UNIVERSITÄT PADERBORN

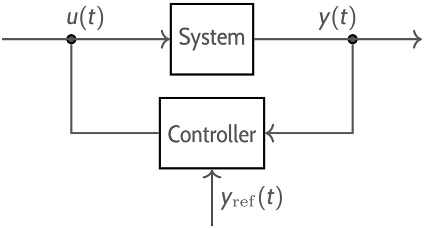
Institute for Mathematics, Paderborn University

FUNNEL CONTROL: AN OVERVIEW AND SOME RECENT TRENDS

Thomas Berger

Darmstadt, March 24, 2023

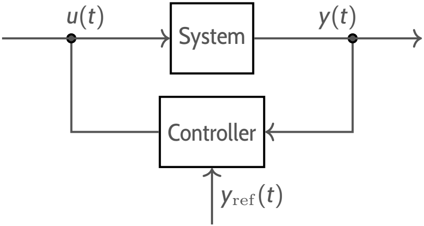
Control objective



$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), u(t)\big), \quad x(t) \in X \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

- **Goal:** simple controller, so that "$y(t)$ tracks $y_{\text{ref}}(t)$"
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters

Control objective



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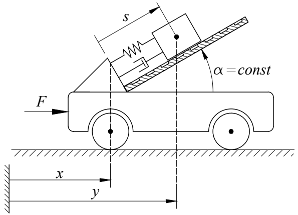
- **Goal:** simple controller, so that "$y(t)$ tracks $y_{\mathrm{ref}}(t)$"
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters
  $\rightarrow$ ODEs and PDEs in the same class!

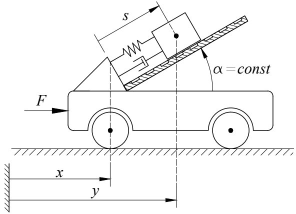


[Seifried, Blajer '13]

angle: $0^\circ \leq \alpha \leq 90^\circ$

spring, damper with nonlinear characteristics: $K(s),\ D(\dot{s})$

$$u(t) = F$$
$$y(t) = x(t) + s(t)\cos\alpha$$



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$$u(t) = F$$
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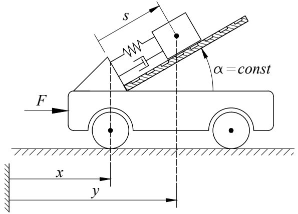
$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\alpha \\ m_2 \cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -K(s) - D(\dot{s}) + m_2 g \sin\alpha \end{pmatrix}$$



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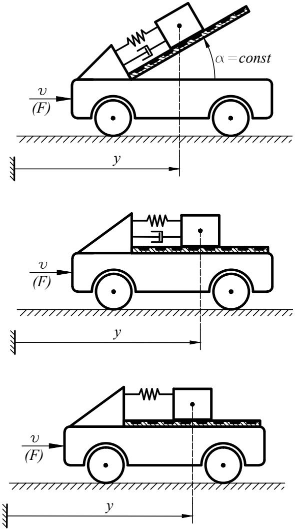
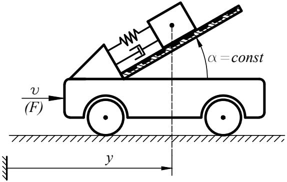
$$\dot{y} = \dot{x} + \dot{s}\cos\alpha$$
$$\ddot{y} = -c_1\big(K(s) + D(\dot{s}) - m_2 g\sin\alpha\big) + \frac{\sin^2\alpha}{m_1 + m_2\sin^2\alpha} u$$



$$0^\circ < \alpha \leq 90^\circ$$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

relative degree $= 2$



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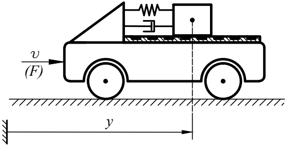
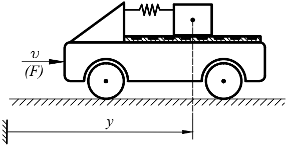
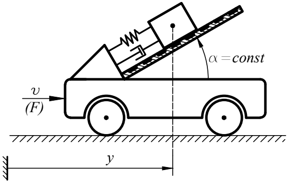


