Regularization of linear descriptor systems

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Regularization of linear systems

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t)} \qquad [E, A, B]$$

$$E, A \in \mathbb{R}^{\ell \times n}, B \in \mathbb{R}^{\ell \times m}$$

Aim: $[E, A, B] \rightsquigarrow [\hat{E}, \hat{A}, \hat{B}]$
s.t. $s\hat{E} - \hat{A}$ is regular

1. approach: state feedback
2. approach: behavioral equivalence
3. approach: 1. + 2. + permutation

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Some basic notions

$sE - A$ is **regular**, if $\ell = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$

regular $sE - A$ has **index** $\nu \in \mathbb{N}_0$, if

$$S(sE - A)T = \begin{bmatrix} sI_r - J & 0 \\ 0 & sN - I_{n-r} \end{bmatrix},$$

$$\text{then} \quad \nu := \begin{cases} 0, & \text{if } r = n, \\ \min \left\{ k \in \mathbb{N}_0 \;\middle|\; N^k = 0 \right\}, & \text{if } r < n \end{cases}$$

augmented Wong sequences:

$$\mathcal{V}_0 := \mathbb{R}^n, \quad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i + \operatorname{im} B),\ i \geq 0, \quad \mathcal{V}^* := \bigcap_{i \geq 0} \mathcal{V}_i$$

$$\mathcal{W}_0 := \{0\}, \quad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i + \operatorname{im} B),\ i \geq 0, \quad \mathcal{W}^* := \bigcup_{i \geq 0} \mathcal{W}_i$$

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Approach 1: State feedback

Here: $\ell = n$, $u(t) = Fx(t)$ for some $F \in \mathbb{R}^{m \times n}$

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = (A + BF)x(t)$$

Thm. [Bunse-Gerstner et al. '92]:
$\exists\, F \in \mathbb{R}^{m \times n}:\ sE - (A+BF)$ is regular and has index ≤ 1
$\iff [E, A, B]$ is impulse controllable ($\operatorname{im} E + A \ker E + \operatorname{im} B = \mathbb{R}^n$)

Thm.: $\exists\, F \in \mathbb{R}^{m \times n}:\ sE - (A+BF)$ is regular
$\iff \dim(E\mathcal{V}^* + \operatorname{im} B) = \dim \mathcal{V}^*$ [Özçaldiran/Lewis '90]
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$\iff \operatorname{rk}_{\mathbb{R}(s)}[sE - A, B] = n$ [Duan/Zhang '03]

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Def.: $[E, A]$ is *autonomous*, if

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) \;\wedge\; Ex(0) = 0 \quad\Longrightarrow\quad x = 0$$

Lemma:

$$sE - A \text{ is regular} \quad\Longleftrightarrow\quad \ell = n \;\wedge\; [E, A] \text{ is autonomous}$$

Thm. [Fletcher '86]:

$$\exists\, F \in \mathbb{R}^{m\times n} : \; [E, A + BF] \text{ is aut.} \quad\Longleftrightarrow\quad \mathrm{rk}_{\mathbb{R}(s)}[sE - A, B] \geq n$$

Thm. [B./Reis '15]:

$$\begin{aligned} &\exists\, F \in \mathbb{R}^{m\times n} : \; [E, A + BF] \text{ is autonomous} \\ \Longleftrightarrow\quad & \mathrm{rk}_{\mathbb{R}(s)}[sE - A, B] \geq n \\ \Longleftrightarrow\quad & \dim(E\mathcal{V}^* + \mathrm{im}\, B) \geq \dim \mathcal{V}^* \\ \Longleftrightarrow\quad & \dim(A\mathcal{W}^* + \mathrm{im}\, B) \geq \dim \mathcal{W}^*. \end{aligned}$$

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Approach 2: Behavioral equivalence

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) \quad \overset{\text{beh.}}{\Longleftrightarrow} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t)$$

$$s \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is regular}$$

Def.: The *behavior* of $[E, A, B]$ is

$$\mathfrak{B}_{[E,A,B]} = \{ \, (x, u) \in \mathcal{C}^\infty \times \mathcal{C}^\infty \; | \; E\dot{x}(t) = Ax(t) + Bu(t) \, \} \, .$$

