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Institute for Mathematics, Paderborn University

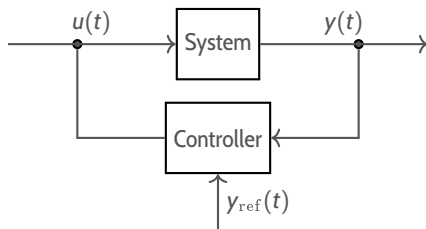
# CONTROL WITH SAFETY GUARANTEES

Thomas Berger

Halle, November 21, 2024



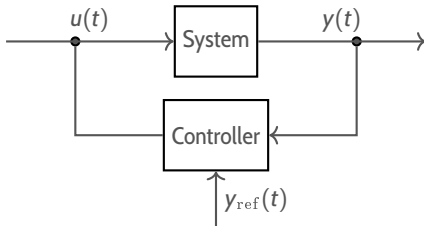
## Control objective



$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), & x(t) &\in X \\ y(t) &= h(x(t)) \end{aligned}$$

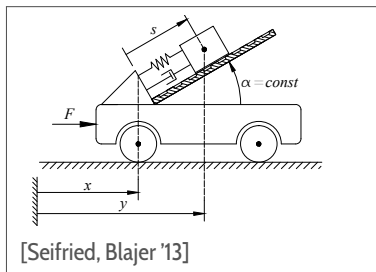
- **Goal:** simple controller, so that “ $y(t)$  tracks  $y_{\text{ref}}(t)$  within **safety region**”
- only uses  $y(t)$ , no knowledge of  $x(t) \in X$  or system parameters

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→ **ODEs and PDEs in the same class!**

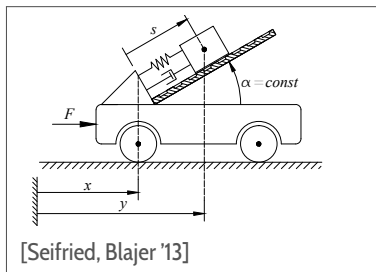


angle:  $0^\circ \leq \alpha \leq 90^\circ$

spring, damper with nonlinear characteristics:  $K(s)$ ,  $D(\dot{s})$

$$u(t) = F$$

$$y(t) = x(t) + s(t) \cos \alpha$$



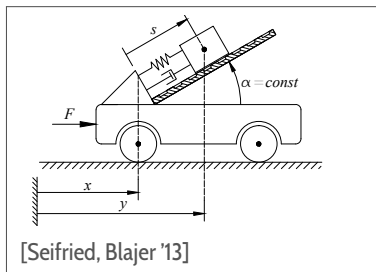
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$$\begin{bmatrix} m_1 + m_2 & m_2 \cos \alpha \\ m_2 \cos \alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -K(s) - D(\dot{s}) + m_2 g \sin \alpha \end{pmatrix}$$



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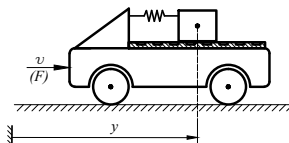
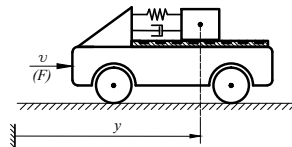
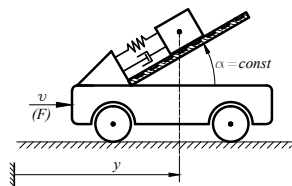
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$$\dot{y} = \dot{x} + \dot{s} \cos \alpha$$

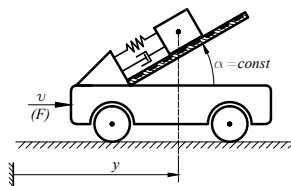
$$\ddot{y} = -c_1(K(s) + D(\dot{s}) - m_2 g \sin \alpha) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$



$$0^\circ < \alpha \leq 90^\circ$$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

