Relativgrad

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ODE-Systeme

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (1)$$
$$\dot{y}(t) = f_2(y(t)) + f_3(x(t)) + \Gamma(y(t))u(t), \qquad y(0) = y^0. \quad (2)$$

$x : \mathbb{R} \to \mathbb{R}^n$, $u, y : \mathbb{R} \to \mathbb{R}^m$; f_1, f_2, f_3 differenzierbar;
$\Gamma : \mathbb{R}^m \to \mathbb{R}^{m \times m}$ differenzierbar

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(1) interne Dynamik

(2) Eingangs-Ausgangs-Verhalten

$$\begin{aligned} \dot{x}(t) &= f_1(x(t), y(t)), & x(0) &= x^0, \quad (1) \\ \dot{y}(t) &= f_2(y(t)) + f_3(x(t)) + \Gamma(y(t))u(t), & y(0) &= y^0. \quad (2) \end{aligned}$$

Relativgrad:

$$\begin{aligned} \dot{z}(t) &= f(z(t)) + g(z(t))u(t) \\ y(t) &= h(z(t)) \end{aligned}$$

hat **strikten Relativgrad** $r \in \mathbb{N} \;:\Longleftrightarrow\;$ [Isidori, 1995]

- $\forall\, \xi \in \mathbb{R}^n \;\forall\, k = 0, \ldots, r-2 : \; L_g L_f^k h(\xi) = 0,$
  $[L_f h = (\partial h / \partial z)\, f]$
- $\forall\, \xi \in \mathbb{R}^n : \; \det L_g L_f^{r-1} h(\xi) \neq 0.$

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (1)$$
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- $\forall\, \xi \in \mathbb{R}^n : \; \det L_g L_f^{r-1} h(\xi) \neq 0.$

(1), (2) hat str. Relativgrad 1 $\Longleftrightarrow \det \Gamma(y) \neq 0 \;\forall\, y \in \mathbb{R}^m$

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Nulldynamik:

$$\begin{aligned} \dot{z}(t) &= f(z(t)) + g(z(t))u(t) \\ y(t) &= h(z(t)) \end{aligned}$$

$$\mathcal{ZD} := \{ \ (z, u) \ \mid \ \dot{z}(t) = f(z(t)) + g(z(t))u(t), \ \ 0 = h(z(t)) \ \} \, ;$$

$$\mathcal{ZD} \textbf{ stabil } :\Longleftrightarrow \ \forall \, (z, u) \in \mathcal{ZD} : \ \lim_{t \to \infty} (z(t), u(t)) = 0$$

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (1)$$
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$$\mathcal{ZD} \ \textbf{stabil} \ :\Longleftrightarrow \ \forall\,(z,u) \in \mathcal{ZD}: \ \lim_{t\to\infty} (z(t), u(t)) = 0$$

$\mathcal{ZD}_{(1),(2)}$ stabil, wenn (1) **input-to-state stable**: [Sontag, 1989]

$$\exists\,\alpha \in \mathcal{KL}, \beta \in \mathcal{K} \ \forall\,(x^0, y) \in \mathbb{R}^n \times \mathcal{C}^0(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$$
$$\forall\,t \geq 0: \ \|x(t; x^0, y)\| \leq \alpha(\|x^0\|, t) + \sup_{s\in[0,t]} \beta(\|y(s)\|),$$

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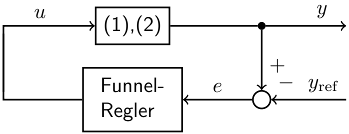

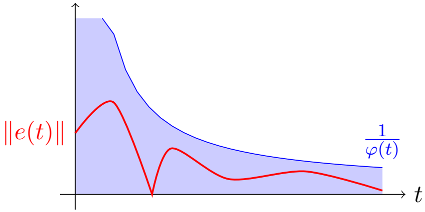

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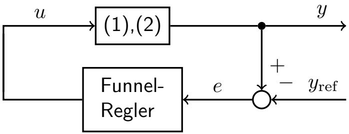

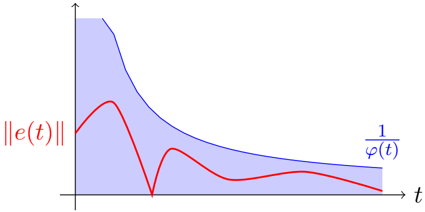

[Ilchmann & Ryan, 2009]: Regelung funktioniert wenn

- strikter Relativgrad 1 und “high-gain Matrix” $\Gamma(y)$ pos. def. für alle $y \in \mathbb{R}^m$
- (1) ist ISS (Nulldynamik ist stabil)

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (1)$$

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$$\Downarrow$$

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$$\tilde{\Gamma}(y(t))\,\dot{y}(t) = f_2(y(t)) + f_3(x(t)) + u(t), \quad y(0) = y^0. \quad (4)$$

$$
\begin{aligned}
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\end{aligned}
$$

$$\{(1),(2) \text{ mit str. Relativgrad } 1\} \overset{\tilde{\Gamma}=\Gamma^{-1}}{\subseteq} \{(3),(4)\}$$

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$$\dot{x}(t) = x(t) + y_1(t)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \dot{y}_3(t) \end{pmatrix} = \begin{pmatrix} 0 \\ y_2(t) \\ y_3(t) \end{pmatrix} + \begin{pmatrix} x(t) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix}$$

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$$y_1 = u_1 - \dot{u}_1 \quad \Longrightarrow \quad \int y_1 = \int u_1 - u_1$$

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$$y_1 = u_1 - \dot{u}_1 \quad \Longrightarrow \quad \int y_1 = \int u_1 - u_1$$

$$G(s) = \begin{bmatrix} s-1 & 0 & 0 \\ 0 & -1 & \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}, \quad \lim_{s\to\infty} \begin{bmatrix} s^{-1} & 0 & 0 \\ 0 & s^{0} & 0 \\ 0 & 0 & s^{1} \end{bmatrix} G(s) \in \mathbb{R}^{3\times 3} \text{ inv.}$$

Theorem (Funnel-Regelung)

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (3)$$

$$\tilde{\Gamma}(y(t))\,\dot{y}(t) = f_2(y(t)) + f_3(x(t)) + u(t), \qquad y(0) = y^0. \quad (4)$$

- (3) ist ISS
- $\tilde{\Gamma}(y) = RG(y)R^\top$, wobei $G(y) > 0 \;\forall\, y \in \mathbb{R}^m$
- $\operatorname{im} K = \ker R^\top$ und $K^\top f_2' K$ beschränkt
- $\hat{k} > \|(K^\top K)^{-1}\| \cdot \sup_{y\in\mathbb{R}^m} \|K^\top f_2'(y) K\|$
- y^0 so dass $K^\top\big(f_2(y^0) + f_3(x^0) - \hat{k}(y^0 - y_{\text{ref}}(0))\big) = 0$

Dann erreicht der *Funnel-Regler*

$$u(t) = -k(t)\,e(t), \qquad \text{wobei} \qquad e(t) = y(t) - y_{\text{ref}}(t)$$

$$k(t) = \hat{k}/\big(1 - \varphi(t)^2\|e(t)\|^2\big),$$

dass: $(x, y, k) \in L^\infty, \;\wedge\; \exists\, \varepsilon > 0 \;\forall\, t > 0 : \; \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon$

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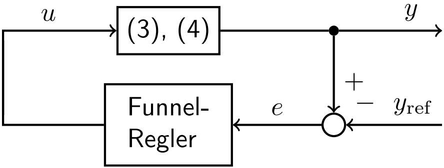

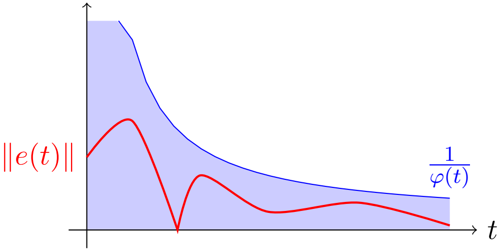

$\hat{k}$-Bedingung

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$$y(t) = [K, R] \begin{bmatrix} (K^\top K)^{-1} K^\top \\ (R^\top R)^{-1} R^\top \end{bmatrix} y(t) = [K, R] \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

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$$R^\top(4)\colon\ (R^\top R) G(y(t)) R^\top [K, R] \begin{pmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \end{pmatrix} = (R^\top R) G(y(t)) (R^\top R) \dot{y}_2(t) = \ldots$$

$$K^\top(4)\colon\ 0 = K^\top \big(f_2'(y(t))\dot{y}(t) + f_3'(x(t)) f_1(x(t), y(t)) - \dot{k}(t) e(t) - k(t)\dot{e}(t) \big)$$

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$$K^\top(4)\colon\ 0 = K^\top\big(f_2'(y(t))\dot{y}(t) + f_3'(x(t))f_1(x(t), y(t)) - \dot{k}(t)e(t) - k(t)\dot{e}(t)\big)$$

$$\Rightarrow\ 0 = \big(\underbrace{K^\top f_2'(y(t))K - k(t)\big(K^\top K + H(t, y_1(t), y_2(t))\big)}_{=:T}\big)\dot{y}_1(t) + \psi(t, y_1(t), y_2(t), \dot{y}_2(t))$$

$\hat{k}$-Bedingung

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (3)$$
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$$H \geq 0,\ k(t) \geq \hat{k} > \|(K^\top K)^{-1}\| \cdot \sup_{y\in\mathbb{R}^m} \|K^\top f_2'(y)K\| \quad \Rightarrow \quad T \text{ inv.}$$

Vergleich ODEs und DAEs

$$\begin{aligned} \dot{x}(t) &= f_1(x(t), y(t)), & x(0) &= x^0, \quad (1) \\ \dot{y}(t) &= f_2(y(t)) + f_3(x(t)) + \Gamma(y(t))u(t), & y(0) &= y^0. \quad (2) \end{aligned}$$

$$\begin{aligned} \dot{x}(t) &= f_1(x(t), y(t)), & x(0) &= x^0, \quad (3) \\ \tilde{\Gamma}(y(t))\,\dot{y}(t) &= f_2(y(t)) + f_3(x(t)) + \qquad u(t), & y(0) &= y^0. \quad (4) \end{aligned}$$

ODEs	DAEs
(1) ist ISS	(3) ist ISS
$\Gamma > 0$ (Relativgrad 1)	$\tilde{\Gamma} \geq 0$ (gemischter Relativgrad)
$\hat{k} = 1$	$\hat{k} > \|(K^\top K)^{-1}\| \cdot \sup_{y \in \mathbb{R}^m} \|K^\top f_2'(y) K\|$
$y^0 \in \mathbb{R}^m$	$K^\top \big(f_2(y^0) + f_3(x^0) - \hat{k}(y^0 - y_{\mathrm{ref}}(0))\big) = 0$