Funnel control for electrical circuits

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Modified Nodal Analysis (MNA)

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)} \quad u = \begin{pmatrix} i_{\mathcal{I}} \\ v_{\mathcal{V}} \end{pmatrix}, \ y = \begin{pmatrix} -v_{\mathcal{I}} \\ -i_{\mathcal{V}} \end{pmatrix}$$

$$sE - A = \begin{bmatrix} s\mathrm{A}_{\mathcal{C}}\mathcal{C}\mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}}\mathcal{G}\mathrm{A}_{\mathcal{R}}^\top & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \\ -\mathrm{A}_{\mathcal{L}}^\top & s\mathcal{L} & 0 \\ -\mathrm{A}_{\mathcal{V}}^\top & 0 & 0 \end{bmatrix}, \quad B = C^\top = \begin{bmatrix} -\mathrm{A}_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I_{n_{\mathcal{V}}} \end{bmatrix}$$

$\mathrm{A}_{\mathcal{C}}, \mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}, \mathrm{A}_{\mathcal{V}}, \mathrm{A}_{\mathcal{I}}$ – element-related incidence matrices

$\mathcal{C}, \mathcal{G}, \mathcal{L}$ – consecutive relations of capacitances, resistances and inductances

passivity: $\mathcal{C} = \mathcal{C}^\top > 0, \mathcal{L} = \mathcal{L}^\top > 0, \mathcal{G} + \mathcal{G}^\top > 0$

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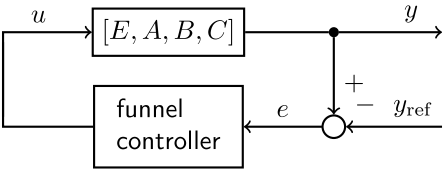
ℓ is $\mathcal{K}$ – **loop** $:\Longleftrightarrow$ ℓ is a loop in the graph that consists of edges contained in $\mathcal{K}$ only

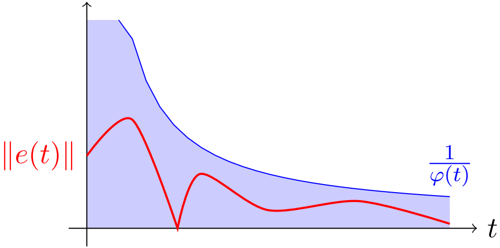
corr. $\mathcal{I}$-loop, $\mathcal{ICL}$-loops, etc.

$\mathcal{L}$ is $\mathcal{K}$ – **cutset** $:\Longleftrightarrow$ deleting $\mathcal{L} \subseteq \mathcal{K}$ results in a disconnected graph and $\mathcal{L}$ is minimal

corr.. $\mathcal{V}$-cutset, $\mathcal{VCL}$-cutset, etc.

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$





Zero dynamics:

$$\mathcal{ZD} := \{\ (x,u,y)\ \mid E\dot{x}(t) = Ax(t) + Bu(t),\ \ y(t) = Cx(t) = 0\ \}$$

$\mathcal{ZD}$ autonomous $:\iff$

$\forall\, w_1, w_2 \in \mathcal{ZD}\ \forall\, I \subseteq \mathbb{R}$ open interval :

$$w_1\big|_I = w_2\big|_I \implies w_1 = w_2$$

$\mathcal{ZD}$ stable $:\iff\ \forall\, w \in \mathcal{ZD}:\ \lim_{t\to\infty} w(t) = 0$

$\lambda \in \mathbb{C}$ is invariant zero $:\iff$

$$\operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} < \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}$$

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Theorem (stable zero dynamics)

Let $[E, A, B, C]$ be a MNA model of a circuit.

$\mathcal{ZD}$ stable $\iff$
- $\mathcal{ZD}$ autonomous
- all invariant zeros $\subseteq \mathbb{C}_-$

$\mathcal{ZD}$ autonomous $\iff$ neither $\mathcal{I}$-loops nor $\mathcal{V}$-cutsets

all invariant zeros $\subseteq \mathbb{C}_-$ $\Longleftarrow$
- neither $\mathcal{IL}$-loops except for $\mathcal{I}$-loops, nor $\mathcal{VCL}$-cutsets except for $\mathcal{VL}$-cutsets
- neither $\mathcal{VC}$-cutsets except for $\mathcal{V}$-cutsets, nor $\mathcal{ICL}$-loops except for $\mathcal{IC}$-loops

Theorem (funnel control - stable zero dynamics)

Let $[E, A, B, C]$ be a MNA model of a circuit with

- $\mathcal{ZD}$ stable,
- $y_{\text{ref}} \in \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.

Then the *funnel controller*

$$u(t) = -k(t)\, e(t), \qquad \text{where} \qquad e(t) = y(t) - y_{\text{ref}}(t)$$

$$k(t) = \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2},$$

applied to $[E, A, B, C]$ achieves that

$$x \in L^\infty,\ k \in L^\infty \quad \wedge \quad \exists\, \varepsilon > 0\ \forall\, t > 0 : \ \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon.$$

$$
\begin{aligned}
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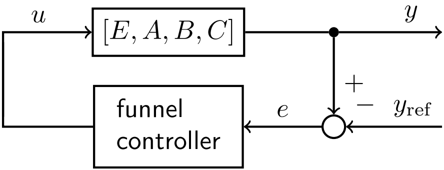
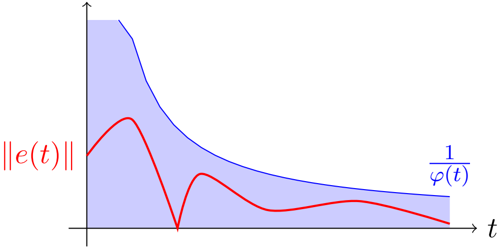
\end{aligned}
$$





Theorem (funnel control - stable invariant zeros)

Let $[E, A, B, C]$ be a MNA model of a circuit with

- all invariant zeros $\subseteq \mathbb{C}_-$
- $y_{\mathrm{ref}} \in \mathcal{B}^\infty\left(\mathbb{R}_{\geq 0}; \operatorname{im} \mathrm{A}_\mathcal{I}^\top \times \ker Z_{\mathcal{CRLI}}^\top \mathrm{A}_\mathcal{V}\right)$, where

$$\operatorname{im} Z_{\mathcal{CRLI}} = \ker \begin{bmatrix} \mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{I} \end{bmatrix}^\top .$$

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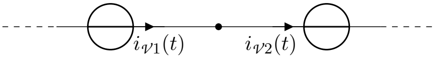
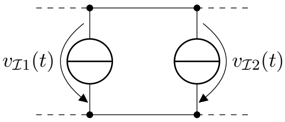
Interpretation of

$$y_{\mathrm{ref}}(t) \in \operatorname{im} \mathrm{A}_{\mathcal{I}}^{\top} \times \ker Z_{\mathcal{CRLI}}^{\top} \mathrm{A}_{\mathcal{V}} \quad \forall\, t \geq 0$$

$\longrightarrow$ y_{ref} satisfies Kirchhoff's laws pointwise!



$$\Rightarrow i_{\mathcal{V}1}(t) = i_{\mathcal{V}2}(t)$$



$$\Rightarrow v_{\mathcal{I}1}(t) = v_{\mathcal{I}2}(t)$$