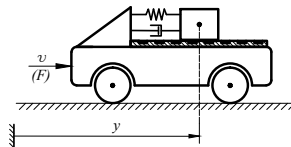
relative degree = 2



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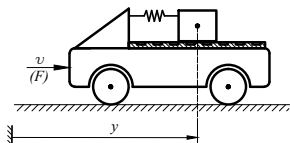
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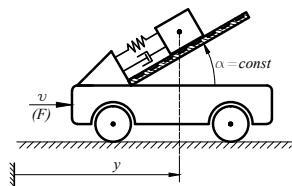
$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

$$y^{(3)} = f_2(s, \dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

relative degree = 3



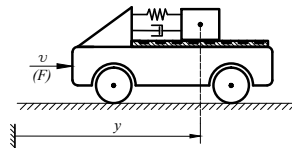




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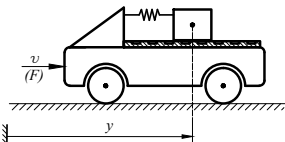
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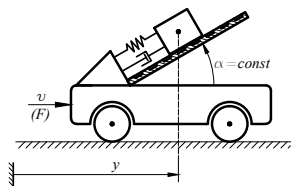
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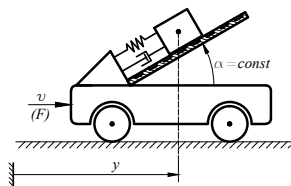
$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

$$y^{(4)} = f_3(s, \dot{s}) + \frac{K'(s)}{m_1 m_2} u$$

relative degree = 4

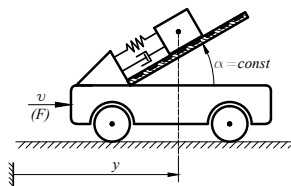


**Internal dynamics:** remaining dynamics when output is fixed



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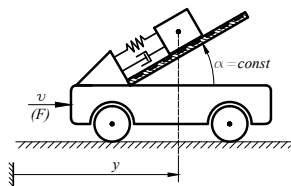
$$\ddot{\eta} = -c_3 K \left( \frac{\eta - y \cos \alpha}{\sin^2 \alpha} \right) - c_3 D \left( \frac{\dot{\eta} - \dot{y} \cos \alpha}{\sin^2 \alpha} \right) + c_4 g \sin \alpha$$



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$$\alpha = 90^\circ, m_2 = 1: \quad \ddot{s} = -K(s) - D(\dot{s}) + g$$

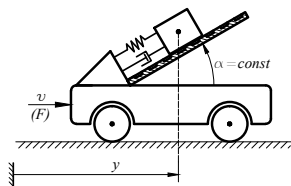


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- Lyapunov function: kinetic + potential energy
- dissipativity:  $D(\dot{s}) \dot{s} \geq 0$



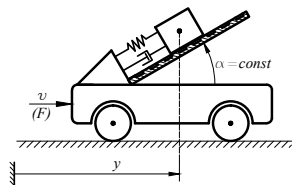
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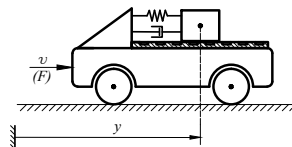
- Lyapunov function: kinetic + potential energy
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$$\Rightarrow s, \dot{s} \in L^\infty \quad (\text{stable internal dynamics})$$



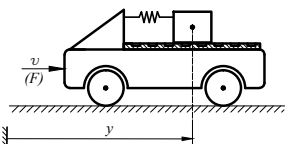
$$0^\circ < \alpha \leq 90^\circ$$

stable internal dynamics



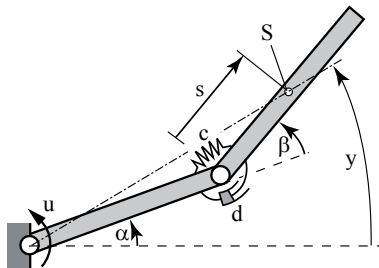
$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

stable internal dynamics



$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

no internal dynamics



[Seifried, Blajer '13]

## Rotational Manipulator Arm

Input: angular velocity of first link

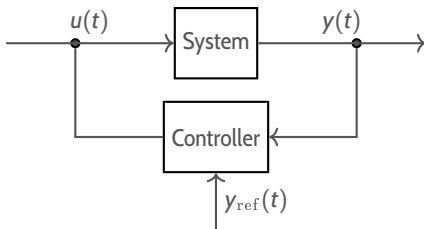
Output: position of  $S$  described by angle  $y$

relative degree = 1

**unstable internal dynamics**



## Reminder



$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), & x(t) &\in X \\ y(t) &= h(x(t)) \end{aligned}$$

