UNIVERSITÄT PADERBORN

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FUNNEL CONTROL IN THE PRESENCE OF DELAYS

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System class

$$\dot{x}_{i,j}(t) = x_{i,j+1}(t),$$

$$\dot{x}_{i,n}(t) = f_i(t, \bar{x}(t)) + \sum_{k=1}^{m} g_{i,k} u_k(t - \tau_u),$$

- system state $\bar{x} = (x_{1,1}, \ldots, x_{1,n}, \ldots, x_{m,1}, \ldots, x_{m,n})^\top$
- control inputs $u_i, i = 1, \ldots, m$
- system outputs $y_i := x_{i,1}, i = 1, \ldots, m$
- input delay $\tau_u > 0$, measurement delay $\tau_s > 0$
- initial history $\bar{x}_n|_{[-\tau_s - \tau_u, 0]} = \varphi \in C([-\tau_s - \tau_u, 0], \mathbb{R}^{nm})$

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Assumption: there exists $d_i \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R})$ s.t.

$$\forall\, (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{mn} : \; |f_i(t, x)| \leq |d_i(t)|(\|x\| + 1)$$

Assumption on nonlinearities

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$$\begin{aligned} & \forall\, t \in [-\tau, 0] :\ u(t) = 0 \\ \implies \quad & \dot{x}(t) = x(t)^2, \quad x(0) = x^0, \quad t \in [0, \tau] \\ \implies \quad & x(t) = \left(\frac{1}{x^0} - t\right)^{-1}, \quad t \in [0, \min\{\tau, 1/x^0\}) \end{aligned}$$

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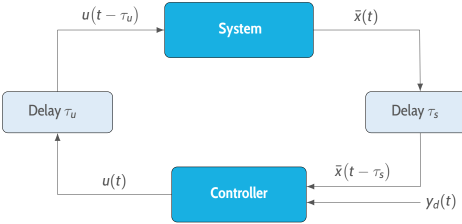
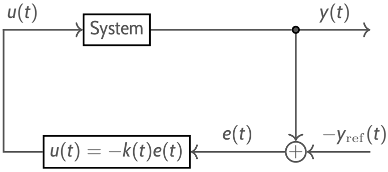
$$\frac{1}{x^0} < \tau \quad \Longrightarrow \quad \text{blow-up of the solutions}$$

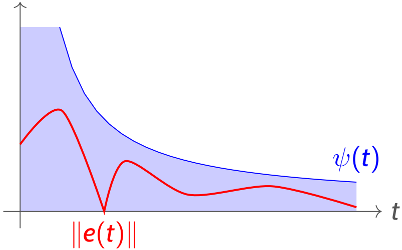
Control objective



- given $\psi_{i,1}$ (positive, bounded, bounded reciprocal) and reference signals $y_{d,i}$, the controller achieves $|y_i(t) - y_{d,i}(t)| < \psi_{i,1}(t)$
- all closed-loop signals are bounded
- the controller does not require knowledge of the system parameters and is of low complexity

Funnel control – without delays





$$k(t) = \frac{1}{1 - \|e(t)/\psi(t)\|^2}$$

[Ilchmann, Ryan, Sangwin '02]:
Works, if

- order $n = 1$
- minimum phase

Funnel control for systems of arbitrary order $n \in \mathbb{N}$

$$\dot{x}_{i,j}(t) = x_{i,j+1}(t), \quad \dot{x}_{i,n}(t) = f_i(t, \bar{x}(t)) + \sum_{k=1}^{m} g_{i,k} u_k(t),$$

$$\begin{aligned} z_{i,1}(t) &= \big(x_{i,1}(t) - y_{d,i}(t)\big)/\psi_{i,1}(t), \\ z_{i,j}(t) &= \big(x_{i,j}(t) + k_{i,j-1}(t) z_{i,j-1}(t)\big)/\psi_{i,j}(t), \\ k_{i,j-1}(t) &= 1/(1 - z_{i,j-1}(t)^2), \quad j = 2, \ldots, n \end{aligned}$$

$$z_n(t) = \big(z_{1,n}(t), \ldots, z_{m,n}(t)\big)^\top,$$

$$u_i(t) = -z_{i,n}(t)/(1 - \|z_n(t)\|^2)$$

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Theorem [B., Lê, Reis '18] (with modifications)

$y_d \in W^{2,\infty}$, $G = (g_{i,k}) \in \mathbb{R}^{m \times m}$ is pos. definite

$\implies u_i, k_{i,j}, x_{i,j} \in L^\infty$ and $|y_i(t) - y_{d,i}(t)| \le \psi_{i,1}(t) - \varepsilon$

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In the presence of delays a modification is necessary!

