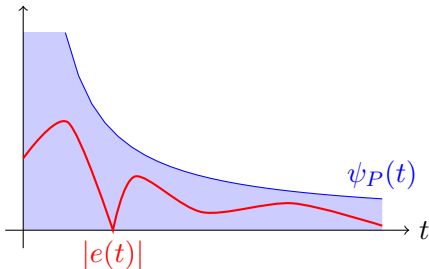
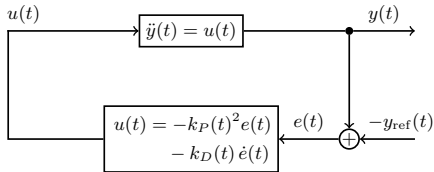


Funnel-Beobachtung

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Freiberg, 12. Mai 2016



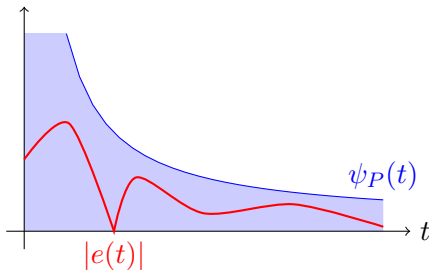
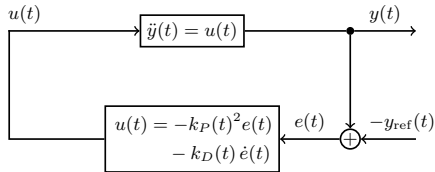
$$k_P(t) = \frac{\psi_P(t)}{\psi_P(t) - |e(t)|},$$

$$k_D(t) = \frac{\psi_D(t)}{\psi_D(t) - |\dot{e}(t)|},$$

$$0 < \delta \leq \frac{d}{dt} \psi_P(t) + \psi_D(t)$$

[HACKL, HOPFE, ILCHMANN,
 MÜLLER, TRENN '13]:
 Das funktioniert!

Problem: Ableitung von y nicht verfügbar!



$$k_P(t) = \frac{\psi_P(t)}{\psi_P(t) - |e(t)|},$$

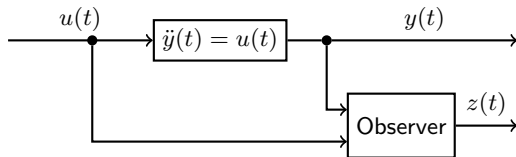
$$k_D(t) = \frac{\psi_D(t)}{\psi_D(t) - |\dot{e}(t)|},$$

$$0 < \delta \leq \frac{d}{dt} \psi_P(t) + \psi_D(t)$$

[HACKL, HOPFE, ILCHMANN,
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 Das funktioniert!

Problem: Ableitung von y nicht verfügbar!

High-gain Beobachter für $\ddot{y} = u$



$$\dot{z}_1 = z_2 + p_1 k (y - z_1),$$

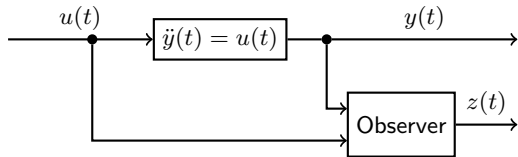
$$\dot{z}_2 = u + p_2 k^2 (y - z_1),$$

$$p_1, p_2 > 0$$

[TORNAMBE '92]: $k > 0$ groß genug

$$\implies z_1 \approx y \wedge z_2 \approx \dot{y}$$

High-gain Beobachter für $\ddot{y} = u$



$$\dot{z}_1 = z_2 + p_1 k(t)(y - z_1),$$

$$\dot{z}_2 = u + p_2 k(t)^2(y - z_1),$$

$$\dot{k}(t) = \gamma d_{\lambda, \hat{\lambda}}^2(y(t) - z_1(t)),$$

$$d_{\lambda, \hat{\lambda}}(z) = \begin{cases} \hat{\lambda} - \lambda, & |z| \geq \hat{\lambda} \\ |z| - \lambda, & \lambda \leq |z| \leq \hat{\lambda} \\ 0, & |z| \leq \lambda \end{cases}$$