- no knowledge of system parameters, only: **known relative degree** and assumption of **stable internal dynamics**
- **Goal:** design simple controller such that “ $y(t)$  tracks  $y_{\text{ref}}(t)$  within **safety region**”

## High-gain based adaptive control

assumption: relative degree = 1, stable internal dynamics

classical (non-adaptive) high-gain controller

$$u(t) = -ky(t), \quad k > 0 \text{ suff. large} \implies y(t) \rightarrow 0$$

drawbacks:  $k$  unnecessary large; restricted to linear systems

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**adaptive high-gain controller (since 1983)**

$$u(t) = -k(t)y(t), \quad \dot{k}(t) = \|y(t)\|^2$$

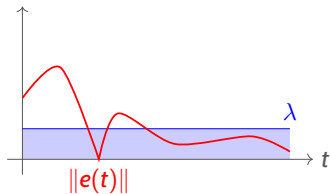
[Byrnes, Ilchmann, Logemann, Mareels, Mårtensson, Morse, Nussbaum, Owens, Prätzel-Wolters, Willems, ...]

drawbacks:  $k(t)$  mon. increasing; restricted to linear systems

## adaptive $\lambda$ -tracker (since 1994)

$$u(t) = -k(t) \underbrace{(y(t) - y_{\text{ref}}(t))}_{=:e(t)},$$

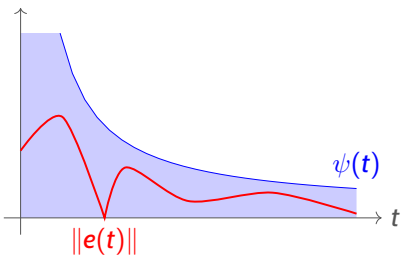
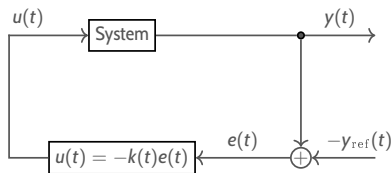
$$\dot{k}(t) = \begin{cases} \frac{\max\{\|e(t)\| - \lambda, 0\}}{\|e(t)\|}, & e(t) \neq 0, \\ 0, & e(t) = 0 \end{cases}$$



[Allgöwer, Ashman, Bullinger, Ilchmann, Logemann, Ryan, Sangwin, ...]

drawbacks:  $k(t)$  mon. increasing; no guaranteed safety

## Funnel control



[Ilchmann, Ryan, Sangwin '02]:  
Works, if

- relative degree = 1
- stable internal dynamics

$$k(t) = \frac{1}{\psi(t) - \|e(t)\|}$$

## Problem: higher relative degree

relative degree 1:

$$\dot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \text{as. stable for } k \gg 0$$

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relative degree 2:

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \text{not as. stable}$$

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relative degree 2:

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \text{not as. stable}$$

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -k_1y(t) - k_2\dot{y}(t) \implies \text{as. stable for } k_1, k_2 \gg 0$$



## Funnel control for systems with higher relative degree

**Funnel control via backstepping:** [Ilchmann, Ryan, Townsend '06 & '07]

drawbacks: escalating controller complexity for relative degree  $\geq 2$ , hence a typically bad controller performance

**relative degree = 2:** [Hackl, Hopfe, Ilchmann, Müller, Trenn '13]

drawbacks: no generalization to arbitrary relative degree

**Bang-bang funnel controller:** [Liberzon & Trenn '13]

drawbacks: restricted to SISO systems, strong compatibility assumptions

**“Prescribed-Performance Control”:** [Bechlioulis & Rovithakis '14]

drawbacks: restricted to systems with trivial internal dynamics

## Funnel control for systems with arbitrary relative degree

[B., Lê, Reis '18] & [B., Ilchmann, Ryan '21]

$$\begin{aligned}e_1(t) &= e(t), & e(t) &= y(t) - y_{\text{ref}}(t), \\e_2(t) &= \dot{e}(t) + k_1(t)e_1(t), \\e_3(t) &= \ddot{e}(t) + k_2(t)e_2(t), \\&\vdots \\e_r(t) &= e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t), \\u(t) &= -k_r(t)e_r(t) \\k_i(t) &= 1/(\psi_i(t) - \|e_i(t)\|), \quad i = 1, \dots, r\end{aligned}$$

