

Disturbance decoupling for DAEs by behavioral feedback

Thomas Berger

Fachbereich Mathematik, Universität Hamburg

Berlin, July 21, 2016

$$\begin{aligned} \frac{d}{dt}Ex(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned}$$

$$[E, A, B, C]$$

$$E, A \in \mathbb{R}^{\ell \times n}, B \in \mathbb{R}^{\ell \times m}, C \in \mathbb{R}^{p \times n}$$

$$\mathfrak{B}_{[E,A,B,C]} = \{ (x, u, y) \mid [E, A, B, C] \text{ is satisfied} \}$$

Some basic notions

$sE - A$ is **regular**, if $\ell = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$

invariant subspace algorithm [WONHAM '85]

$$\mathcal{V}_0 := \ker C, \quad \mathcal{V}_{i+1} := A^{-1}(\mathcal{V}_i + \text{im } B) \cap \ker C, \quad i \geq 0$$

controllability subspace algorithm [WONHAM '85]

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generalized Wong sequences:

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$sE - A$ regular: disturbance decoupled $\iff C(sE - A)^{-1}Q = 0$

$$\begin{aligned} \Phi_{[E,A,B,C]} : C^\infty &\rightarrow \mathcal{P}(C^\infty), \\ u &\mapsto \left\{ y \in C^\infty \mid \exists x \in C^\infty : (x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \right\} \end{aligned}$$

Def. ($B = 0$): $[E, A, Q, C]$ is DD $:\iff$

$$\forall d_1, d_2 \in C^\infty : \Phi_{[E,A,Q,C]}(d_1) = \Phi_{[E,A,Q,C]}(d_2)$$

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Thm.:

$$[E, A, Q, C] \text{ is DD} \iff \text{im } Q \subseteq E\mathcal{V}_{[E,A,0,C]}^* + A\mathcal{W}_{[E,A,0,C]}^*$$

Thm. [WONHAM '85]:

$$\exists F : [I, A + BF, Q, C] \text{ is DD} \iff \text{im } Q \subseteq \mathcal{V}_{[I,A,B,C]}^*$$

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- $\dim(E\mathcal{R}^* + \text{im } Q) \leq \dim \mathcal{R}^*$
 where $\mathcal{R}^* = \mathcal{V}_{[E,A,[B,Q],C]}^* \cap \mathcal{W}_{[E,A,[B,Q],C]}^*$

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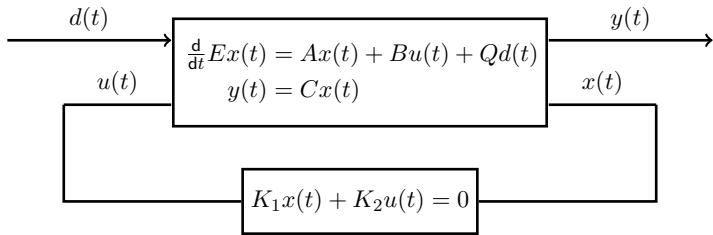
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Behavioral feedback



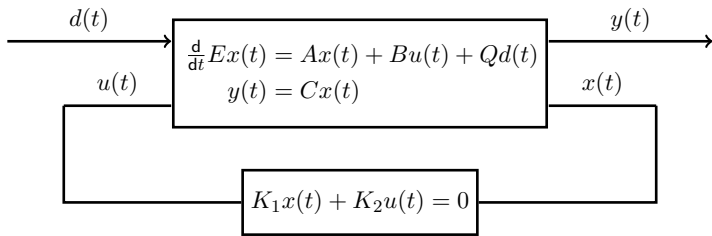
closed-loop system:

$$[E^K, A^K, Q^K, C^K] = \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_1 & K_2 \end{bmatrix}, \begin{bmatrix} Q \\ 0 \end{bmatrix}, [C, 0] \right]$$

Def.: $K = [K_1, K_2]$ is *compatible* for $[E, A, B, C]$, if

$$\forall (x, u, y) \in \mathfrak{B}_{[E, A, B, C]} \exists (\tilde{x}, \tilde{u}) \in \mathfrak{B}_{[E^K, A^K, 0, 0]} : Ex(0) = E\tilde{x}(0)$$

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Still a lot of freedom in the choice of K !

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Open problem due to [LEBRET '94]

Disturbance decoupling does not guarantee disturbance rejection!

