On the regularization of linear descriptor systems

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Regularization of linear systems

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t)} \qquad [E, A, B]$$

$$E, A \in \mathbb{R}^{\ell \times n}, B \in \mathbb{R}^{\ell \times m}$$

Aim: $[E, A, B] \rightsquigarrow [\hat{E}, \hat{A}, \hat{B}]$
s.t. $s\hat{E} - \hat{A}$ is regular

1. approach: state feedback
2. approach: behavioral equivalence
3. approach: 1. + 2. + permutation

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Some basic notions

$sE - A$ is **regular**, if $\ell = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$

regular $sE - A$ has **index** $\nu \in \mathbb{N}_0$, if

$$S(sE - A)T = \begin{bmatrix} sI_r - J & 0 \\ 0 & sN - I_{n-r} \end{bmatrix},$$

$$\text{then} \quad \nu := \begin{cases} 0, & \text{if } r = n, \\ \min\left\{ k \in \mathbb{N}_0 \;\middle|\; N^k = 0 \right\}, & \text{if } r < n \end{cases}$$

augmented Wong sequences:

$$\mathcal{V}_0 := \mathbb{R}^n, \quad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i + \operatorname{im} B),\ i \geq 0, \quad \mathcal{V}^* := \bigcap_{i \geq 0} \mathcal{V}_i$$

$$\mathcal{W}_0 := \{0\}, \quad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i + \operatorname{im} B),\ i \geq 0, \quad \mathcal{W}^* := \bigcup_{i \geq 0} \mathcal{W}_i$$

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Approach 1: State feedback

Here: $\ell = n, \quad u(t) = Fx(t)$ for some $F \in \mathbb{R}^{m \times n}$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = (A + BF)x(t)$$

Thm. [Bunse-Gerstner et al. '92]:
$\exists\ F \in \mathbb{R}^{m \times n}: \ sE - (A + BF)$ is regular and has index ≤ 1
$\iff [E, A, B]$ is impulse controllable $(\operatorname{im} E + A \ker E + \operatorname{im} B = \mathbb{R}^n)$

Thm.: $\exists\ F \in \mathbb{R}^{m \times n}: \ sE - (A + BF)$ is regular
$\iff \dim(E\mathcal{V}^* + \operatorname{im} B) = \dim \mathcal{V}^*$ [Özçaldiran/Lewis '90]
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$\iff \operatorname{rk}_{\mathbb{R}(s)} [sE - A, B] = n$ [Duan/Zhang '03]

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Def.: $[E, A]$ is *autonomous*, if

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) \;\wedge\; Ex(0) = 0 \quad\Longrightarrow\quad x = 0$$

Lemma:

$$sE - A \text{ is regular} \quad\Longleftrightarrow\quad \ell = n \;\wedge\; [E, A] \text{ is autonomous}$$

Thm. [Fletcher '86]:

$$\exists\, F \in \mathbb{R}^{m\times n}:\; [E, A + BF] \text{ is aut.} \quad\Longleftrightarrow\quad \mathrm{rk}_{\mathbb{R}(s)}[sE - A, B] \geq n$$

Thm. [B./Reis '15]:

$$\begin{aligned} & \exists\, F \in \mathbb{R}^{m\times n}:\; [E, A + BF] \text{ is autonomous} \\ \Longleftrightarrow\quad & \mathrm{rk}_{\mathbb{R}(s)}[sE - A, B] \geq n \\ \Longleftrightarrow\quad & \dim(E\mathcal{V}^* + \operatorname{im} B) \geq \dim \mathcal{V}^* \\ \Longleftrightarrow\quad & \dim(A\mathcal{W}^* + \operatorname{im} B) \geq \dim \mathcal{W}^*. \end{aligned}$$

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Approach 2: Behavioral equivalence

Example:

$$\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}\dot{x}(t) = \begin{bmatrix}0 & 0\\1 & 0\\0 & 1\end{bmatrix}x(t) \overset{\text{beh.}}{\iff} \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\dot{x}(t) = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}x(t)$$

$$s\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \text{ is regular}$$

Def.: The *behavior* of $[E, A, B]$ is

$$\mathfrak{B}_{[E,A,B]} = \{\ (x,u) \in \mathcal{C}^\infty \times \mathcal{C}^\infty \ \mid E\dot{x}(t) = Ax(t) + Bu(t)\ \}.$$

Two systems $[E_i, A_i, B_i]$, $i = 1, 2$, are *behaviorally equivalent* ($[E_1, A_1, B_1] \simeq_{\mathfrak{B}} [E_2, A_2, B_2]$), if

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Def.: $[E,A,B]$ is *minimal*, if

$$\forall\, k \in \{1,\dots,\ell\}\ \forall\, [\tilde E, \tilde A, \tilde B] \text{ with } k \text{ rows}:$$
$$\left([E,A,B] \simeq_{\mathfrak{B}} \left[\begin{bmatrix} \tilde E \\ 0_{\ell-k,n} \end{bmatrix}, \begin{bmatrix} \tilde A \\ 0_{\ell-k,n} \end{bmatrix}, \begin{bmatrix} \tilde B \\ 0_{\ell-k,m} \end{bmatrix} \right] \implies k = \ell \right).$$

Thm. : $[E,A,B]$ is minimal

$\iff \operatorname{rk}_{\mathbb{R}(s)}[sE-A,B] = \ell$ [Polderman/Willems '98]

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Approach 3: Combination of 1 and 2 + permutation

Feedback transformation of $[E, A, B]$:

$$S\,[sE - A, -B]\,\begin{bmatrix} T & 0 \\ F & V \end{bmatrix}$$

Lemma: $[E_1, A_1, B_1] \simeq_{\mathfrak{B}} [E_2, A_2, B_2] \iff \exists\, U(s) \in \mathbf{Gl}_\ell(\mathbb{R}[s]):$

$$[sE_1 - A_1, -B_1] = U(s)[sE_2 - A_2, -B_1]$$

Behavioral and feedback transformation of $[E, A, B]$:

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Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Approaches 1 and 2 do not work!

x_2, x_3 are free $\rightsquigarrow$ x_2, x_3 are inputs in the behavioral framework

$u = 0$ $\rightsquigarrow$ u is a state in the behavioral framework

Permutation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{u}(t) \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1(t) \\ u(t) \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_2(t) \\ x_3(t) \end{pmatrix}$$

$$s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ is regular and has index 1}$$

$\rightarrow$ general (numerical) procedure derived by [Campbell et al. '12]

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$\exists\, U(s) \in \mathbf{Gl}_\ell(\mathbb{R}[s])$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$, $F \in \mathbb{R}^{m\times n}$ and a permutation matrix $P \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that

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where $s\hat{E}-\hat{A} \in \mathbb{R}[s]^{\hat{n}\times\hat{n}}$ is regular and has index at most 1

- $[\hat{E},\hat{A},\hat{B}]$ is minimal
- explicit construction of transformation matrices using feedback canonical form [Loiseau et al. '91]
- the subsystem $[\hat{E},\hat{A},\hat{B}]$ is fully determined by the augmented Wong sequences

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