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FUNNEL CONTROL FOR THE MONODOMAIN EQUATIONS

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Monodomain equations

- (simple) model for the electric activity of the human heart to describe defibrillation processes
- “reentry waves” can be interpreted as fibrillation processes that should be terminated by external control

$$\partial_t v = \operatorname{div}(D\nabla v) + I_{ion}(v,w) + I_{s,i} + BI_{s,e}, \quad (\nu^\top \cdot D\nabla v + \alpha v)|_{\partial\Omega} = 0,$$
$$\partial_t w = c_5 v - c_4 w$$

- D – coercive diffusion matrix
- B – input operator
- v – transmembrane electric potential
- w – cellular state variable
- I_{ion} – ionic current
- $I_{s,i}$ – intracellular stimulation current
- $I_{s,e}$ – extracellular stimulation current
- α – nonnegative weight function

FitzHugh-Nagumo model

$$p_3(v) = -c_1 v + c_2 v^2 - c_3 v^3, \qquad I_{ion}(v, w) = p_3(v) - w$$

output: $y = B'v$, *input:* $I_{s,e}(t)$

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$$\textit{output:} \quad y = B'v, \qquad \textit{input:} \quad I_{s,e}(t)$$

$$\mathfrak{a} : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{R},\ (z_1, z_2) \mapsto \langle \nabla z_1, D\nabla z_2 \rangle + \langle z_1, \alpha z_2 \rangle_{L^2(\partial\Omega)}$$

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$\Longrightarrow$ $\exists!$ operator $A : \mathcal{D}(A) \subset L^2(\Omega) \to L^2(\Omega)$ with

$$\mathcal{D}(A) = \left\{ z_2 \in W^{1,2}(\Omega) \,\middle|\, \exists y \in L^2(\Omega)\, \forall z_1 \in W^{1,2}(\Omega) : \mathfrak{a}(z_1, z_2) = -\langle z_1, y \rangle \right\}$$

$$\forall z_1 \in W^{1,2}(\Omega)\, \forall z_2 \in \mathcal{D}(A) : \ \mathfrak{a}(z_1, z_2) = -\langle z_1, A z_2 \rangle$$

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A is called *Robin elliptic operator*; it is *closed* and *self-adjoint* if $\partial\Omega$, D and α are sufficiently smooth:

$$Av = \operatorname{div}(D\nabla v),\ \ \mathcal{D}(A) = \left\{ v \in W^{2,2}(\Omega) \,\middle|\, (\nu^\top \cdot D\nabla v + \alpha v)|_{\partial\Omega} = 0 \right\}$$

FitzHugh-Nagumo model

FHN model:

$$\begin{aligned}
\partial_t v(t) &= Av(t) + p_3(v)(t) - w(t) + I_{s,i}(t) + BI_{s,e}(t)\\
\partial_t w(t) &= c_5 v(t) - c_4 w(t)\\
y(t) &= B' v(t)
\end{aligned}$$

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solutions via weak formulation:
$v, w \in C([0,T]; L^2(\Omega)), v \in L^2(0,T; W^{1,2}(\Omega))$ s.t.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle v(t), \theta \rangle &= -\mathfrak{a}(v(t), \theta) + \big\langle p_3(v)(t) - w(t) + I_{s,i}(t), \theta \big\rangle + \big\langle I_{s,e}(t), B'\theta \big\rangle_{\mathbb{R}^m} \\ \frac{\mathrm{d}}{\mathrm{d}t} \langle w(t), \psi \rangle &= \langle c_5 v(t) - c_4 w(t), \psi \rangle \\ y(t) &= B' v(t) \end{aligned}$$

Input-output configurations

- $q \in L^2(\Omega)$ and distributed input of the form $Bz = z \cdot q$, i.e. $B \in \mathcal{L}(\mathbb{R}, L^2(\Omega))$

$$\Longrightarrow \quad y(t) = B'v(t) = \int_\Omega q(\xi) \cdot (v(t))(\xi)\, \mathrm{d}\xi$$

e.g. $q = \mathbb{1}_\omega$ for $\omega \subseteq \Omega$

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- $q \in W^{1/2,2}(\partial\Omega)'$ and input operator $Bw \in W^{1,2}(\Omega)'$, $w \in \mathbb{R}$, with

$$(Bw)(z) = \langle q \cdot w, \mathrm{tr}(z) \rangle_{W^{1/2,2}(\partial\Omega)', W^{1/2,2}(\partial\Omega)}$$