$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

$$y^{(3)} = f_2(s,\dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

relative degree $= 3$





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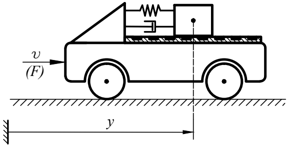

$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

$$y^{(3)} = f_2(s, \dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

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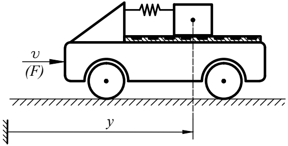

$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

$$y^{(4)} = f_3(s, \dot{s}) + \frac{K'(s)}{m_1 m_2} u$$

relative degree $= 4$

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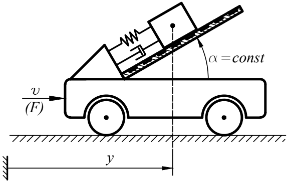
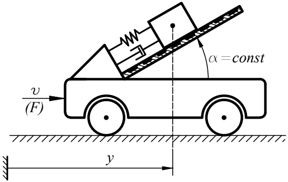
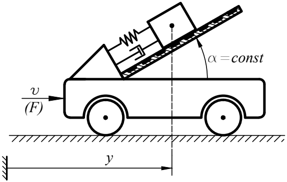


Internal dynamics: remaining dynamics when output is fixed



Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K \left(\frac{\eta - y \cos \alpha}{\sin^2 \alpha} \right) - c_3 D \left(\frac{\dot{\eta} - \dot{y} \cos \alpha}{\sin^2 \alpha} \right) + c_4 g \sin \alpha$$



Internal dynamics: remaining dynamics when output is fixed

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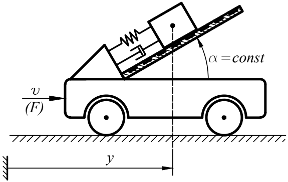
$$\alpha = 90^\circ, m_2 = 1: \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$



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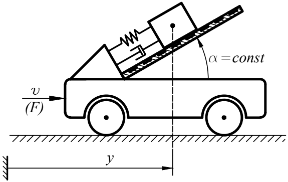
$$\alpha = 90^\circ, m_2 = 1: \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$

- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\,\dot{s} \geq 0$



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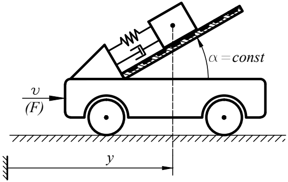
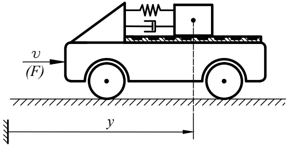
- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\,\dot{s} \geq 0$

$\Longrightarrow \; s, \dot{s} \in L^\infty$ **(stable internal dynamics)**



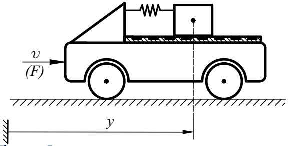
$0^\circ < \alpha \leq 90^\circ$

stable internal dynamics



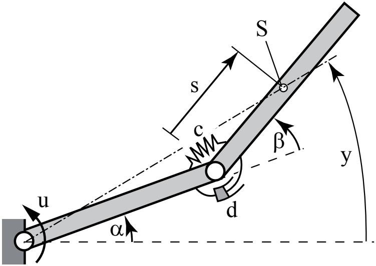
$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$

stable internal dynamics



$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$

no internal dynamics



[Seifried, Blajer '13]

Rotational Manipulator Arm

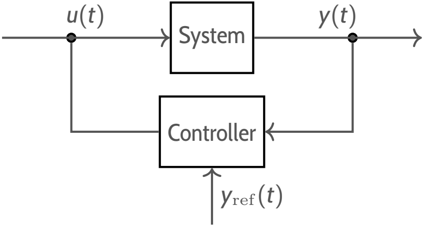
Input: angular velocity of first link

Output: position of S described by angle y

relative degree $= 1$

unstable internal dynamics

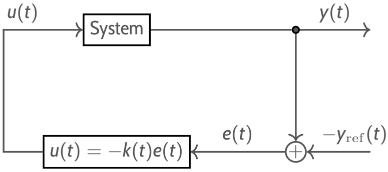
Reminder

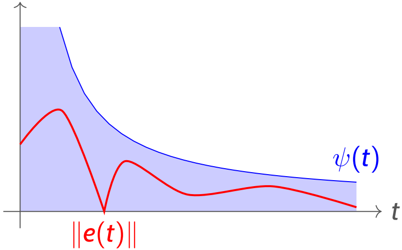


$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), u(t)\big), \quad x(t) \in X \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