## System class

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t), u(t))$$

$d \in L^\infty$ ,  $T : C \rightarrow L_{\text{loc}}^\infty$  causal, locally Lipschitz, BIBO,  $f \in C$  with HG property

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### Linear prototype:

$\dot{x}(t) = Ax(t) + Bu(t)$ ,  $y(t) = Cx(t)$  with

(A1)  $\text{rk}_{\mathbb{C}} \begin{bmatrix} \lambda I - A & B \\ C & O \end{bmatrix} = n + m$  for all  $\lambda \in \mathbb{C}$  with  $\text{Re } \lambda \geq 0$ ;

(A2)  $CB = CAB = \dots = CA^{r-2}B = O$  and  $CA^{r-1}B > O$

## System class

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t), u(t))$$

$d \in L^\infty$ ,  $T : C \rightarrow L_{\text{loc}}^\infty$  causal, locally Lipschitz, BIBO,  $f \in C$  with HG property

### Linear prototype:

Is equivalent to  $\dot{z}(t) = \hat{A}z(t) + \hat{B}u(t)$ ,  $y(t) = \hat{C}z(t)$  with

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & I_m & 0 \\ R_1 & R_2 & \dots & R_r & S \\ P & 0 & \dots & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ 0 \end{bmatrix}, \quad \hat{C} = [I_m \ 0 \ \dots \ 0 \ 0], \quad \sigma(Q) \subseteq \mathbb{C}_-$$

$$\begin{aligned} \implies y^{(r)}(t) &= R_1 y(t) + \dots + R_r y^{(r-1)}(t) + Se^{Qt} \eta(0) \\ &+ \int_0^t Se^{Q(t-\tau)} P y(\tau) d\tau + CA^{r-1} B u(t) \end{aligned}$$

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$d \in L^\infty$ ,  $T : C \rightarrow L^\infty_{\text{loc}}$  causal, locally Lipschitz, BIBO,  $f \in C$  with HG property

### Linear prototype:

$$\implies y^{(r)}(t) = T(y, \dot{y}, \dots, y^{(r-1)})(t) + \Gamma u(t)$$

with  $\Gamma = CA^{r-1}B$  and

$$\begin{aligned} & T(y, \dots, y^{(r-1)})(t) \\ &= R_1 y(t) + \dots + R_r y^{(r-1)}(t) + Se^{Qt} \eta(0) + \int_0^t Se^{Q(t-\tau)} P y(\tau) d\tau \end{aligned}$$

## Funnel control for systems with arbitrary relative degree $r \in \mathbb{N}$

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t), u(t))$$

$$e_1(t) = e(t), \quad e(t) = y(t) - y_{\text{ref}}(t),$$

$$e_2(t) = \dot{e}(t) + k_1(t)e_1(t),$$

$$\vdots$$

$$e_r(t) = e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t),$$

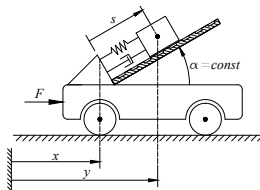
$$u(t) = -k_r(t)e_r(t)$$

$$k_i(t) = 1/(\psi_i(t) - \|e_i(t)\|), \quad i = 1, \dots, r$$

### Theorem [B., Lê, Reis '18] & [B., Ilchmann, Ryan '21]

$$y_{\text{ref}} \in W^{r, \infty} \implies u, k_i, y^{(i)} \in L^\infty \text{ and } \|e_i(t)\| \leq \psi_i(t) - \varepsilon_i$$

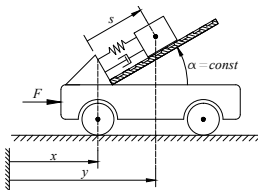
## Control of multibody systems – jointly with R. Seifried (TU Hamburg, Germany)



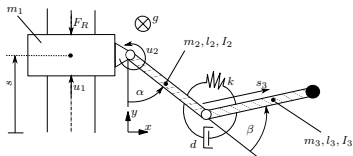
[B., Lê, Reis '18]:  $u(t) = u_{FC}(t)$



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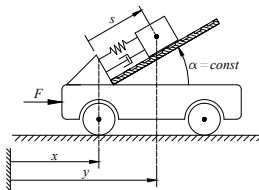


