

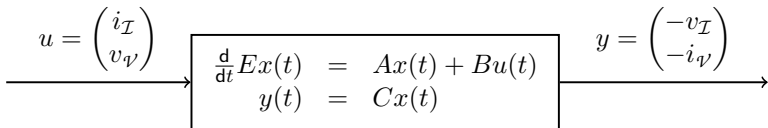
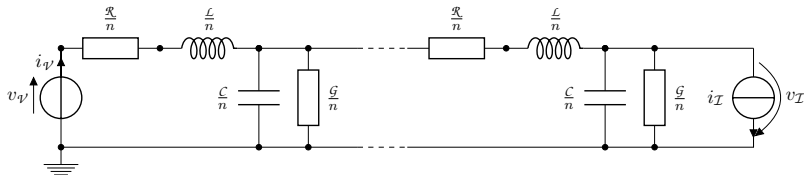
# Funnel control for electrical circuits

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# Example: Transmission Line



$$E = E^T \geq 0, \quad A + A^T \geq 0, \quad B = C^T$$

## Modified Nodal Analysis (MNA)

$$\boxed{\frac{d}{dt}Ex(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t)} \quad u = \begin{pmatrix} i_{\mathcal{I}} \\ v_{\mathcal{V}} \end{pmatrix}, \quad y = \begin{pmatrix} -v_{\mathcal{I}} \\ -i_{\mathcal{V}} \end{pmatrix}$$

$$sE - A = \begin{bmatrix} sA_C C A_C^T + A_{\mathcal{R}} G A_{\mathcal{R}}^T & A_{\mathcal{L}} & A_{\mathcal{V}} \\ -A_{\mathcal{L}}^T & s\mathcal{L} & 0 \\ -A_{\mathcal{V}}^T & 0 & 0 \end{bmatrix}, \quad B = C^T = \begin{bmatrix} -A_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I_{n_{\mathcal{V}}} \end{bmatrix}$$

$A_C, A_{\mathcal{R}}, A_{\mathcal{L}}, A_{\mathcal{V}}, A_{\mathcal{I}}$  – element-related incidence matrices  
 $C, G, \mathcal{L}$  – constitutive relations of capacitances, resistances and inductances

**passivity:**  $C = C^T > 0, \mathcal{L} = \mathcal{L}^T > 0, G + G^T > 0$

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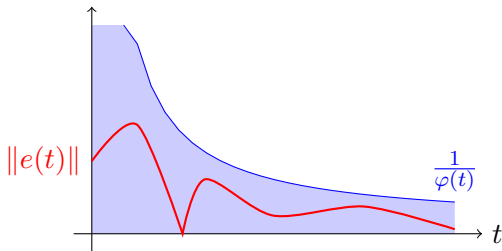
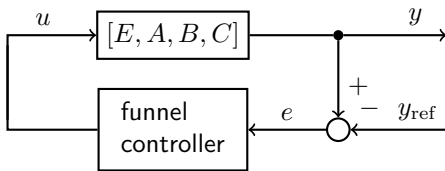
**passivity:**  $C = C^{\top} > 0, \mathcal{L} = \mathcal{L}^{\top} > 0, G + G^{\top} > 0$

$\ell$  is  $\mathcal{K}$  – **loop**  $:\Leftrightarrow$   $\ell$  is a loop in the graph that consists  
 of edges contained in  $\mathcal{K}$  only  
 corr.  $\mathcal{I}$ -loop,  $\mathcal{ICL}$ -loops, etc.

$\mathcal{L}$  is  $\mathcal{K}$  – **cutset**  $:\Leftrightarrow$  deleting  $\mathcal{L} \subseteq \mathcal{K}$  results in a disconnected  
 graph and  $\mathcal{L}$  is minimal  
 corr.  $\mathcal{V}$ -cutset,  $\mathcal{VCL}$ -cutset, etc.

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$



## Zero dynamics:

$$\mathcal{ZD} := \{ (x, u, y) \mid E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) = 0 \}$$

$\mathcal{ZD}$  autonomous  $:\Leftrightarrow$

$\forall w_1, w_2 \in \mathcal{ZD} \forall I \subseteq \mathbb{R}$  open interval :

$$w_1|_I = w_2|_I \implies w_1 = w_2$$

$\mathcal{ZD}$  stable  $:\Leftrightarrow \forall w \in \mathcal{ZD} : \lim_{t \rightarrow \infty} w(t) = 0$

$\lambda \in \mathbb{C}$  is invariant zero  $:\Leftrightarrow$

$$\text{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} < \text{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}$$

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## Theorem (stable zero dynamics)

Let  $[E, A, B, C]$  be a MNA model of a circuit.

$$\mathcal{ZD} \text{ stable} \iff \left\{ \begin{array}{l} \bullet \mathcal{ZD} \text{ autonomous} \\ \bullet \text{ all invariant zeros} \subseteq \mathbb{C}_- \end{array} \right.$$

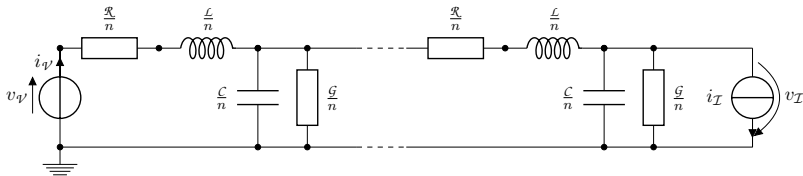
$$\mathcal{ZD} \text{ autonomous} \iff \text{neither } \mathcal{I}\text{-loops nor } \mathcal{V}\text{-cutsets}$$

$$\text{all invariant zeros} \subseteq \mathbb{C}_- \iff \left\{ \begin{array}{l} \bullet \text{ neither } \mathcal{I}\mathcal{L}\text{-loops except for } \mathcal{I}\text{-loops, nor } \mathcal{V}\mathcal{C}\mathcal{L}\text{-cutsets except for } \mathcal{V}\mathcal{L}\text{-cutsets} \\ \bullet \text{ neither } \mathcal{V}\mathcal{C}\text{-cutsets except for } \mathcal{V}\text{-cutsets, nor } \mathcal{I}\mathcal{C}\mathcal{L}\text{-loops except for } \mathcal{I}\mathcal{C}\text{-loops} \end{array} \right.$$

$\mathcal{ZD}$  autonomous  $\iff$  neither  $\mathcal{I}$ -loops nor  $\mathcal{V}$ -cutsets

all invariant zeros  $\subseteq \mathbb{C}_-$   $\iff$

- neither  $\mathcal{I}\mathcal{L}$ -loops except for  $\mathcal{I}$ -loops, nor  $\mathcal{V}\mathcal{C}\mathcal{L}$ -cutsets except for  $\mathcal{V}\mathcal{L}$ -cutsets
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Transmission line has stable zero dynamics!

## Theorem (funnel control - stable zero dynamics)

Let  $[E, A, B, C]$  be a MNA model of a circuit with

- $\mathcal{ZD}$  stable,
- $y_{\text{ref}} \in \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ .

Then the *funnel controller*

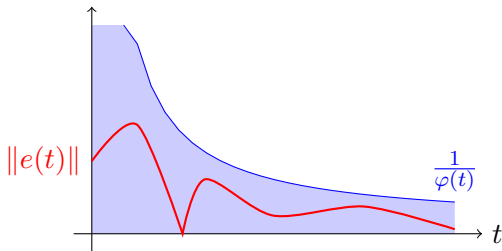
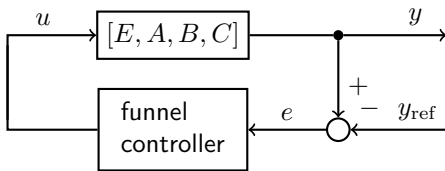
$$\begin{aligned}
 u(t) &= -k(t) e(t), & \text{where } e(t) &= y(t) - y_{\text{ref}}(t) \\
 k(t) &= \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2},
 \end{aligned}$$

applied to  $[E, A, B, C]$  achieves that

$$x \in L^\infty, k \in L^\infty \quad \wedge \quad \exists \varepsilon > 0 \quad \forall t > 0 : \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon.$$

$$E\dot{x}(t) = Ax(t) + Bu(t)$$

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## Theorem (funnel control - stable invariant zeros)

Let  $[E, A, B, C]$  be a MNA model of a circuit with

- all invariant zeros  $\subseteq \mathbb{C}_-$
- $y_{\text{ref}} \in \mathcal{B}^\infty \left( \mathbb{R}_{\geq 0}; \text{im } A_{\mathcal{I}}^\top \times \ker Z_{\mathcal{CRLI}}^\top A_{\mathcal{V}} \right)$ , where

$$\text{im } Z_{\mathcal{CRLI}} = \ker \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{I}} \end{bmatrix}^\top.$$

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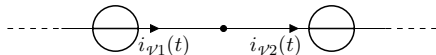
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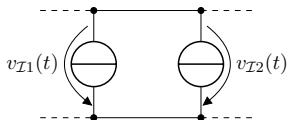
## Interpretation of

$$y_{\text{ref}}(t) \in \text{im } A_I^T \times \ker Z_{CRLI}^T A_V \quad \forall t \geq 0$$

→  $y_{\text{ref}}$  satisfies Kirchhoff's laws pointwise!



$$\Rightarrow i_{\nu_1}(t) = i_{\nu_2}(t)$$



$$\Rightarrow v_{I_1}(t) = v_{I_2}(t)$$



# Simulation - sawtooth wave signal