[BULLINGER, ILCHMANN,
 ALLGÖWER '98]:

$z, k \in L^\infty$ und

$\text{dist}(y - z_1, [-\lambda, \lambda]) \rightarrow 0$

Betrachte modifizierten Beobachter für $\ddot{y} = u$

$$\dot{z}_1 = z_2 + (q_1 + p_1 k(t))(y - z_1), \quad k(t) = \frac{\psi_0(t)}{\psi_0(t) - |y(t) - z_1(t)|}$$

$$\dot{z}_2 = u + (q_2 + p_2 k(t))(y - z_1),$$

$$e_1 := y - z_1, \quad e_2 := \dot{y} - z_2, \quad k(t) = \frac{\psi_0(t)}{\psi_0(t) - |e_1(t)|}$$

$$\dot{e}_1 = e_2 - (q_1 + p_1 k(t))e_1,$$

$$\dot{e}_2 = -(q_2 + p_2 k(t))e_1,$$

$$A := \begin{bmatrix} -q_1 & 1 \\ -q_2 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \quad \text{s.t.} \quad A^\top P + PA = -I$$

$$p_1 = 1, \quad p_2 = -\frac{p_{12}}{p_{22}}; \quad \dot{e} = Ae - k(t) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1$$

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$$\dot{z}_1 = z_2 + (q_1 + p_1 k(t))(y - z_1), \quad k(t) = \frac{\psi_O(t)}{\psi_O(t) - |y(t) - z_1(t)|}$$

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$$p_1 = 1, \quad p_2 = -\frac{p_{12}}{p_{22}}; \quad \dot{e} = Ae - k(t) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1$$

$$\dot{e} = Ae - k(t) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1$$

$$\begin{aligned} \frac{d}{dt} e^\top P e &= e^\top (A^\top P + PA) e - k(t) \left(e_1 \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}^\top P e + e^\top P \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1 \right) \\ &= -\|e\|^2 - k(t) \left(p_{11} - \frac{p_{12}^2}{p_{22}} \right) e_1^2 \leq -\|e\|^2 \end{aligned}$$

$\implies e_1, e_2 \in L^\infty[0, \omega)$, Standard Funnel-Argument : $k \in L^\infty$

$\implies \omega = \infty \wedge e(t) \rightarrow 0 \wedge k(t) \rightarrow 1$

$$\dot{e} = Ae - k(t) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1$$

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$\implies e_1, e_2 \in L^\infty[0, \omega)$, Standard Funnel-Argument : $k \in L^\infty$

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Erweiterung

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) + \Gamma(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t))u(t)$$

$T : \mathcal{C} \rightarrow \mathcal{L}_{\text{loc}}^{\infty}$ kausal, lokal Lipschitz, BIBO

Beispiele:

Erweiterung

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) + \Gamma(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t))u(t)$$

$T : \mathcal{C} \rightarrow \mathcal{L}_{\text{loc}}^{\infty}$ kausal, lokal Lipschitz, BIBO

Beispiele:

$\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ mit

(A1) $\text{rk}_{\mathbb{C}} \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} = n + m$ für alle $\lambda \in \mathbb{C}$ mit $\text{Re } \lambda \geq 0$;

(A2) $CB = CAB = \dots = CA^{r-2}B = 0$ und $CA^{r-1}B \in \mathbf{GL}_m(\mathbb{R})$.

Erweiterung

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) + \Gamma(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t))u(t)$$

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Ist äquivalent zu $\frac{d}{dt}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t)$, $y(t) = \hat{C}\hat{x}(t)$ mit

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & I_m & 0 \\ R_1 & R_2 & \dots & R_r & S \\ P & 0 & \dots & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ 0 \end{bmatrix}, \quad \hat{C} = [I_m \ 0 \ \dots \ 0 \ 0]$$

Erweiterung

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) + \Gamma(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t))u(t)$$

$T : \mathcal{C} \rightarrow \mathcal{L}_{\text{loc}}^{\infty}$ kausal, lokal Lipschitz, BIBO

Beispiele:

$$\frac{d}{dt}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \quad y(t) = \hat{C}\hat{x}(t) \quad \text{mit}$$