[B., Lê, Reis '18]:  $u(t) = u_{\text{FC}}(t)$

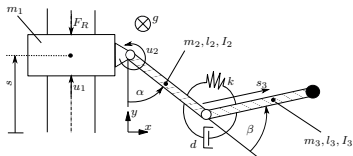


[B., Otto, Reis, Seifried '19]:  
 $u(t) = u_{\text{FC}}(t) + u_{\text{FF}}(t)$

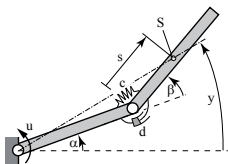
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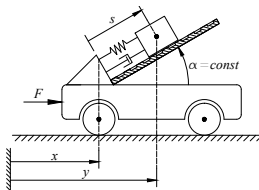


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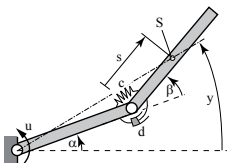


[B., Lanza '20]:  $u(t) = u_{FC}(t)$   
unstable internal dynamics

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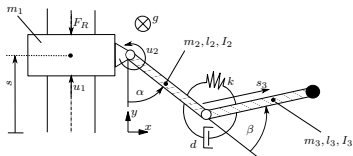


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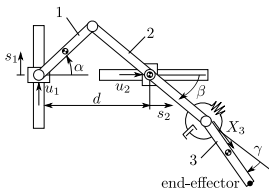


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unstable internal dynamics

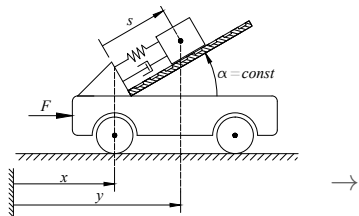
Thomas Berger



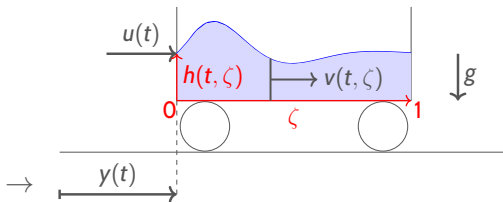
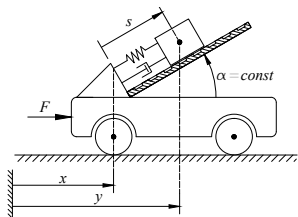
[B., Otto, Reis, Seifried '19]:  
 $u(t) = u_{FC}(t) + u_{FF}(t)$



[B., Drücker, Lanza, Reis, Seifried '21]  
 $u(t) = u_{FC}(t) + u_{FF}(t)$   
unstable internal dynamics, DAE formulation



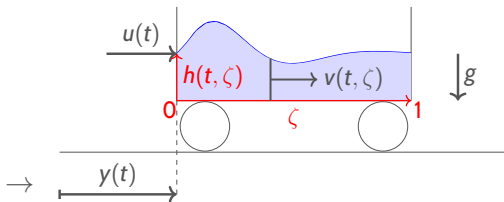
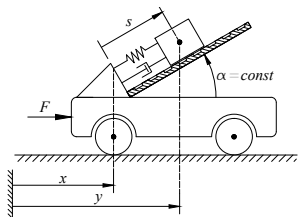
$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma u(t)$$



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$$\ddot{y}(t) = \hat{T}(y, \dot{y})(t) + \hat{\gamma} u(t)$$

[B., Puche, Schwenninger '22]



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[B., Puche, Schwenninger '22]

**Finite and infinite dimensional systems in the same class!**

## Funnel control for $\infty$ -dimensional systems

$$y^{(r)}(t) = f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t), u(t))$$

allows for a “simple” class of  $\infty$ -dimensional systems

→ internal dynamics described by PDE

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→ internal dynamics described by PDE

$r = 1$ : [Ilchmann, Ryan, Sangwin '02, etc.]

