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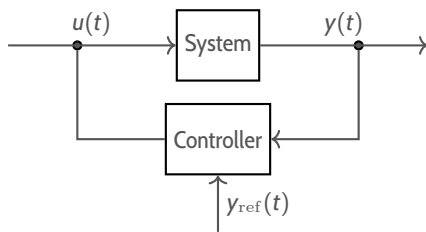
# INPUT-CONSTRAINED FUNNEL CONTROL

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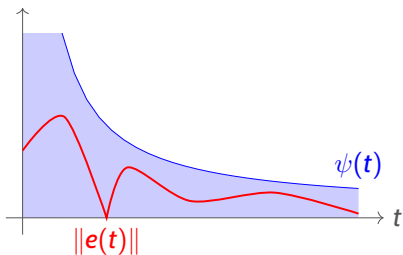
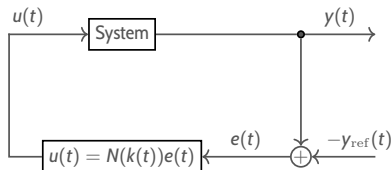
## Control objective



$$\begin{aligned}\dot{x}(t) &= f(t, x(t), u(t)), & x(0) &= x^0 \in \mathbb{R}^n \\ y(t) &= h(x(t))\end{aligned}$$

- **Goal:** simple controller, so that “ $y(t)$  tracks  $y_{\text{ref}}(t)$ ”
- only uses  $y(t)$ , no knowledge of  $x^0$  or system parameters
- only: **known relative degree and stable internal dynamics**

## Funnel control



$$k(t) = \frac{1}{1 - \|e(t)\|^2 / \psi(t)^2},$$

$N \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$  a surjection  
(e.g.  $N(s) = s \sin s$ )

[Ilchmann, Ryan, Sangwin '02]:  
Works, if

- relative degree = 1
- stable internal dynamics
- high-gain property

## Funnel control for systems with arbitrary relative degree

$$e_1(t) = e(t) = y(t) - y_{\text{ref}}(t),$$

$$e_2(t) = \dot{e}(t) + k_1(t)e_1(t),$$

$$e_3(t) = \ddot{e}(t) + k_2(t)e_2(t),$$

$$\vdots$$

$$e_r(t) = e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t),$$

$$u(t) = N(k_r(t))e_r(t)$$

$$k_i(t) = 1/(1 - \|e_i(t)\|^2/\psi_i(t)^2), \quad i = 1, \dots, r$$

### Theorem [B., Lê, Reis '18] & [B., Ilchmann, Ryan '21]

$$y_{\text{ref}} \in W^{r,\infty} \implies u, k_i, y^{(i)} \in L^\infty \text{ and } \|e(t)\| \leq \psi(t) - \varepsilon$$

## Funnel control under input constraints

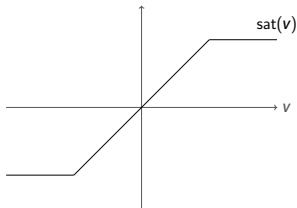
$$u(t) = \text{sat}(v(t))$$

such that for some  $M > 0$

- $\text{sat} \in C(\mathbb{R}^m, \mathbb{R}^m)$  bounded
  - $\forall v \in \mathbb{R}^m : \|v\| \leq M \implies \text{sat}(v) = v$
- } ( $\text{sat}_M$ )

- $\text{sat}$  with ( $\text{sat}_M$ ) is assumed arbitrary but *known*

○



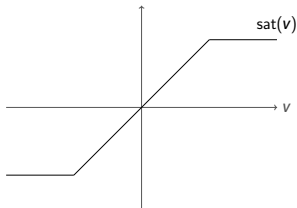
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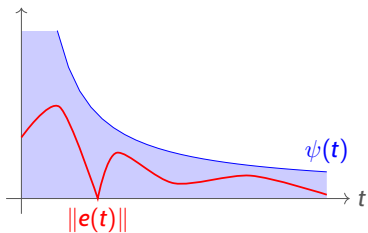
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- $$\left. \vphantom{\begin{matrix} \bullet \\ \bullet \end{matrix}} \right\} (\text{sat}_M)$$

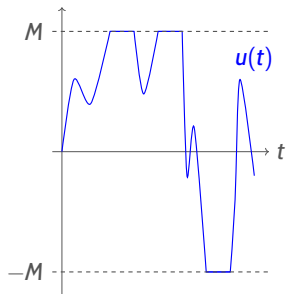
- $\text{sat}$  with  $(\text{sat}_M)$  is assumed arbitrary but *known*
- input constraints and output constraints are conflicting!