- no knowledge of system parameters, only: **known relative degree** and assumption of **stable internal dynamics**
- **Goal:** design simple controller such that "$y(t)$ tracks $y_{\text{ref}}(t)$"

Funnel control





$$k(t) = \frac{1}{\psi(t) - \|e(t)\|}$$

[Ilchmann, Ryan, Sangwin '02]:
Works, if

- relative degree $= 1$
- stable internal dynamics

Funnel control for systems with higher relative degree

Funnel control via backstepping: [Ilchmann, Ryan, Townsend '06 & '07]

drawbacks: escalating controller complexity for relative degree ≥ 2, hence a typically bad controller performance

relative degree $= 2$: [Hackl, Hopfe, Ilchmann, Müller, Trenn '13]

drawbacks: no generalization to arbitrary relative degree

Bang-bang funnel controller: [Liberzon & Trenn '13]

drawbacks: restricted to SISO systems, strong compatibility assumptions

"Prescribed-Performance Control": [Bechlioulis & Rovithakis '14]

drawbacks: restricted to systems with trivial internal dynamics

Funnel control for systems with arbitrary relative degree $r \in \mathbb{N}$

$$\begin{aligned} e_1(t) &= e(t), \qquad e(t) = y(t) - y_{\mathrm{ref}}(t), \\ e_2(t) &= \dot{e}(t) + k_1(t)e_1(t), \\ &\vdots \\ e_r(t) &= e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t), \end{aligned}$$

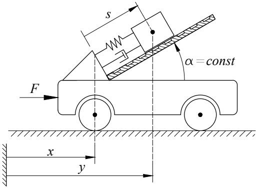
$$u(t) = -k_r(t)e_r(t)$$

$$k_i(t) = 1/(\psi_i(t) - \|e_i(t)\|), \quad i = 1, \ldots, r$$

Theorem [B., Lê, Reis '18] & [B., Ilchmann, Ryan '21]

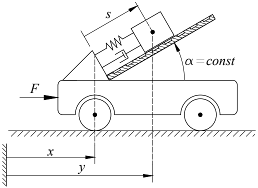
$$y_{\mathrm{ref}} \in W^{r,\infty} \implies u, k_i, y^{(i)} \in L^\infty \text{ and } \|e_i(t)\| \le \psi_i(t) - \varepsilon_i$$

Control of multibody systems – jointly with R. Seifried (TU Hamburg, Germany)



[B., Lê, Reis '18]: $u(t) = u_{\mathrm{FC}}(t)$

Control of multibody systems – jointly with R. Seifried (TU Hamburg, Germany)



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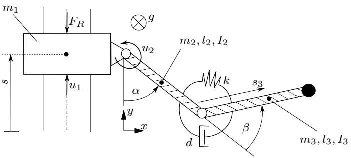
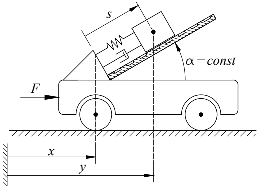
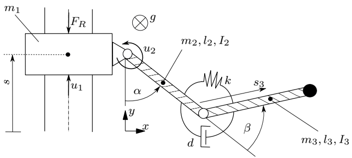


[B., Otto, Reis, Seifried '19]:
$u(t) = u_{\mathrm{FC}}(t) + u_{\mathrm{FF}}(t)$

Control of multibody systems – jointly with R. Seifried (TU Hamburg, Germany)



[B., Lê, Reis '18]: $u(t) = u_{\mathrm{FC}}(t)$



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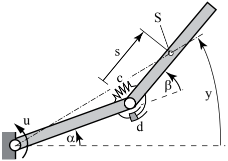