$$(Ty)(t) = A_1 y(t) + A_2 \int_0^t \mathcal{T}(t-s) A_3 y(s) ds$$

- $(\mathcal{T}(t))_{t \geq 0}$  exp. stable  $C^0$ -semigroup on real Hilbert space  $X$  with generator  $A_4 : \mathcal{D}(A_4) \subseteq X \rightarrow X$  (finite dimensional:  $\mathcal{T}(t) = e^{A_4 t}$ )
- $(A_4, A_3, A_2)$  “regular well-posed”,  $A_1 \in \mathbb{R}^{m \times m}$



## Funnel control for $\infty$ -dimensional systems

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$r \in \mathbb{N}$ : [Ilchmann, Selig, Trunk '16]

Byrnes-Isidori form for linear  $\infty$ -dimensional systems

$$\begin{aligned}\dot{\eta}(t) &= A_4\eta(t) + A_3y(t), \\ y^{(r)}(t) &= R_1y(t) + \dots + R_r y^{(r-1)}(t) + A_2\eta(t) + \gamma u(t)\end{aligned}$$

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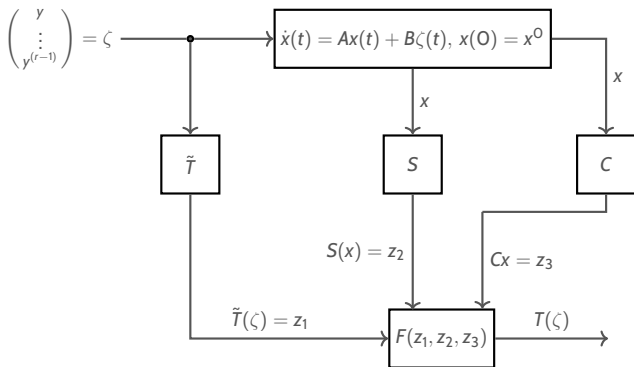
with  $T(y, \dots, y^{(r-1)})(t)$

$$= R_1 y(t) + \dots + R_r y^{(r-1)}(t) + A_2 \int_0^t \mathcal{T}(t-s) A_3 y(s) ds$$

## Funnel control for $\infty$ -dimensional systems

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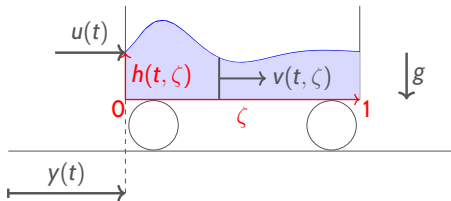
$r \in \mathbb{N}$ : [B., Puche, Schwenninger '20]



- $(A, B, C)$  regular well-posed
- $f \mapsto \mathcal{L}^{-1}(G) * f$  bounded
- $S$  nonlinear,  $\mathcal{D}(S) = X$ ,  $(A, B, S)$  BIBO stable

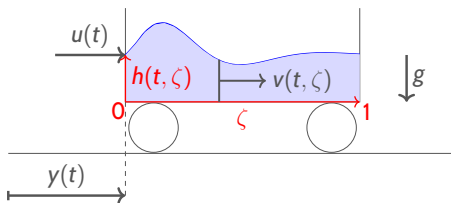
## Rolling water tank – [B., Puche, Schwenninger '22]

$$\begin{aligned} \partial_t h + \partial_\zeta(hv) &= 0, \\ \partial_t v + \partial_\zeta \left( \frac{v^2}{2} + gh \right) \\ + hS \left( \frac{v}{h} \right) &= -\ddot{y} \end{aligned}$$



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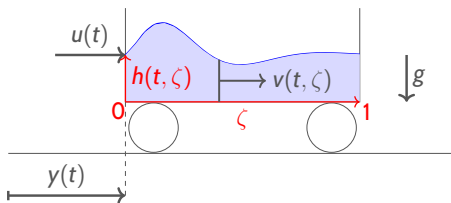
Linearized model:

$$\partial_t z_1 = -h_0 \partial_\zeta z_2, \quad \partial_t z_2 = -g \partial_\zeta z_1 - \mu z_2 - \ddot{y}, \quad z_2(t, 0) = z_2(t, 1) = 0$$

$$\begin{aligned} \ddot{y}(t) &= \frac{g}{2m_T} (z_1(t, 1) - z_1(t, 0)) (2h_0 + z_1(t, 1) + z_1(t, 0)) \\ &+ \frac{\mu h_0}{m_T} \int_0^1 z_2(t, \zeta) d\zeta + \frac{\mu}{m_T} \int_0^1 z_1(t, \zeta) z_2(t, \zeta) d\zeta + \frac{u(t)}{m_T} \end{aligned}$$