Funnel control for systems with delays

$$\dot{x}_{i,j}(t) = x_{i,j+1}(t), \quad \dot{x}_{i,n}(t) = f_i(t, \bar{x}(t)) + \sum_{k=1}^{m} g_{i,k} u_k(t - \tau_u),$$

$$z_{i,1}(t) = \big(x_{i,1}(t-\tau_s) - y_{d,i}(t-\tau_s) + l_{i,1}(t)\big)/\psi_{i,1}(t-\tau_s),$$

$$z_{i,j}(t) = \Big(x_{i,j}(t-\tau_s) + k_{i,j-1}(t) z_{i,j-1}(t) + \textstyle\sum_{k=1}^{j} \binom{j-1}{j-k} (-\alpha)^{j-k} l_{i,k}(t)\Big)/\psi_{i,2}(t-\tau_s),$$

$$k_{i,j-1}(t) = 1/(1 - z_{i,j-1}(t)^2), \quad j = 2, \ldots, n$$

$$z_n(t) = \big(z_{1,n}(t), \ldots, z_{m,n}(t)\big)^\top,$$

$$u_i(t) = -z_{i,n}(t)/(1 - \|z_n(t)\|^2)$$

$$\dot{l}_{i,j}(t) = l_{i,j+1}(t) - \alpha l_{i,j}(t), \quad l_{i,j}(0) = 0, \quad j = 1, \ldots, n-1,$$

$$\dot{l}_{i,n}(t) = -\alpha l_{i,n}(t) + \textstyle\sum_{k=1}^{m} s_{i,k}\big(u_k(t) - u_k(t-\tau_s-\tau_u)\big), \; l_{i,n}(0) = 0$$

Funnel control for systems with delays

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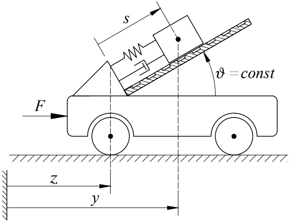
Theorem

$y_d \in W^{2,\infty}$, $\alpha > 0$, $S = (s_{i,k}) \in \mathbb{R}^{m \times m}$ pos. definite s.t.

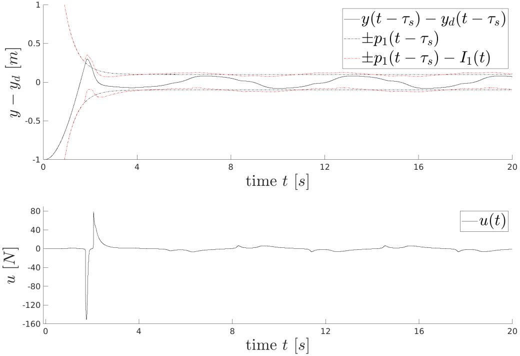
$$\|G - S\| + C(\tau_s, \tau_u) < \lambda_{\min}(S)$$

$\implies u_i, k_{i,j}, x_{i,j}, l_{i,j} \in L^\infty$ and $|y_i(t) - y_{d,i}(t) + l_{i,1}(t)| \leq \psi_{i,1}(t) - \varepsilon$

Simulation



$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\vartheta \\ m_2 \cos\vartheta & m_2 \end{bmatrix} \begin{pmatrix} \ddot{z} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -ks - d\dot{s} + m_2 g \sin\vartheta \end{pmatrix}$$



System parameters: $m_1 = 4, m_2 = 1, k = 2, d = 1, \vartheta = \pi/4$
Delays and controller parameters: $\tau_s = 0.05, \tau_u = 0.05, \alpha = 1, s_{1,1} = 1/9$

Comparison with the result of Bikas & Rovithakis [IEEE-TAC, 2023]

- no Lipschitz assumption on $f_i \longrightarrow$ blow-up possible
- no correction terms $I_{i,j}$ (only one term $\int_{t-\tau_s-\tau_u}^{t} u_i(s)\mathrm{d}s$ appearing in $z_{i,n}(t)$, but error in the proof)
- algorithm is unstable in simulations

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Extension of system class in joint work:

$$\begin{aligned}
\dot{x}_{i,j}(t) &= x_{i,j+1}(t), \\
\dot{x}_{i,n}(t) &= f_i(t, \bar{x}(t), \eta(t)) + \sum_{k=1}^{m} g_{i,k}(t, \bar{x}(t), \eta(t)) u_k(t - \tau_u(t)), \\
\dot{\eta}(t) &= h(t, \bar{x}(t), \eta(t))
\end{aligned}$$

Outlook

- extension to general nonlinear systems
- relax assumptions on input matrix G and delays τ_s, τ_u
- allow for time-varying delays $\tau_s(t)$
- replace Lipschitz assumption on f_i by a proper choice of the initial history