Nulldynamik und Stabilisierbarkeit linearer DAEs

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Schloss Eringerfeld, 4. März 2013

$$
\left.\begin{array}{rcl}
\frac{\mathsf{d}}{\mathsf{d}t}Ex(t) & = & Ax(t) + Bu(t) \\
y(t) & = & Cx(t) \\
E, A \in \mathbb{R}^{l\times n}, & & B \in \mathbb{R}^{l\times m}, \quad C \in \mathbb{R}^{p\times n}
\end{array}\right\} =: \Sigma_{l,n,m,p}
$$

$$
\mathfrak{B} := \left\{ \ (x,u,y) \in \mathcal{C}^0 \ \middle| \ Ex \in \mathcal{C}^1 \text{ und } (x,u,y) \text{ löst } [E,A,B,C] \ \right\}
$$

Nulldynamik:

$$\mathcal{ZD} := \left\{ (x,u,y) \in \mathfrak{B} \;\middle|\; \begin{array}{rcl} \frac{\mathsf{d}}{\mathsf{d}t} Ex &=& Ax + Bu \\ 0 = \quad y &=& Cx \end{array} \right\}$$

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$\mathcal{ZD}$ **autonom** $:\Longleftrightarrow$

$\forall\, w_1, w_2 \in \mathcal{Z}D\; \forall\, I \subseteq \mathbb{R}$ offenes Intervall :

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Lem.: $\mathcal{ZD}$ stabil $\implies$ $\mathcal{ZD}$ autonom

$[E, A, B, C]$ stabilisierbar im behavior-Sinne $:\Longleftrightarrow$

$$\forall\,(x,u,y) \in \mathfrak{B}\ \exists\,(\hat{x},\hat{u},\hat{y}) \in \mathfrak{B} :$$
$$(x,u)|_{(-\infty,0]} = (\hat{x},\hat{u})|_{(-\infty,0]} \ \wedge\ \lim_{t\to\infty} (\hat{x}(t),\hat{u}(t)) = 0.$$

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$$\forall\,\lambda\in\overline{\mathbb{C}}_+:\;\operatorname{rk}_{\mathbb{C}}[\lambda E - A, -B] = \operatorname{rk}_{\mathbb{R}(s)}[sE - A, -B]$$

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Proposition:

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$\Longrightarrow$ $[E, A, B, C]$ stabilisierbar im behavior-Sinne

$\mathcal{V} \subseteq \mathbb{R}^n$ heißt **(E, A, B)-invariant** $:\Longleftrightarrow \; A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B$

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Wähle $V \in \mathbb{R}^{n \times k}$ mit $\operatorname{rk} V = k$, so dass

$$\operatorname{im} V = \sup \{ \; \mathcal{V} \; | \; \mathcal{V} \text{ ist } (E, A, B)\text{-invariant } \wedge \; \mathcal{V} \subseteq \ker C \; \}$$

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Proposition:

$$\mathcal{ZD} \text{ autonom} \quad \Longleftrightarrow \quad \begin{cases} \textbf{(A1)} & \operatorname{rk} B = m \\ \textbf{(A2)} & \ker E \cap \operatorname{im} V = \{0\} \\ \textbf{(A3)} & \operatorname{im} B \cap \operatorname{im} EV = \{0\} \end{cases}$$

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Theorem („Normalform“)

- $[E,A,B,C] \in \Sigma_{l,n,m,p}$ mit $\mathcal{ZD}$ autonom
- $\operatorname{im} V = \sup\{\ \mathcal{V} \mid \mathcal{V}$ ist (E,A,B)-invariant $\wedge\ \mathcal{V} \subseteq \ker C\ \}$

$\Longrightarrow\ \exists\ S \in \mathbf{Gl}_l(\mathbb{R}), W \in \mathbb{R}^{n\times(n-k)}:\ [V,W] \in \mathbf{Gl}_n(\mathbb{R}) \wedge$

$$SE[V,W] = \begin{bmatrix} I_k & E_2 \\ 0 & E_4 \\ 0 & E_6 \end{bmatrix},\ SA[V,W] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ 0 & A_6 \end{bmatrix},\ SB = \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix},$$

$$C[V,W] = [0, C_2]$$

$[\hat{E}, \hat{A}, \hat{B}, \hat{C}] \in \Sigma_{\hat{l},\hat{n},p,m}$ heißt **Inverses** von $[E, A, B, C] \in \Sigma_{l,n,m,p} \;:\Longleftrightarrow$

$$\forall\,(u,y) \in \mathcal{C}^0: \qquad \begin{array}{rl} & \exists\, x \in \mathcal{C}^0:\; (x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \\ \Longleftrightarrow & \exists\, \hat{x} \in \mathcal{C}^0:\; (\hat{x},y,u) \in \mathfrak{B}_{[\hat{E},\hat{A},\hat{B},\hat{C}]} \end{array}$$