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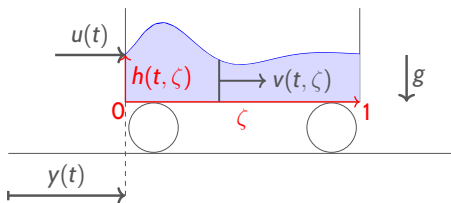
Linearized model:

$$\dot{z}(t) = Az(t) + Ab\dot{y}(t), \quad Az = - \begin{pmatrix} h_0 \partial_\zeta z_2 \\ g \partial_\zeta z_1 + \mu z_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\begin{aligned} z_1(t, 1) - z_1(t, 0) &= Cz(t) = C\mathcal{T}(t)z(0) + C \int_0^t \mathcal{T}(t-s)Ab\dot{y}(s) ds \\ &= c(t) + ((h_{L^1} + h_\delta) * \dot{y})(t) \end{aligned}$$

## Rolling water tank – [B., Puche, Schwenninger '22]

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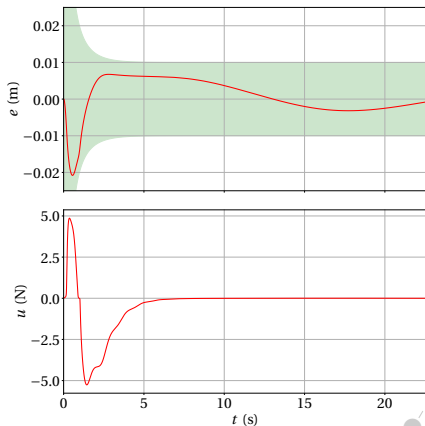
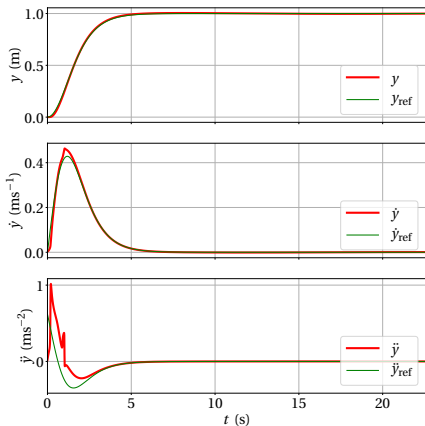
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$$\ddot{y}(t) = T(\dot{y})(t) + \frac{u(t)}{m_T}$$

**Lemma:**  $\|h_\delta\| = \delta_0 + 2 \sum_{k \in \mathbb{N}} (-1)^k e^{-k\mu/\sqrt{h_0 g}} \delta_{k/\sqrt{h_0 g}} \implies T \text{ is BIBO}$

## Simulation





## Funnel control for $\infty$ -dimensional systems

“Hard”  $\infty$ -dimensional systems – there is NO concept of relative degree

- boundary controlled heat equation [Reis, Selig '15]

$$\partial_t x(t) = \Delta x(t), \quad u(t) = (\nu^\top \cdot \nabla x(t))|_{\partial\Omega},$$

$$y(t) = \int_{\partial\Omega} x(t)(\zeta) d\zeta$$

- general class of boundary control problems based on  $m$ -dissipative operators [Puche, Reis, Schwenninger '21, Puche '19]

$$\dot{x}(t) = \mathfrak{A}x(t), \quad x(0) = x_0 \in \mathcal{D}(\mathfrak{A}) \subseteq X,$$

$$u(t) = \mathfrak{B}x(t), \quad y(t) = \mathfrak{C}x(t)$$

e.g. lossy transmission line, wave equation, diffusion equation

- Fokker-Planck equation [B. '21]