[B., Lanza '20]: $u(t) = u_{\mathrm{FC}}(t)$
unstable internal dynamics

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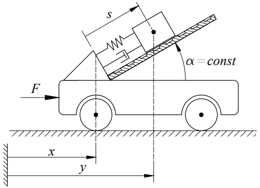

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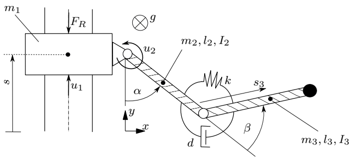

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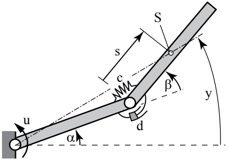

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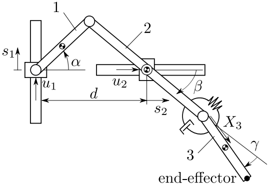
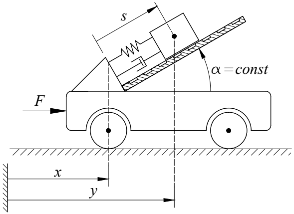
unstable internal dynamics

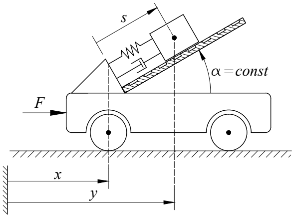


[B., Drücker, Lanza, Reis, Seifried '21]
$u(t) = u_{\mathrm{FC}}(t) + u_{\mathrm{FF}}(t)$
unstable internal dynamics, DAE formulation

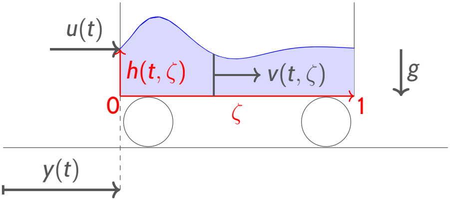


$\rightarrow$

$$\ddot{y}(t) = T(y,\dot{y})(t) + \gamma\, u(t)$$



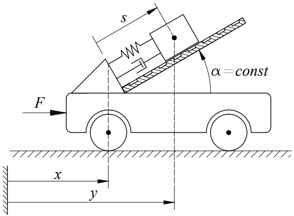
$\rightarrow$



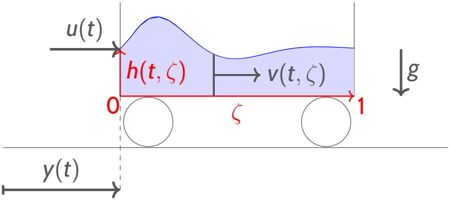
$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma\, u(t)$$

$$\ddot{y}(t) = \hat{T}(y, \dot{y})(t) + \hat{\gamma}\, u(t)$$

[B., Puche, Schwenninger '22]



$\rightarrow$



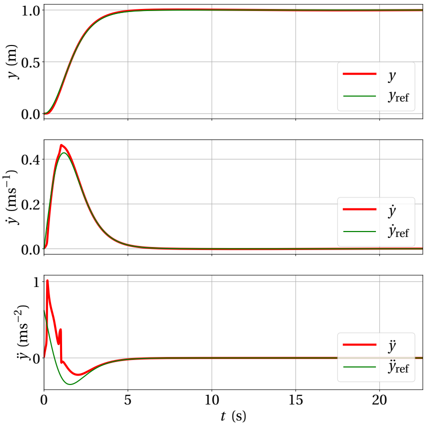
$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma\, u(t)$$

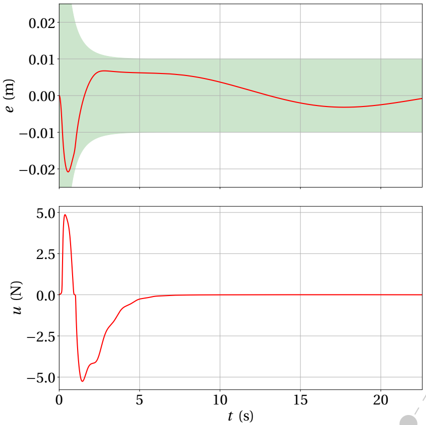
$$\ddot{y}(t) = \hat{T}(y, \dot{y})(t) + \hat{\gamma}\, u(t)$$

[B., Puche, Schwenninger '22]

Finite and infinite dimensional systems in the same class!