Two systems $[E_i, A_i, B_i]$, $i = 1, 2$, are *behaviorally equivalent* ($[E_1, A_1, B_1] \simeq_{\mathfrak{B}} [E_2, A_2, B_2]$), if

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Def.: $[E, A, B]$ is *minimal*, if

$$\forall\, k \in \{1, \ldots, \ell\}\ \forall\, [\tilde{E}, \tilde{A}, \tilde{B}] \text{ with } k \text{ rows}:$$
$$\left([E, A, B] \simeq_{\mathfrak{B}} \left[\begin{bmatrix} \tilde{E} \\ 0_{\ell-k,n} \end{bmatrix}, \begin{bmatrix} \tilde{A} \\ 0_{\ell-k,n} \end{bmatrix}, \begin{bmatrix} \tilde{B} \\ 0_{\ell-k,m} \end{bmatrix} \right] \implies k = \ell \right).$$

Thm. : $[E, A, B]$ is minimal

$\iff \operatorname{rk}_{\mathbb{R}(s)}[sE - A, B] = \ell$ [Polderman/Willems '98]

$\iff E\mathcal{V}^* + A\mathcal{W}^* + \operatorname{im} B = \mathbb{R}^\ell$ [B./Reis '15]

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$$\begin{aligned} & [E, A, B] \text{ is minimal} \\ \iff\ & \mathrm{rk}_{\mathbb{R}(s)}[sE - A, B] = \ell \quad \text{[Polderman/Willems '98]} \\ \iff\ & E\mathcal{V}^* + A\mathcal{W}^* + \operatorname{im} B = \mathbb{R}^\ell \quad \text{[B./Reis '15]} \end{aligned}$$

Approach 3: Combination of 1 and 2 + permutation

Feedback transformation of $[E, A, B]$:

$$S\,[sE - A, -B]\,\begin{bmatrix} T & 0 \\ F & V \end{bmatrix}$$

Lemma: $[E_1, A_1, B_1] \simeq_{\mathfrak{B}} [E_2, A_2, B_2] \iff \exists\, U(s) \in \mathbf{Gl}_\ell(\mathbb{R}[s]) :$

$$[sE_1 - A_1, -B_1] = U(s)[sE_2 - A_2, -B_1]$$

Behavioral and feedback transformation of $[E, A, B]$:

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Example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Approaches 1 and 2 do not work!

x_2, x_3 are free $\rightsquigarrow$ x_2, x_3 are inputs in the behavioral framework

$u = 0$ $\rightsquigarrow$ u is a state in the behavioral framework

Permutation:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{x}_1(t) \\ \dot{u}(t) \end{pmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x_1(t) \\ u(t) \end{pmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x_2(t) \\ x_3(t) \end{pmatrix}$$

$$s \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \text{ is regular and has index 1}$$

$\rightarrow$ general (numerical) procedure derived by [Campbell et al. '12]

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$\exists\, U(s) \in \mathbf{Gl}_\ell(\mathbb{R}[s])$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$, $F \in \mathbb{R}^{m \times n}$ and a permutation matrix $P \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that

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where $s\hat{E} - \hat{A} \in \mathbb{R}[s]^{\hat{n} \times \hat{n}}$ is regular and has index at most 1

- $[\hat{E}, \hat{A}, \hat{B}]$ is minimal
- explicit construction of transformation matrices using feedback canonical form [Loiseau et al. '91]
- the subsystem $[\hat{E}, \hat{A}, \hat{B}]$ is fully determined by the augmented Wong sequences

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$$[sE - A, -B]$$

$$S_1 B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$S_1[sE - A, -B] = \left[\begin{array}{c|c} sE_1 - A_1 & 0 \\ sE_2 - A_2 & -B_2 \end{array}\right]$$

$$S_1 B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

$$S_1[sE-A,-B] = \left[\begin{array}{c|c} sE_1 - A_1 & 0 \\ sE_2 - A_2 & -B_2 \end{array}\right]$$