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$$\begin{array}{l} [E, A, B, C] \\ \text{mit } \operatorname{rk} B = q \le m \end{array} \quad \Longrightarrow \quad \begin{bmatrix} S_1 \\ S_2 \end{bmatrix} BT = \begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$$

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$$\left.\begin{array}{rcl} \tfrac{\mathrm{d}}{\mathrm{d}t} S_1 E x &=& S_1 A x + \tilde{u}_1 \\ \tfrac{\mathrm{d}}{\mathrm{d}t} S_2 E x &=& S_2 A x \\ y &=& C x. \end{array}\right\} \quad \overset{\hat{x}=(x,w,z)}{\underset{u=T\binom{\tilde{u}_1}{\tilde{u}_2}}{\Longleftrightarrow}} \quad \left\{\begin{array}{rcl} \tfrac{\mathrm{d}}{\mathrm{d}t} S_1 E x &=& w \\ \tfrac{\mathrm{d}}{\mathrm{d}t} S_2 E x &=& S_2 A x \\ 0 &=& -Cx + y \\ \tilde{u}_1 &=& -S_1 A x + w \\ \tilde{u}_2 &=& z. \end{array}\right.$$

$[\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]$ **links-invertierbar** $:\Longleftrightarrow$

$$\forall\, (x_1, u_1, y_1), (x_2, u_2, y_2) \in \mathfrak{B} : \qquad \Big[y_1 = y_2 \ \wedge \ Ex_1(0) = Ex_2(0) = 0 \Big] \implies u_1 = u_2$$

$[\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]$ **rechts-invertierbar** $:\Longleftrightarrow$

$$\forall\, y \in \mathcal{C}^\infty \ \exists\, (x, u) \in \mathcal{C}^0 : \ (x, u, y) \in \mathfrak{B}$$

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Lem.: $\mathcal{ZD}$ autonom $\implies$ $[E, A, B, C]$ links-invertierbar

Klar: $[E, A, B, C]$ rechts-invertierbar $\implies$ $\operatorname{rk} C = p$

Theorem (System inversion form)

- $[E, A, B, C] \in \Sigma_{l,n,m,p}$ mit $\operatorname{rk} C = p$
- $\mathcal{ZD}$ autonom

$\Longrightarrow \quad \exists\, S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R})$: $[SET, SAT, SB, CT]$ hat die Form

$$
\begin{aligned}
\dot{x}_1 &= Qx_1 + A_{12}y - E_{13}\dot{x}_3 \\
E_{22}\dot{y} &= A_{22}y - E_{23}\dot{x}_3 + A_{21}x_1 + u \\
x_3 &= \textstyle\sum_{k=0}^{\nu-1} N^k E_{32} y^{(k+1)} \\
0 &= A_{42}y - E_{42}\dot{y} - E_{43}\dot{x}_3
\end{aligned}
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Lem.: $\mathcal{ZD}$ stabil $\quad \Longleftrightarrow \quad \sigma(Q) \subseteq \mathbb{C}_-$

Proposition

- $[E, A, B, C] \in \Sigma_{l,n,m,p}$
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$[E, A, B, C]$ rechts-invertierbar $\iff$ $\operatorname{rk} C = p \wedge$

$$
\boxed{
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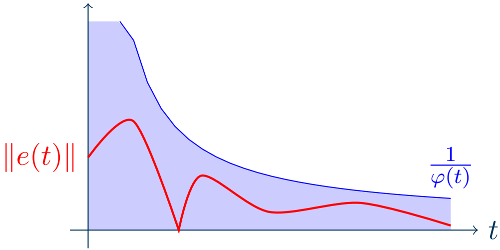
\end{array}
}
$$

Ausblick: funnel control

- $[E, A, B, C] \in \Sigma_{l,n,m,m}$
- $\mathcal{ZD}$ stabil
- $[E, A, B, C]$ rechts-invertierbar
- „Relativgrad-Annahme“:

$$\exists\, \Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)[0, I_m]^\top \quad \wedge \quad \Gamma = \Gamma^\top \geq 0,$$

wobei $L(s)$ Linksinverse von $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ über $\mathbb{R}(s)$ ist



Zusammenfassung

- autonome Nulldynamik erlaubt Entkopplung
- stabile Nulldynamik impliziert Stabilisierbarkeit
- ein inverses System existiert immer
- Bedingungen für Invertierbarkeit bei autonomer Nulldynamik