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \dot{0} & \cdots & 0 & I_m & \dot{0} \\ R_1 & R_2 & \cdots & R_r & S \\ P & 0 & \cdots & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \vdots \\ \dot{0} \\ CA^{r-1}B \\ 0 \end{bmatrix}, \quad \hat{C} = [I_m \quad 0 \quad \cdots \quad 0 \quad 0]$$

$$y^{(r)}(t) = R_1 y(t) + \dots + R_r y^{(r-1)}(t) + Se^{Qt}\eta(0) + \int_0^t Se^{Q(t-\tau)}Py(\tau)d\tau + CA^{r-1}Bu(t).$$

Erweiterung

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) + \Gamma(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t))u(t)$$

$T : \mathcal{C} \rightarrow \mathcal{L}_{\text{loc}}^{\infty}$ kausal, lokal Lipschitz, BIBO

Beispiele:

$$y^{(r)}(t) = T(y, \dot{y}, \dots, y^{(r-1)})(t) + \Gamma u(t)$$

mit $\Gamma = CA^{r-1}B$ und

$$\begin{aligned} T(y, \dots, y^{(r-1)})(t) \\ = R_1 y(t) + \dots R_r y^{(r-1)}(t) + Se^{Qt}\eta(0) + \int_0^t Se^{Q(t-\tau)} P y(\tau) d\tau. \end{aligned}$$

Erweiterung

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) + \Gamma(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t))u(t)$$

$T : \mathcal{C} \rightarrow \mathcal{L}_{\text{loc}}^{\infty}$ kausal, lokal Lipschitz, BIBO

Beispiele:

$$\dot{x}_1(t) = x_2(t), \dots, \dot{x}_{r-1}(t) = x_r(t), \quad \hat{x}(t) = (x_1(t)^\top, \dots, x_r(t)^\top)^\top$$

$$\dot{x}_r(t) = g_1(\hat{x}(t), \eta(t)) + g_2(\hat{x}(t), \eta(t))u(t),$$

$$\dot{\eta}(t) = g_3(\hat{x}(t), \eta(t)),$$

$$y(t) = x_1(t)$$

Erweiterung

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) + \Gamma(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t))u(t)$$

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Beispiele:

$$\dot{x}_1(t) = x_2(t), \dots, \dot{x}_{r-1}(t) = x_r(t), \quad \hat{x}(t) = (x_1(t)^{\top}, \dots, x_r(t)^{\top})^{\top}$$

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$$\dot{\eta}(t) = g_3(\hat{x}(t), \eta(t)),$$

$$y(t) = x_1(t)$$

$$T(y, \dots, y^{(r-1)})(t) := (y(t)^{\top}, \dots, y^{(r-1)}(t)^{\top}, \eta(t; \eta^0, y, \dots, y^{(r-1)})^{\top})^{\top}$$

$$y^{(r)}(t) = g_1(T(y, \dots, y^{(r-1)})(t)) + g_2(T(y, \dots, y^{(r-1)})(t))u(t)$$

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) + \Gamma(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t))u(t)$$

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) + (q_1 + p_1 k(t))(y(t) - z_1(t)), \\ \dot{z}_2(t) &= z_3(t) + (q_2 + p_2 k(t))(y(t) - z_1(t)), \\ &\vdots \\ \dot{z}_{r-1}(t) &= z_r(t) + (q_{r-1} + p_{r-1} k(t))(y(t) - z_1(t)), \\ \dot{z}_r(t) &= \tilde{\Gamma}u(t) + (q_r + p_r k(t))(y(t) - z_1(t)), \\ k(t) &= \frac{1}{1 - \varphi(t)^2 \|y(t) - z_1(t)\|^2} \end{aligned}$$

mit Design-Parametern $p_i > 0$, $q_i > 0$, $\tilde{\Gamma} \in \mathbb{R}^{m \times m}$ und der Funnel-Funktion $\varphi : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$