Staircase form [Beelen/Van Dooren '88a]:

$$S_2(sE_1 - A_1)T_2 = \begin{bmatrix} sE_\eta - A_\eta & 0 & 0 & 0 \\ * & sE_\infty - A_\infty & 0 & 0 \\ * & * & sE_f - A_f & 0 \\ * & * & * & sE_\varepsilon - A_\varepsilon \end{bmatrix}$$

1. $E_\eta, A_\eta \in \mathbb{R}^{l_\eta \times n_\eta}, l_\eta > n_\eta$, are such that $\mathrm{rk}_\mathbb{C}(\lambda E_\eta - A_\eta) = n_\eta$ and $\mathrm{rk}\, E_\eta = n_\eta$;
2. $E_\infty, A_\infty \in \mathbb{R}^{n_\infty \times n_\infty}$, A_∞ is invertible and $A_\infty^{-1} E_\infty$ is nilpotent;
3. $E_f, A_f \in \mathbb{R}^{n_f \times n_f}$ and E_f is invertible;
4. $E_\varepsilon, A_\varepsilon \in \mathbb{R}^{l_\varepsilon \times n_\varepsilon}, l_\varepsilon < n_\varepsilon$, are such that $\mathrm{rk}_\mathbb{C}(\lambda E_\varepsilon - A_\varepsilon) = l_\varepsilon$ and $\mathrm{rk}\, E_\varepsilon = l_\varepsilon$.

$$S[sE-A,-B]\begin{bmatrix}T & 0\\ 0 & I\end{bmatrix} = \left[\begin{array}{cccc|c} sE_\eta - A_\eta & 0 & 0 & 0 & 0\\ * & sE_\infty - A_\infty & 0 & 0 & 0\\ * & * & sE_f - A_f & 0 & 0\\ * & * & * & sE_\varepsilon - A_\varepsilon & 0\\ * & * & * & * & -B_2 \end{array}\right]$$

Staircase form [Beelen/Van Dooren '88a]:

$$S_2(sE_1 - A_1)T_2 = \begin{bmatrix} sE_\eta - A_\eta & 0 & 0 & 0\\ * & sE_\infty - A_\infty & 0 & 0\\ * & * & sE_f - A_f & 0\\ * & * & * & sE_\varepsilon - A_\varepsilon \end{bmatrix}$$

$$S[sE-A,-B]\begin{bmatrix}T & 0\\ 0 & I\end{bmatrix} = \left[\begin{array}{cccc|c} sE_\eta - A_\eta & 0 & 0 & 0 & 0\\ * & sE_\infty - A_\infty & 0 & 0 & 0\\ * & * & sE_f - A_f & 0 & 0\\ * & * & * & sE_\varepsilon - A_\varepsilon & 0\\ * & * & * & * & -B_2\end{array}\right]$$

Unimodular embedding [Beelen/Van Dooren '88b]:

$$U_1(s) := \begin{bmatrix} K & sE_\eta - A_\eta & 0\\ 0 & * & sE_\infty - A_\infty\end{bmatrix}$$

$$U(s)^{-1}[sE-A,-B]\begin{bmatrix}T & 0\\ 0 & I\end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0\\ -I_{n_\eta+n_\infty} & 0 & 0 & 0\\ sE_{31}-A_{31} & sE_f-A_f & 0 & 0\\ sE_{41}-A_{41} & * & sE_\varepsilon-A_\varepsilon & 0\\ sE_{51}-A_{51} & * & * & -B_2\end{array}\right]$$

Unimodular embedding [Beelen/Van Dooren '88b]:

$$U_1(s) := \begin{bmatrix}K & sE_\eta-A_\eta & 0\\ 0 & * & sE_\infty-A_\infty\end{bmatrix}$$

$$U(s)^{-1}[sE-A,-B]\begin{bmatrix}T & 0\\ 0 & I\end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0\\ -I_{n_\eta+n_\infty} & 0 & 0 & 0\\ sE_{31}-A_{31} & sE_f - A_f & 0 & 0\\ sE_{41}-A_{41} & * & sE_\varepsilon - A_\varepsilon & 0\\ sE_{51}-A_{51} & * & * & -B_2\end{array}\right]$$

$$U_2(s) = \begin{bmatrix} I_{l_\eta-n_\eta} & 0 & 0 & 0 & 0\\ 0 & I_{n_\eta+n_\infty} & 0 & 0 & 0\\ 0 & -sE_{31}+A_{31} & I_{n_f} & 0 & 0\\ 0 & -sE_{41}+A_{41} & 0 & I_{l_\varepsilon} & 0\\ 0 & -sE_{51}+A_{51} & 0 & 0 & I_r\end{bmatrix}$$

$$U(s)^{-1}[sE-A,-B]\begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -I_{n_\eta+n_\infty} & 0 & 0 & 0 \\ 0 & sE_f - A_f & 0 & 0 \\ 0 & * & sE_\varepsilon - A_\varepsilon & 0 \\ 0 & * & * & -B_2 \end{array}\right]$$

$$U_2(s) = \begin{bmatrix} I_{l_\eta - n_\eta} & 0 & 0 & 0 & 0 \\ 0 & I_{n_\eta+n_\infty} & 0 & 0 & 0 \\ 0 & -sE_{31}+A_{31} & I_{n_f} & 0 & 0 \\ 0 & -sE_{41}+A_{41} & 0 & I_{l_\varepsilon} & 0 \\ 0 & -sE_{51}+A_{51} & 0 & 0 & I_r \end{bmatrix}$$

$$U(s)^{-1}[sE-A,-B]\begin{bmatrix}T & 0\\ 0 & I\end{bmatrix} = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0\\ -I_{n_\eta+n_\infty} & 0 & 0 & 0\\ 0 & sE_f - A_f & 0 & 0\\ 0 & * & sE_\varepsilon - A_\varepsilon & 0\\ 0 & * & * & -B_2 \end{array}\right]$$

$$E_\varepsilon T_3 = [\Sigma_1, 0], \quad B_2 V_3 = [\Sigma_2, 0]$$

Σ_1, Σ_2 invertible

$$U(s)^{-1}[sE-A,-B]\begin{bmatrix} T & 0 \\ 0 & V \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ -I_{n_\eta+n_\infty} & 0 & 0 & 0 & 0 & 0 \\ 0 & sE_f - A_f & 0 & 0 & 0 & 0 \\ 0 & * & s\Sigma_1 - A_{43} & -A_{44} & 0 & 0 \\ 0 & * & * & sE_{54} - A_{54} & -\Sigma_2 & 0 \end{array}\right]$$

$$E_\varepsilon T_3 = [\Sigma_1, 0], \quad B_2 V_3 = [\Sigma_2, 0]$$

Σ_1, Σ_2 invertible

$$U(s)^{-1}[sE-A,-B]\begin{bmatrix} T & 0 \\ 0 & V \end{bmatrix}$$

$$= \left[\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ -I_{n_\eta+n_\infty} & 0 & 0 & 0 & 0 & 0 \\ 0 & sE_f - A_f & 0 & 0 & 0 & 0 \\ 0 & * & s\Sigma_1 - A_{43} & -A_{44} & 0 & 0 \\ 0 & * & * & sE_{54} - A_{54} & -\Sigma_2 & 0 \end{array}\right]$$

$$S_4E_{54}T_4 = \begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_4\Sigma_2V_4 = \begin{bmatrix} \Sigma_{21} & 0 \\ * & \Sigma_{22} \end{bmatrix}$$

$\Sigma_3, \Sigma_{21}, \Sigma_{22}$ invertible

$$U(s)^{-1}[sE-A,-B]\begin{bmatrix} T & 0 \\ 0 & V \end{bmatrix}$$

$$= \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -I_{n_\eta+n_\infty} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & sE_f - A_f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & s\Sigma_1 - A_{43} & -\tilde{A}_{44} & -\tilde{A}_{45} & 0 & 0 & 0 \\ 0 & * & * & s\Sigma_3 - \tilde{A}_{54} & -\tilde{A}_{55} & -\Sigma_{21} & 0 & 0 \\ 0 & * & * & -\tilde{A}_{64} & -\tilde{A}_{65} & * & -\Sigma_{22} & 0 \end{array}\right]$$

$$S_4 E_{54} T_4 = \begin{bmatrix} \Sigma_3 & 0 \\ 0 & 0 \end{bmatrix}, \quad S_4 \Sigma_2 V_4 = \begin{bmatrix} \Sigma_{21} & 0 \\ * & \Sigma_{22} \end{bmatrix}$$

$\Sigma_3, \Sigma_{21}, \Sigma_{22}$ invertible

$$U(s)^{-1}[sE-A,-B]\begin{bmatrix}T & 0\\ 0 & V\end{bmatrix}$$

$$=\left[\begin{array}{ccccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
-I_{n_\eta+n_\infty} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & sE_f-A_f & 0 & 0 & 0 & 0 & 0 & 0\\
0 & * & s\Sigma_1-A_{43} & -\tilde{A}_{44} & -\tilde{A}_{45} & 0 & 0 & 0\\
0 & * & * & s\Sigma_3-\tilde{A}_{54} & -\tilde{A}_{55} & -\Sigma_{21} & 0 & 0\\
0 & * & * & -\tilde{A}_{64} & -\tilde{A}_{65} & * & -\Sigma_{22} & 0
\end{array}\right]$$

$$P=\begin{bmatrix}
I_{n_\eta+n_\infty+n_f+l_\varepsilon+q} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & I_{n_\varepsilon-l_\varepsilon-q} & 0\\
0 & 0 & I_q & 0 & 0\\
0 & I_{r-q} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & I_{m-r}
\end{bmatrix}$$

$$U(s)^{-1}[sE-A,-B]\begin{bmatrix} T & 0\\ 0 & V\end{bmatrix} P$$

$$= \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -I_{n_\eta+n_\infty} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & sE_f - A_f & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & * & s\Sigma_1 - A_{43} & -\tilde{A}_{44} & 0 & 0 & -\tilde{A}_{45} & 0\\ 0 & * & * & s\Sigma_3 - \tilde{A}_{54} & 0 & -\Sigma_{21} & -\tilde{A}_{55} & 0\\ 0 & * & * & -\tilde{A}_{64} & -\Sigma_{22} & * & -\tilde{A}_{65} & 0 \end{array}\right]$$

$$U(s)^{-1}[sE - A, -B] \begin{bmatrix} T & 0 \\ 0 & V \end{bmatrix} P$$

$$= \left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -I_{n_\eta + n_\infty} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & sE_f - A_f & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & s\Sigma_1 - A_{43} & -\tilde{A}_{44} & 0 & 0 & -\tilde{A}_{45} & 0 \\ 0 & * & * & s\Sigma_3 - \tilde{A}_{54} & 0 & -\Sigma_{21} & -\tilde{A}_{55} & 0 \\ 0 & * & * & -\tilde{A}_{64} & -\Sigma_{22} & * & -\tilde{A}_{65} & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0_{(l_\eta - n_\eta)\times \hat{n}} & 0_{(l_\eta - n_\eta)\times m} \\ sE_{\mathrm{reg}} - A_{\mathrm{reg}} & -B_{\mathrm{reg}} \end{array}\right]$$

$sE_{\mathrm{reg}} - A_{\mathrm{reg}} \in \mathbb{R}[s]^{\hat{n}\times\hat{n}}$ is regular and has index at most one

Computational aspects

- the algorithm is numerically stable
- the computational complexity, given $E, A \in \mathbb{R}^{\ell \times n}$, $B \in \mathbb{R}^{\ell \times m}$ is
$$O\big(\ell^2(\ell + n + m) + n^2(\ell + n) + m^3\big)$$
- the inverse $V(s) = U(s)^{-1}$ can be computed [Beelen/Van Dooren '88b]; this requires inversion of a triangular matrix and has a cost of
$$O(q\ell^3),$$
where $q = \deg V(s)$