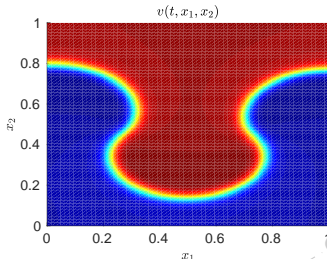
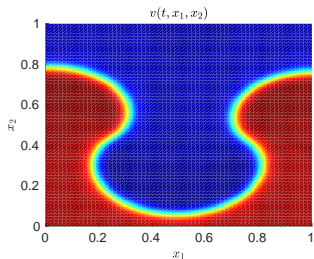
**Monodomain equations** [B., Breiten, Puche, Reis '21] – (simple) model for the electric activity of the human heart to describe defibrillation processes

$$\begin{aligned}\partial_t v(t) &= \nabla \cdot (D \nabla v(t)) + p_3(v)(t) - w(t) + I_{s,i}(t) + B I_{s,e}(t), \\ \partial_t w(t) &= cv(t) - dw(t), \quad y(t) = B'v(t)\end{aligned}$$

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*Control objective:* “reentry waves”, which can be interpreted as fibrillation processes, should be terminated



## An optimization-based approach: Funnel MPC [B., Kästner, Worthmann '20]

OCP:      minimize  $\int_{\hat{t}}^{\hat{t}+T} \ell(t, x(t), u(t)) dt$   
             $u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)$

subject to     $\dot{x}(t) = f(t, x(t), u(t)),$   
                   $x(\hat{t}) = \hat{x}, \quad \|u\|_\infty \leq M$

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$$\underset{u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)}{\text{minimize}} \int_{\hat{t}}^{\hat{t}+T} \ell(t, x(t), u(t)) dt$$

subject to

$$\dot{x}(t) = f(t, x(t), u(t)),$$

$$x(\hat{t}) = \hat{x}, \quad \|u\|_\infty \leq M$$

Idea:

$$\ell(t, x, u) = \|h(x) - y_{\text{ref}}(t)\|^2 + \lambda \|u\|^2$$

$$\rightarrow \ell(t, x, u) = \frac{1}{\psi(t) - \|h(x) - y_{\text{ref}}(t)\|} + \lambda \|u\|^2$$

## An optimization-based approach: Funnel MPC

### Funnel MPC Algorithm

Choose time shift  $\delta > 0$ , prediction horizon  $T \geq \delta$ , initialize  $\hat{t} := t_0$

1. Obtain a measurement of the state at  $\hat{t}$ , set  $\hat{x} = x(\hat{t})$
2. Compute a solution  $u^* \in L^\infty([\hat{t}, \hat{t} + T], \mathbb{R}^m)$  of the OCP
3. Apply the feedback law  $\mu(t, \hat{x}) = u^*(t), t \in [\hat{t}, \hat{t} + \delta)$
4. Set  $\hat{t} := \hat{t} + \delta$  and go to 1.

## An optimization-based approach: Funnel MPC

### Theorem [B., Dennstädt, Ilchmann, Worthmann '22]

relative degree = 1, stable internal dynamics  $\implies \exists M > 0$ : Funnel MPC is initially and recursively feasible such that  $\|u(t)\| \leq M$  and  $\|e(t)\| \leq \psi(t)$

## An optimization-based approach: Funnel MPC

### Theorem [B., Dennstädt, Ilchmann, Worthmann '22]

relative degree = 1, stable internal dynamics  $\implies \exists M > 0$ : Funnel MPC is initially and recursively feasible such that  $\|u(t)\| \leq M$  and  $\|e(t)\| \leq \psi(t)$

1. no terminal costs or conditions required
2. no additional state constraints in the OCP
3. independent of the length of the prediction horizon  $T > 0$



## An optimization-based approach: Funnel MPC

- extension to arbitrary relative degree in [B., Dennstädt '22 & '24]

### Future goals:

- extend control scheme so that robustness is inherited from funnel control (first results for relative degree one systems in [B., Dennstädt, Lanza, Worthmann '24])
- combine with machine learning for continuous model adaptation and improvement of controller performance (results for relative degree one systems submitted [Lanza, Dennstädt, B., Worthmann '23])
- application to magnetic levitation systems (first results in [Oppeneiger, Lanza, Schell, Dennstädt, Schaller, Zamzow, B., Worthmann '24])

## Magnetic levitation system – cooperation with TU Ilmenau and Max Bögl



Bildquelle: <https://transportsystemboegl.com>, Pressemitteilung vom 26.04.2024

## Magnetic levitation system – cooperation with TU Ilmenau and Max Bögl