Simulation


1.0
0.5
0.0
y (m)
y
yref
0.4
0.2
0.0
ẏ (ms−1)
ẏ
ẏref
1
0
ÿ (ms−2)
ÿ
ÿref
0
5
10
15
20
t (s)


0.02
0.01
0.00
−0.01
−0.02
e (m)
5.0
2.5
0.0
−2.5
−5.0
u (N)
0
5
10
15
20
t (s)

Funnel control for ∞-dimensional systems

"Simple" ∞-dimensional systems – there is a concept of relative degree

- [Ilchmann, Ryan, Sangwin '02], [Ilchmann, Selig, Trunk '16], [B., Puche, Schwenninger '20]

Funnel control for ∞-dimensional systems

“Simple” ∞-dimensional systems – there is a concept of relative degree

- [Ilchmann, Ryan, Sangwin ’02], [Ilchmann, Selig, Trunk ’16], [B., Puche, Schwenninger ’20]

“Hard” ∞-dimensional systems – there is NO concept of relative degree

- boundary controlled heat equation [Reis, Selig ’15]

$$\partial_t x(t) = \Delta x(t), \qquad u(t) = \left(\nu^\top \cdot \nabla x(t)\right)|_{\partial\Omega},$$
$$y(t) = \int_{\partial\Omega} x(t)(\zeta)\,\mathrm{d}\zeta$$

- general class of boundary control problems based on m-dissipative operators [Puche, Reis, Schwenninger ’21], [Puche ’19]
- Fokker-Planck equation [B. 21’] $\to$ *video clip*

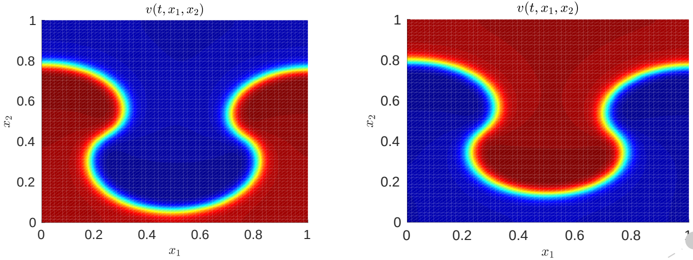
Monodomain equations [B., Breiten, Puche, Reis '21] – (simple) model for the electric activity of the human heart to describe defibrillation processes

$$\partial_t v(t) = \nabla \cdot \big(D\nabla v(t)\big) + p_3(v)(t) - w(t) + I_{s,i}(t) + BI_{s,e}(t),$$
$$\partial_t w(t) = cv(t) - dw(t), \quad y(t) = B'v(t)$$

Monodomain equations [B., Breiten, Puche, Reis '21] – (simple) model for the electric activity of the human heart to describe defibrillation processes

$$\partial_t v(t) = \nabla \cdot \left(D \nabla v(t)\right) + p_3(v)(t) - w(t) + I_{s,i}(t) + B I_{s,e}(t),$$
$$\partial_t w(t) = c v(t) - d w(t), \quad y(t) = B' v(t)$$

Control objective: "reentry waves", which can be interpreted as fibrillation processes, should be terminated



Recent research: Funnel MPC [B., Kästner, Worthmann '20]

OCP:

$$\underset{u\in L^\infty([\widehat{t},\widehat{t}+T],\mathbb{R}^m)}{\text{minimize}} \int_{\widehat{t}}^{\widehat{t}+T} \ell(t,x(t),u(t))\,\mathrm{d}t$$

$$\text{subject to} \quad \dot{x}(t) = f\big(t,x(t),u(t)\big),$$

$$x(\widehat{t}) = \widehat{x}, \qquad \|u\|_\infty \le M$$

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Idea:

$$\ell(t, x, u) = \|h(x) - y_{\mathrm{ref}}(t)\|^2 + \lambda \|u\|^2$$

$$\longrightarrow \quad \ell(t, x, u) = \frac{1}{\psi(t) - \|h(x) - y_{\mathrm{ref}}(t)\|} + \lambda \|u\|^2$$