$$\begin{aligned}
 \dot{z}_1 &= z_2 + (q_1 + p_1 k(t))(y - z_1) \\
 \dot{z}_2 &= z_3 + (q_2 + p_2 k(t))(y - z_1) \\
 &\vdots \\
 \dot{z}_{r-1} &= z_r + (q_{r-1} + p_{r-1} k(t))(y - z_1) \\
 \dot{z}_r &= \tilde{\Gamma} u + (q_r + p_r k(t))(y - z_1)
 \end{aligned}$$

$$A := \begin{bmatrix} -q_1 & 1 & & \\ \vdots & & \ddots & \\ -q_{r-1} & & & 1 \\ -q_r & & & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} \quad P_{11} \in \mathbb{R}, \quad P_{22} \in \mathbb{R}^{(r-1) \times (r-1)}$$

$$\text{s.t. } A^\top P + PA = -I \quad \text{and} \quad \begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} = \begin{pmatrix} 1 \\ -P_{22}^{-1} P_{12}^\top \end{pmatrix}$$

Theorem:

$$y, \dots, y^{(r-1)} \in L^\infty \implies z_1, \dots, z_r, k \in L^\infty \wedge |y(t) - z_1(t)| < \psi_0(t)$$

$$\begin{aligned}
 \dot{z}_1 &= z_2 + (q_1 + p_1 k(t))(y - z_1) \\
 \dot{z}_2 &= z_3 + (q_2 + p_2 k(t))(y - z_1) \\
 &\vdots \\
 \dot{z}_{r-1} &= z_r + (q_{r-1} + p_{r-1} k(t))(y - z_1) \\
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 \end{aligned}$$

$$A := \begin{bmatrix} -q_1 & 1 & & \\ \vdots & & \ddots & \\ -q_{r-1} & & & 1 \\ -q_r & & & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} \quad P_{11} \in \mathbb{R}, \quad P_{22} \in \mathbb{R}^{(r-1) \times (r-1)}$$

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Theorem:

$$y, \dots, y^{(r-1)} \in L^\infty \implies z_1, \dots, z_r, k \in L^\infty \wedge |y(t) - z_1(t)| < \psi_O(t)$$

Simulation - Bioreaktor

$$\dot{m}(t) = \frac{a_1 m(t) s(t)}{a_2 m(t) + s(t)} - m(t) u(t),$$

$$\dot{s}(t) = -\frac{a_1 a_3 m(t) s(t)}{a_2 m(t) + s(t)} + (a_4 - s(t)) u(t),$$

$$y(t) = m(t),$$

m - Konzentration Mikroorganismen, $m(0) = 0.075$

s - Konzentration Substrat, $s(0) = 0.03$

u - Substrateinflussrate

$$u(t) = \begin{cases} 0.08, & t \in [0, 30) \\ 0.02, & t \in [30, 50) \\ 0.08, & t \geq 50, \end{cases} \quad \begin{aligned} \tilde{\Gamma} &= 0, \quad q_1 = 1, \quad q_2 = 0.2, \\ p_1 &= 1, \quad p_2 = \frac{1}{11} \\ \varphi(t) &= \frac{1}{2} t e^{-t} + \frac{100}{\pi} \arctan t. \end{aligned}$$

Simulation - Bioreaktor

$$\dot{m}(t) = \frac{a_1 m(t) s(t)}{a_2 m(t) + s(t)} - m(t) u(t),$$

$$\dot{s}(t) = -\frac{a_1 a_3 m(t) s(t)}{a_2 m(t) + s(t)} + (a_4 - s(t)) u(t),$$

$$y(t) = m(t),$$

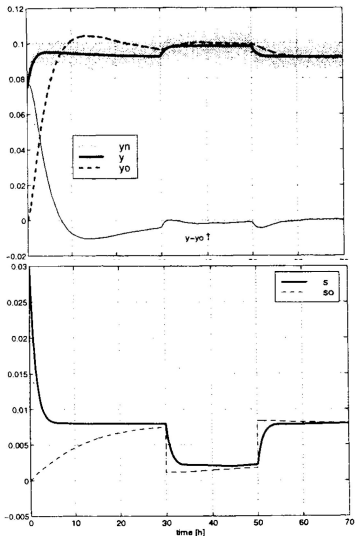
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λ -strip Beobachter [BIA '98]:



Funnel-Beobachter:

