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Institut für Mathematik, Universität Paderborn

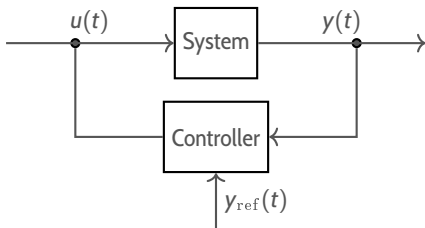
THE JOY OF FUNNEL CONTROL

Thomas Berger

GAMM Annual Meeting, March 17, 2021

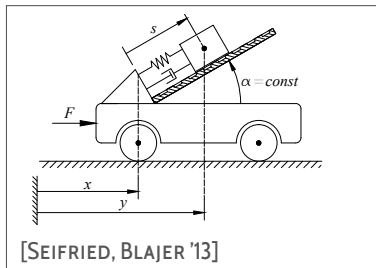


Control objective



$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), & x(t) &\in X \\ y(t) &= h(x(t)) \end{aligned}$$

- **Goal:** simple controller, so that “ $y(t)$ tracks $y_{\text{ref}}(t)$ ”
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters

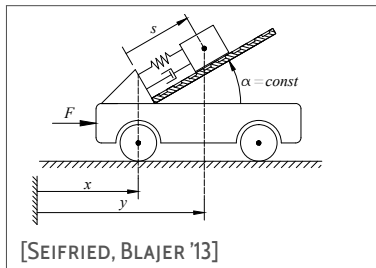


angle: $0^\circ \leq \alpha \leq 90^\circ$

spring, damper with nonlinear characteristics: $K(s)$, $D(\dot{s})$

$$u(t) = F$$

$$y(t) = x(t) + s(t) \cos \alpha$$



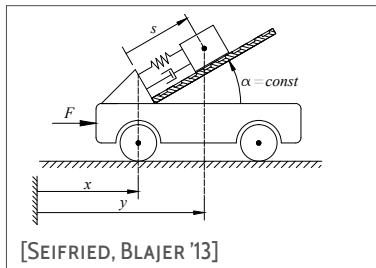
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spring, damper with nonlinear characteristics: $K(s)$, $D(\dot{s})$

$$u(t) = F$$

$$y(t) = x(t) + s(t) \cos \alpha$$

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos \alpha \\ m_2 \cos \alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -K(s) - D(\dot{s}) + m_2 g \sin \alpha \end{pmatrix}$$



angle: $0^\circ \leq \alpha \leq 90^\circ$

spring, damper with nonlinear characteristics: $K(s)$, $D(\dot{s})$

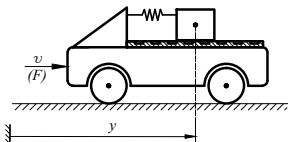
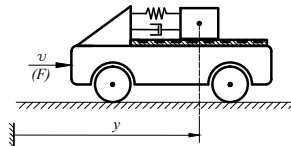
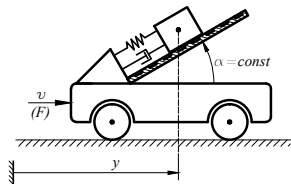
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$$\dot{y} = \dot{x} + \dot{s} \cos \alpha$$

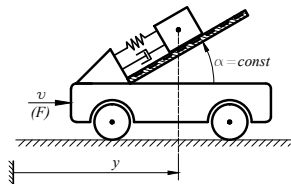
$$\ddot{y} = -c_1(K(s) + D(\dot{s}) - m_2 g \sin \alpha) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$



$$0^\circ < \alpha \leq 90^\circ$$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

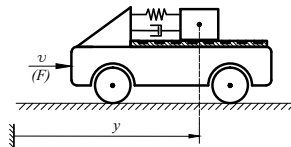
relative degree = 2



$$0^\circ < \alpha \leq 90^\circ$$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

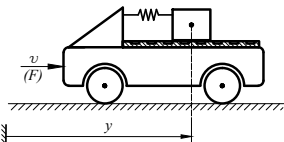
relative degree = 2

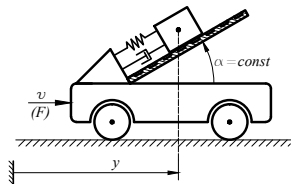


$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

$$y^{(3)} = f_2(s, \dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

relative degree = 3

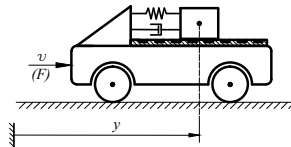




$$0^\circ < \alpha \leq 90^\circ$$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

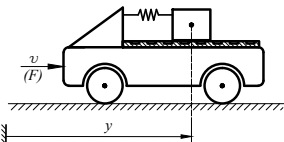
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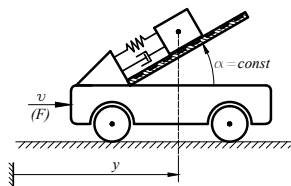
relative degree = 3



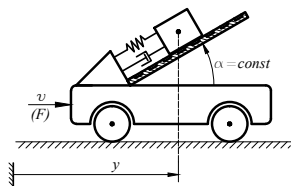
$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

$$y^{(4)} = f_3(s, \dot{s}) + \frac{K'(s)}{m_1 m_2} u$$

relative degree = 4

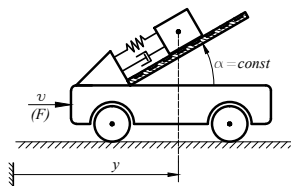


Internal dynamics: remaining dynamics
when output is fixed



Internal dynamics: remaining dynamics
when output is fixed

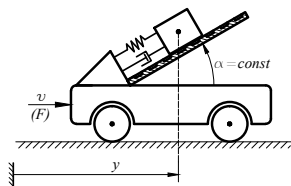
$$\ddot{\eta} = -c_3 K \left(\frac{\eta - y \cos \alpha}{\sin^2 \alpha} \right) - c_3 D \left(\frac{\dot{\eta} - \dot{y} \cos \alpha}{\sin^2 \alpha} \right) + c_4 g \sin \alpha$$



Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K \left(\frac{\eta - y \cos \alpha}{\sin^2 \alpha} \right) - c_3 D \left(\frac{\dot{\eta} - \dot{y} \cos \alpha}{\sin^2 \alpha} \right) + c_4 g \sin \alpha$$

$$\alpha = 90^\circ, m_2 = 1: \quad \ddot{s} = -K(s) - D(\dot{s}) + g$$

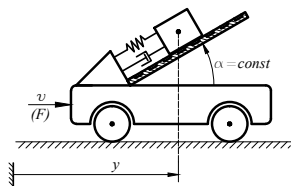


Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K \left(\frac{\eta - y \cos \alpha}{\sin^2 \alpha} \right) - c_3 D \left(\frac{\dot{\eta} - \dot{y} \cos \alpha}{\sin^2 \alpha} \right) + c_4 g \sin \alpha$$

$$\alpha = 90^\circ, m_2 = 1: \quad \ddot{s} = -K(s) - D(\dot{s}) + g$$

- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\dot{s} \geq 0$



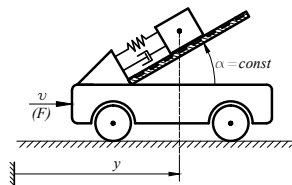
Internal dynamics: remaining dynamics when output is fixed

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$$\alpha = 90^\circ, m_2 = 1: \quad \ddot{s} = -K(s) - D(\dot{s}) + g$$

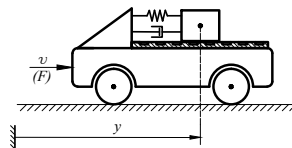
- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\dot{s} \geq 0$

$$\Rightarrow s, \dot{s} \in L^\infty \quad (\text{stable internal dynamics})$$



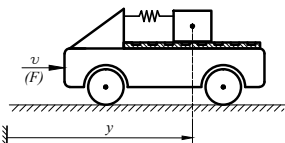
$$0^\circ < \alpha \leq 90^\circ$$

stable internal dynamics



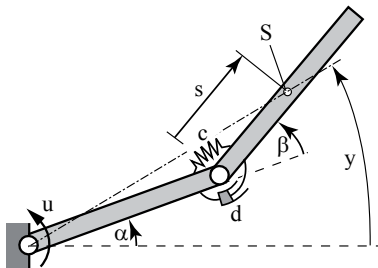
$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

stable internal dynamics



$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

no internal dynamics



[SEIFRIED, BLAJER '13]

Rotational Manipulator Arm

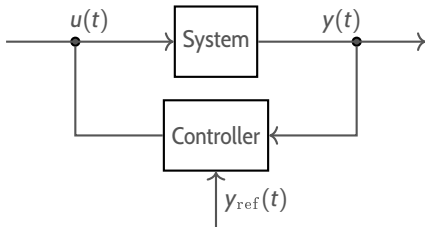
Input: angular velocity of first link

Output: position of S described by angle y

relative degree = 1

unstable internal dynamics

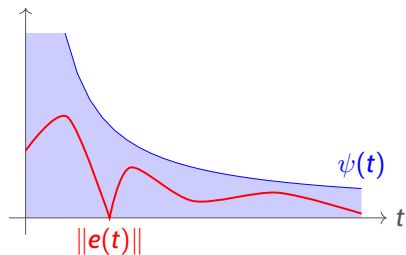
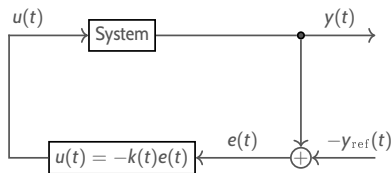
Reminder



$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), & x(t) &\in X \\ y(t) &= h(x(t)) \end{aligned}$$

- no knowledge of system parameters, only: **known relative degree** and assumption of **stable internal dynamics**
- Goal:** design simple controller such that “ $y(t)$ tracks $y_{\text{ref}}(t)$ ”

Funnel control



[ILCHMANN, RYAN, SANGWIN '02]:
Works, if

- relative degree = 1
- stable internal dynamics

$$k(t) = \frac{1}{\psi(t) - \|e(t)\|}$$

Funnel control for systems with arbitrary relative degree

$$e_1(t) = e(t)/\psi(t), \quad e(t) = y(t) - y_{\text{ref}}(t),$$

$$e_2(t) = \dot{e}(t)/\psi(t) + k_1(t)e_1(t),$$

$$e_3(t) = \ddot{e}(t)/\psi(t) + k_2(t)e_2(t),$$

$$\vdots$$

$$e_r(t) = e^{(r-1)}(t)/\psi(t) + k_{r-1}(t)e_{r-1}(t),$$

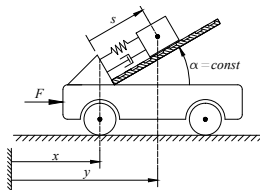
$$u(t) = -k_r(t)e_r(t)$$

$$k_i(t) = 1/(\psi(t) - \|e_i(t)\|), \quad i = 1, \dots, r$$

Theorem [B., LÊ, REIS '18] & [B., ILCHMANN, RYAN '21]

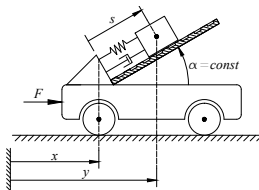
$$y_{\text{ref}} \in W^{r,\infty} \implies u, k_i, y^{(i)} \in L^\infty \text{ and } \|e(t)\| \leq \psi(t) - \varepsilon$$

DFG project – jointly with R. Seifried (TU Hamburg)

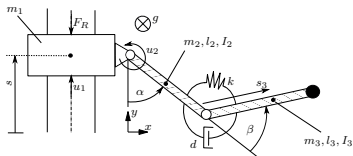


[B., Lê, REIS '18]: $u(t) = u_{\text{FC}}(t)$

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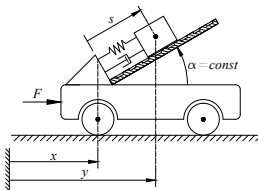


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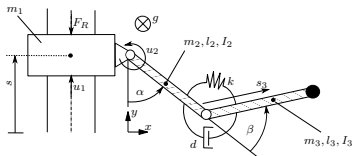


[B., OTTO, REIS, SEIFRIED '19]:
 $u(t) = u_{\text{FC}}(t) + u_{\text{FF}}(t)$

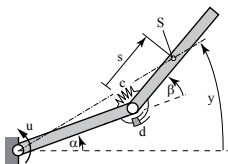
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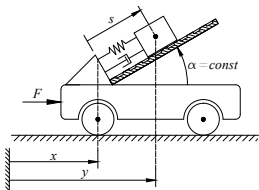


[B., OTTO, REIS, SEIFRIED '19]:
 $u(t) = u_{FC}(t) + u_{FF}(t)$

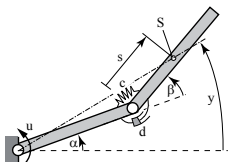


[B., LANZA '20]: $u(t) = u_{FC}(t)$
unstable internal dynamics

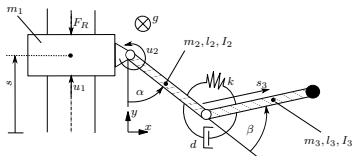
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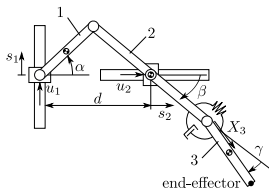
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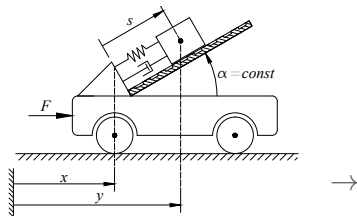
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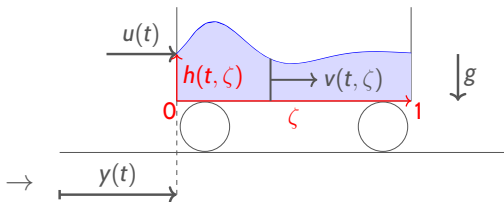
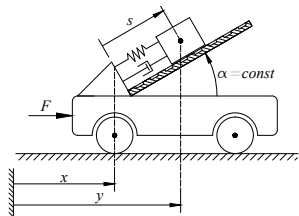
[B., OTTO, REIS, SEIFRIED '19]:
 $u(t) = u_{FC}(t) + u_{FF}(t)$



[B., DRÜCKER, LANZA, REIS, SEIFRIED '21]
 $u(t) = u_{FC}(t) + u_{FF}(t)$
unstable internal dynamics, DAE formulation



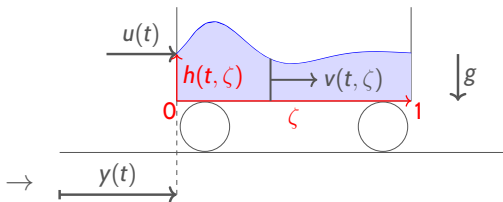
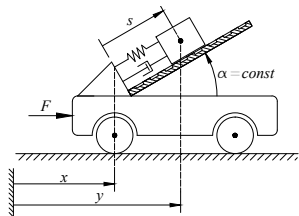
$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma u(t)$$



$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma u(t)$$

$$\ddot{y}(t) = \hat{T}(y, \dot{y})(t) + \hat{\gamma} u(t)$$

[B., PUCHE, SCHWENNINGER '20]



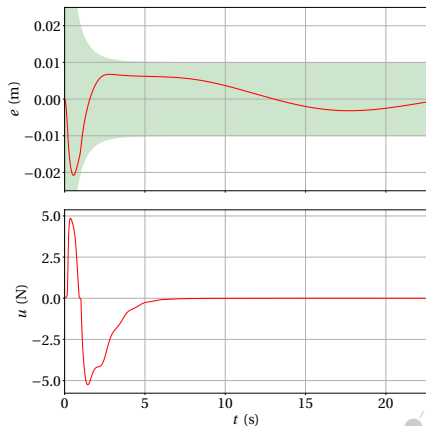
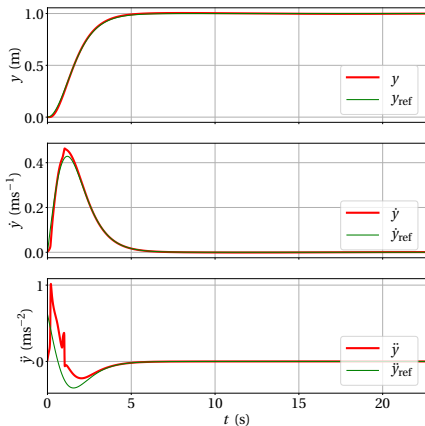
$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma u(t)$$

$$\ddot{y}(t) = \hat{T}(y, \dot{y})(t) + \hat{\gamma} u(t)$$

[B., PUCHE, SCHWENNINGER '20]

Finite and infinite dimensional systems in the same class!

Simulation



Funnel control for ∞ -dimensional systems

“Simple” ∞ -dimensional systems – there is a concept of relative degree

- [ILCHMANN, RYAN, SANGWIN '02], [ILCHMANN, SELIG, TRUNK '16], [B., PUCHE, SCHWENNINGER '20]

Funnel control for ∞ -dimensional systems

“Simple” ∞ -dimensional systems – there is a concept of relative degree

- [ILCHMANN, RYAN, SANGWIN '02], [ILCHMANN, SELIG, TRUNK '16], [B., PUCHE, SCHWENNINGER '20]

“Hard” ∞ -dimensional systems – there is NO concept of relative degree

- boundary controlled heat equation [REIS, SELIG '15]

$$\partial_t x(t) = \Delta x(t), \quad u(t) = (\nu^\top \cdot \nabla x(t))|_{\partial\Omega},$$

$$y(t) = \int_{\partial\Omega} x(t)(\zeta) d\zeta$$

- general class of boundary control problems based on m -dissipative operators [PUCHE, REIS, SCHWENNINGER '20], [PUCHE '19]
- Fokker-Planck equation [B. 20'] \rightarrow *video clip*

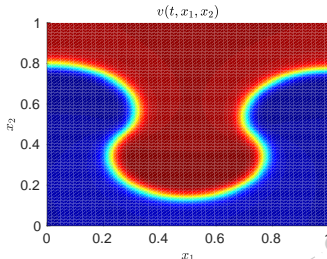
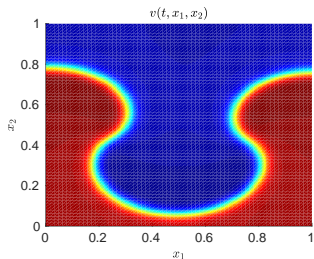
Monodomain equations [B., BREITEN, PUCHE, REIS '21] – (simple) model for the electric activity of the human heart to describe defibrillation processes

$$\begin{aligned}\partial_t v(t) &= \nabla \cdot (D \nabla v(t)) + p_3(v)(t) - w(t) + I_{s,i}(t) + B I_{s,e}(t), \\ \partial_t w(t) &= c v(t) - d w(t), \quad y(t) = B' v(t)\end{aligned}$$

Monodomain equations [B., BREITEN, PUCHE, REIS '21] – (simple) model for the electric activity of the human heart to describe defibrillation processes

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Control objective: “reentry waves”, which can be interpreted as fibrillation processes, should be terminated



Outlook: The future of funnel control

[B., KÄSTNER, WORTHMANN '20]: Funnel-MPC

$$\underset{u \in L^\infty(\widehat{t}, \widehat{t}+T, \mathbb{R}^m)}{\text{minimize}} \int_{\widehat{t}}^{\widehat{t}+T} \frac{\lambda_1}{\psi(t) - \|h(x(t)) - y_{\text{ref}}(t)\|} + \lambda_2 \|u(t)\|^2 dt$$

$$\text{subject to } \dot{x}(t) = f(t, x(t), u(t)), \\ x(\widehat{t}) = \widehat{x}, \quad \|u\|_\infty \leq M$$

Outlook: The future of funnel control

[B., KÄSTNER, WORTHMANN '20]: **Funnel-MPC**

$$\underset{u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)}{\text{minimize}} \int_{\hat{t}}^{\hat{t}+T} \frac{\lambda_1}{\psi(t) - \|h(x(t)) - y_{\text{ref}}(t)\|} + \lambda_2 \|u(t)\|^2 dt$$

$$\begin{aligned} \text{subject to } \dot{x}(t) &= f(t, x(t), u(t)), \\ x(\hat{t}) &= \hat{x}, \quad \|u\|_\infty \leq M \end{aligned}$$

Future goals:

- prove initial and recursive feasibility, while $\|e(t)\| < \psi(t)$ is guaranteed
- extend control scheme so that robustness is inherited from funnel control
- combine with machine learning for continuous model adaptation and improvement of controller performance