Example:

$$\dot{x}_1(t) = x_2(t) + d(t), \quad y(t) = x_1(t)$$

is DD since $\Phi(d_1) = \Phi(d_2) = \mathcal{C}^\infty$ for all $d_1, d_2 \in \mathcal{C}^\infty$

but: y still depends on d

$$y(t) = x_1(0) + \int_0^t d(s) - x_2(s) \, ds$$

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state feedback $u = f_1x_1 + f_2x_2 + f_3x_3$, when $y = 0$

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$$\Rightarrow d_2 = (1 - f_2)x_2 \wedge \dot{x}_2 = d_1,$$

$$f_2 = 1 \Rightarrow d_2 = 0,$$

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additional assumption: $\frac{d}{dt}E^K z(t) = A^K z(t)$ has unique solutions

i.e., $[E^K, A^K]$ is autonomous $\iff \ker_{\mathbb{R}[s]}(sE^K - A^K) = \{0\}$

Aim: Characterize existence of compatible K s.t.

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Let inv. S, T be s.t.

$$S[sE - A, -B]T = \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \end{bmatrix}$$

where $\text{rk}(\lambda E_{22} - A_{22}) = n_3$ for all $\lambda \in \mathbb{C}$;

$$SQ = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \quad [C, 0]T = [C_1, C_2, C_3], \quad KT = [K_1, K_2, K_3]$$

Lemma:

K is compatible for $[E, A, B, C]$ s.t. $[E^K, A^K]$ is autonomous
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$$\forall d \in \mathcal{C}^\infty \exists (x_1, x_2, x_3) \in \mathcal{C}^\infty : \quad \begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + Q_1d \\ E_{22}\dot{x}_3 &= \quad \quad \quad A_{22}x_3 + Q_2d \\ 0 &= C_1x_1 + C_2x_2 + C_3x_3 \\ 0 &= K_1x_1 + K_2x_2 + K_3x_3 \end{aligned}$$

By Lemma: \exists inv. V s.t.: $V[K_1, K_2, K_3] = \begin{bmatrix} Z_1 & I_{n_2} & Z_2 \\ 0 & 0 & Z_3 \end{bmatrix}$

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$\implies \exists F = [Z_1, Z_2]:$

$$\left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \\ C_1 & C_3 \end{bmatrix} - \begin{bmatrix} A_{12} \\ 0 \\ C_2 \end{bmatrix} F, \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix}, 0 \right] \text{ is DD}$$

equivalently, by [BANASZUK ET AL. '90], with

$$[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{Q}] = \left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \\ C_1 & C_3 \end{bmatrix}, \begin{bmatrix} A_{12} \\ 0 \\ C_2 \end{bmatrix}, \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix} \right],$$

- $\text{im } \tilde{Q} \subseteq \tilde{E}\mathcal{V}_{[\tilde{E}, \tilde{A}, \tilde{B}, 0]}^* + \tilde{A}\mathcal{W}_{[\tilde{E}, \tilde{A}, \tilde{B}, 0]}^* + \text{im } \tilde{B},$
- $\dim \left(\tilde{E}\tilde{\mathcal{R}}^* + \text{im } \tilde{Q} \right) \leq \dim \tilde{\mathcal{R}}^*$
 where $\tilde{\mathcal{R}}^* = \mathcal{V}_{[\tilde{E}, \tilde{A}, [\tilde{B}, \tilde{Q}], 0]}^* \cap \mathcal{W}_{[\tilde{E}, \tilde{A}, [\tilde{B}, \tilde{Q}], 0]}^*$

$\implies \exists F = [Z_1, Z_2]:$

$$\left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \\ C_1 & C_3 \end{bmatrix} - \begin{bmatrix} A_{12} \\ 0 \\ C_2 \end{bmatrix} F, \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix}, 0 \right] \text{ is DD}$$

equivalently, by [BANASZUK ET AL. '90], with

$$[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{Q}] = \left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \\ C_1 & C_3 \end{bmatrix}, \begin{bmatrix} A_{12} \\ 0 \\ C_2 \end{bmatrix}, \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix} \right],$$

- $\text{im } \tilde{Q} \subseteq \tilde{E}\mathcal{V}_{[\tilde{E}, \tilde{A}, \tilde{B}, 0]}^* + \tilde{A}\mathcal{W}_{[\tilde{E}, \tilde{A}, \tilde{B}, 0]}^* + \text{im } \tilde{B},$
- $\dim \left(\tilde{E}\tilde{\mathcal{R}}^* + \text{im } \tilde{Q} \right) \leq \dim \tilde{\mathcal{R}}^*$
 where $\tilde{\mathcal{R}}^* = \mathcal{V}_{[\tilde{E}, \tilde{A}, [\tilde{B}, \tilde{Q}], 0]}^* \cap \mathcal{W}_{[\tilde{E}, \tilde{A}, [\tilde{B}, \tilde{Q}], 0]}^*$