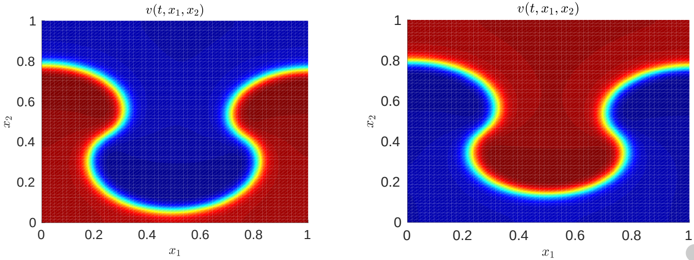
corresponds to a Robin boundary control

$$(\nu^\top \cdot D\nabla v(t) + av(t))|_{\partial\Omega} = q \cdot I_{s,e}(t)$$

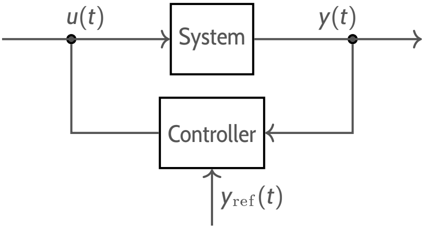
$$\implies y(t) = B'v(t) = \int_{\partial\Omega} q(\xi) \cdot (v(t))(\xi)\, \mathrm{d}\sigma \quad \text{(Dirichlet boundary obs.)}$$

Control objective

- under certain initial conditions, reentry phenomena and spiral waves may occur [Kunisch, Nagaiah, Wagner '11]
- reentry phenomena resemble a dysfunctional heart rhythm which impedes the intracellular stimulation current $I_{s,i}$
- such situations can be interpreted as fibrillation processes of the heart that should be terminated by an external control



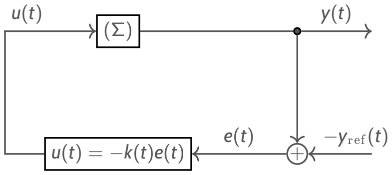
Control objective

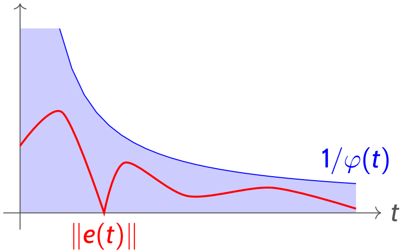


$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), u(t)\big), \quad x(t) \in X \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

- **Goal:** simple controller, so that "$y(t)$ tracks $y_{\mathrm{ref}}(t)$"
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters

Funnel control





$$(\Sigma): \; \dot{y}(t) = F\big(T(y)(t)\big) + G\big(T(y)(t)\big)\, u(t)$$

$$k(t) = \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}$$

[ILCHMANN, RYAN, SANGWIN '02]: Feasible, if T is

- causal and loc. Lipschitz
- BIBO stable

Funnel control for ∞-dimensional systems

- systems which have a relative degree: [Ilchmann, Selig, Trunk '16], [B., Puche, Schwenninger '20]
- boundary controlled heat equation [Reis, Selig '15]

$$\partial_t x(t) = \Delta x(t), \qquad u(t) = \left(\nu^\top \cdot \nabla x(t)\right)\big|_{\partial\Omega},$$
$$y(t) = \int_{\partial\Omega} \big(x(t)\big)(\zeta)\,\mathrm{d}\zeta$$

- general class of boundary control systems based on m-dissipative operators [Puche, Reis, Schwenninger '20], [Puche '19]

$$\dot{x}(t) = \mathfrak{A}x(t), \qquad x(0) = x_0 \in \mathcal{D}(\mathfrak{A}) \subseteq X,$$
$$u(t) = \mathfrak{B}x(t), \qquad y(t) = \mathfrak{C}x(t)$$

e.g. lossy transmission line, wave equation

Funnel controller for the monodomain equations

Goal: controller independent of $\Omega, B, D, a, c_i > 0$ and $I_{s,i}$

$$\varphi \in \Phi_\gamma := \left\{ \varphi \in W^{1,\infty}([0,\infty);\mathbb{R}) \;\middle|\; \begin{array}{c} \varphi|_{[0,\gamma]} = 0, \\ \forall \delta > 0 : \inf_{t \geq \gamma+\delta} \varphi(t) > 0 \end{array} \right\}$$

$\rightarrow$ φ is a *design parameter*!

Controller:

$$I_{s,e}(t) = -\frac{k_0}{1-\varphi(t)^2\|y(t)-y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}}(y(t)-y_{\mathrm{ref}}(t)),$$

$$y(t) = B'v(t),$$

$$\varphi \in \Phi_\gamma,\; y_{\mathrm{ref}} \in W^{1,\infty}([0,\infty);\mathbb{R}^m)$$

$$\forall\, t \in [0,\gamma] : \; I_{s,e}(t) = -k_0(y(t)-y_{\mathrm{ref}}(t))$$

Closed-loop system

$$
\begin{aligned}
\partial_t v(t) &= Av(t) + p_3(v)(t) - w(t) + I_{s,i}(t) \\
&\qquad - \frac{k_0 B(B'v(t) - y_{\mathrm{ref}}(t))}{1 - \varphi(t)^2 \|B'v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}}, \\
\partial_t w(t) &= c_5 v(t) - c_4 w(t)
\end{aligned}
$$

Closed-loop system

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nonlinear time-varying PDE!

Theorem [B., Breiten, Puche, Reis '21]

- $\Omega \subset \mathbb{R}^d$, $d \leq 3$, bounded domain with Lipschitz boundary $\partial\Omega$
- $D \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ coercive; $a \in L^\infty(\partial\Omega)$
- $B \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega)')$, $\ker B = \{0\}$
- $I_{s,i} \in L^\infty(0, \infty; L^2(\Omega))$, $v_0, w_0 \in L^2(\Omega)$
- $y_{\mathrm{ref}} \in W^{1,\infty}(0, \infty; \mathbb{R}^m)$, $\gamma > 0$ and $\varphi \in \Phi_\gamma$

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- $y_{\rm ref} \in W^{1,\infty}(0,\infty;\mathbb{R}^m)$, $\gamma > 0$ and $\varphi \in \Phi_\gamma$

$\Longrightarrow$ $\exists!$ solution s.t. $v, w, \dot{w} \in BC([0,\infty); L^2(\Omega))$ and

$$v \in BUC([\delta,\infty); W^{1,2}(\Omega)) \cap C^{0,1/2}([\delta,\infty); L^2(\Omega)),$$
$$y, I_{s,e} \in BUC([\delta,\infty);\mathbb{R}^m),$$
$$\forall\, t \geq \gamma : \varphi(t)^2 \|B'v(t) - y_{\rm ref}(t)\|^2_{\mathbb{R}^m} \leq 1 - \varepsilon$$

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$$\forall\, t \geq \gamma : \varphi(t)^2 \|B'v(t) - y_{\rm ref}(t)\|^2_{\mathbb{R}^m} \leq 1 - \varepsilon$$

$$\implies \quad v(\gamma) \in W^{1,2}(\Omega)$$

Funnel control for the monodomain equations

a) $B \in \mathcal{L}(\mathbb{R}^m, W^{r,2}(\Omega)'),\ r \in (0,1)$

$$\implies \quad v \in C^{0,1-r/2}([\delta,\infty); L^2(\Omega)),\ \ y, I_{s,e} \in C^{0,1-r}([\delta,\infty); \mathbb{R}^m)$$

b) $B \in \mathcal{L}(\mathbb{R}^m, L^2(\Omega))$

$$\implies \quad \forall\, \lambda \in (0,1):\ v \in C^{0,\lambda}([\delta,\infty); L^2(\Omega)),\ \ y, I_{s,e} \in C^{0,\lambda}([\delta,\infty); \mathbb{R}^m)$$

c) $B \in \mathcal{L}(\mathbb{R}^m, W^{1,2}(\Omega))$

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based on results by [LUNARDI '95] on optimal regularity in parabolic problems

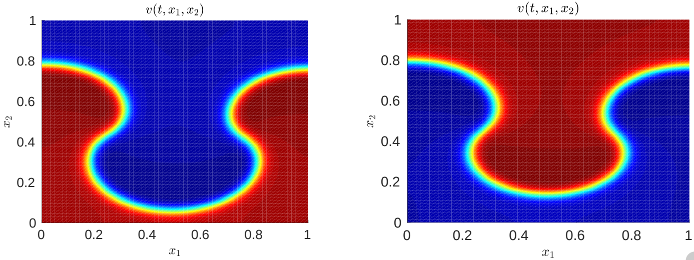
Monodomain equations – Simulation

FHN model:

$$\partial_t v(t) = Av(t) + p_3(v)(t) - w(t) + I_{s,i}(t) + BI_{s,e}(t),$$
$$\partial_t w(t) = c_5 v(t) - c_4 w(t), \quad y(t) = B'v(t)$$

Example: standard test example for termination of reentry waves

$\Omega = (0,1)^2$; $(v(0), w(0)) = (v_0^*, w_0^*)$ – snapshot of a reentry wave, resembling a dysfunctional heart rhythm



Monodomain equations – Simulation

Task: design stimulation current $I_{s,e}$, which restores a natural heart rhythm

Reference $y_{\mathrm{ref}} = B' v_{\mathrm{ref}}$ corresponds to $(v_{\mathrm{ref}}, w_{\mathrm{ref}})$ with $(v_{\mathrm{ref}}(0), w_{\mathrm{ref}}(0)) = (0, 0)$, $I_{s,e} = 0$ and

$$I_{s,i}(t) = 101 \cdot q(\xi)\,\big(\mathbb{1}_{[49,51]}(t) + \mathbb{1}_{[299,301]}(t)\big),$$

$$q(\xi) = \begin{cases} 1, & \text{falls } (\xi_1 - \frac{1}{2})^2 + (\xi_2 - \frac{1}{2})^2 \le 0.0225, \\ 0, & \text{otherwise} \end{cases}$$

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Control: input operator B corresponds to **boundary control**

$$I_{s,e}(t) = -\frac{0.75}{1 - \varphi(t)^2 \|B' v(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^2}^2}\left(B' v(t) - y_{\mathrm{ref}}(t)\right)$$

Monodomain equations – Simulation