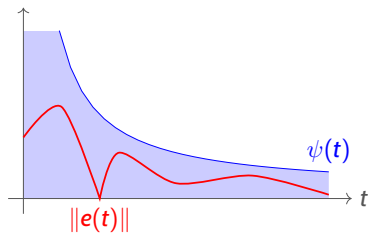
## Funnel control under input constraints



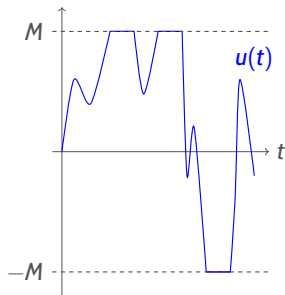
trade-off  
↔



## Funnel control under input constraints



trade-off  
↔



- design controller which “negotiates” this trade-off
- here: input constraints are strict, output constraints are flexible



## Funnel control under input constraints – known results

**so far only:** output constraints are strict, input constraints are flexible  
→ funnel control works, if  $M$  in  $(\text{sat}_M)$  suff. large

[Hopfe, Ilchmann, Ryan '10 a & b]: relative degree 1

[Hackl, Hopfe, Ilchmann, Müller, Trenn '13]: relative degree 2

[Liberzon & Trenn '13] (bang-bang funnel control): arbitrary relative degree

## Funnel control under input constraints – controller design

$$e_1(t) = e(t) = y(t) - y_{\text{ref}}(t),$$

$$e_2(t) = \dot{e}(t) + k_1(t)e_1(t),$$

$$\vdots$$

$$e_r(t) = e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t),$$

$$k_i(t) = 1/(1 - \|e_i(t)\|^2/\psi_i(t)^2),$$

$$v(t) = N(k_r(t))e_r(t)$$

$$\dot{\psi}_1(t) = -\alpha_1\psi_1(t) + \beta_1$$

$$+ \rho_1 \left( \psi_2(t) - \frac{\beta_2}{\alpha_2} \right),$$

$$\vdots$$

$$\dot{\psi}_{r-1}(t) = -\alpha_{r-1}\psi_{r-1}(t) + \beta_{r-1}$$

$$+ \rho_{r-1} \left( \psi_r(t) - \frac{\beta_r}{\alpha_r} \right),$$

$$\dot{\psi}_r(t) = -\alpha_r\psi_r(t) + \beta_r + \psi_r(t) \frac{\kappa(v(t))}{\|e_r(t)\|},$$

$$\kappa(v(t)) = \|v(t) - \text{sat}(v(t))\|$$

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$$v(t) = N(k_r(t))e_r(t)$$

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- $\alpha_1 > \alpha_2 > \dots > \alpha_r > 0$ ,  $p_i > 1$ ,  $\beta_i > 0$ ,  $\psi_i(0) > \frac{\beta_i}{\alpha_i}$
- Note:  $\|e_r(t)\| \ll 1 \implies \kappa(v(t)) = 0$
- $\psi_1, \dots, \psi_r$  are *not* bounded in general

## Lipschitz condition

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$$\implies \dot{y}(t) \geq y(t)^2 - M \implies y(t) \geq z(t),$$

$$z(t) = \sqrt{M} \frac{\sqrt{M} + 1 + (1 - \sqrt{M})e^{2\sqrt{M}t}}{\sqrt{M} + 1 - (1 - \sqrt{M})e^{2\sqrt{M}t}}$$

$M \geq 1 \implies z(\cdot)$  globally defined

$M < 1 \implies z(\cdot)$  blows up at  $\omega = \frac{1}{2\sqrt{M}} \ln \left( \frac{1 + \sqrt{M}}{1 - \sqrt{M}} \right)$

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$\rightarrow$  some kind of Lipschitz condition required

## Funnel control under input constraints – results

### Theorem 1 [B. '22]

- relative degree  $r$
- Lipschitz condition
- $y_{\text{ref}} \in \mathcal{C}^r$

$\implies \exists$  global solution  $(y, \psi_1, \dots, \psi_r)$  with  $\|e_i(t)\| \leq \varepsilon_i \psi_i(t)$  for  $i = 1, \dots, r - 1$  and  $\|e_r(t)\| < \psi_r(t)$

$\implies$  if  $v(t) = \text{sat}(v(t))$  for  $t \in [t_0, t_1]$ , then

$$\psi_i(t) \leq \frac{\beta_i}{\alpha_i} + \sum_{j=i}^r c_{ij}(t_0) e^{-\alpha_j(t-t_0)}$$



## Funnel control under input constraints – results

### Theorem 2 [B. '22]

- relative degree  $r$
- **Lipschitz condition** stable internal dynamics and high-gain property
- ~~$y_{\text{ref}} \in \mathcal{C}^r$~~   $y_{\text{ref}} \in W^{r,\infty}$

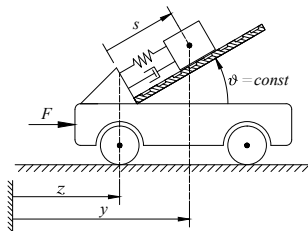
$\implies \forall \varepsilon \in (0, 1) \forall K > 0 \exists M = M(\varepsilon, K)$  such that

- $\forall$  sat with  $(\text{sat}_M)$ ,
- $\forall y^0$  with  $\|e_i(0)\| \leq \varepsilon \psi_i(0), i = 1, \dots, r$ ,
- $\forall y_{\text{ref}} \in W^{r,\infty}$  with  $\|y_{\text{ref}}^{(i)}\|_{\infty} \leq K, i = 0, \dots, r$ ,

$\exists$  global solution  $(y, \psi_1, \dots, \psi_r)$  with  $y \in W^{r,\infty}, \psi_i, k_i \in L^{\infty}$ ,

$\lim_{t \rightarrow \infty} \psi_i(t) \leq \frac{\beta_i}{\alpha_i}$  and  $\|v(t)\| \leq M$

## Simulation



angle:  $\vartheta = \pi/4$

$m_1 = 4, m_2 = 1, k = 2, d = 1$

$z(0) = s(0) = \dot{z}(0) = \dot{s}(0) = 0$

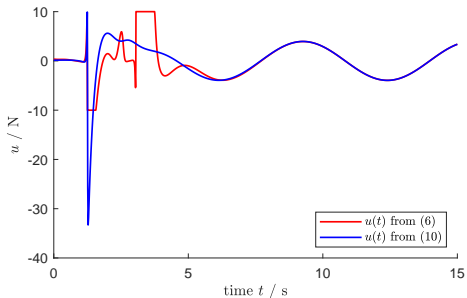
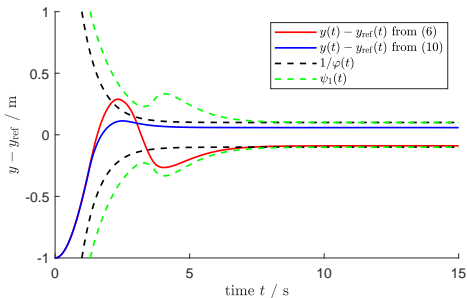
$y_{\text{ref}}(t) = \cos t,$

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos \vartheta \\ m_2 \cos \vartheta & m_2 \end{bmatrix} \begin{pmatrix} \ddot{z}(t) \\ \ddot{s}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t) + d\dot{s}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix},$$

$$y(t) = z(t) + s(t) \cos \vartheta$$

**Controller parameters:**  $\alpha_1 = 1.5, \alpha_2 = 0.9 \cdot \alpha_1, \beta_1 = 0.15, \beta_2 = 0.5 \cdot \alpha_2,$   
 $p_1 = 1.1, \psi_1^0 = 4.1, \psi_2^0 = 2, N(s) = -s^2 \cos s$

## Simulation



**blue curve:** funnel controller from [B., Ilchmann, Ryan '21] with

$$\varphi(t) = (4e^{-3t/2} + 0.1)^{-1}$$

**red curve:** input-constrained funnel controller from [B. '22]