Controlled invariance for nonlinear descriptor systems

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Linear systems

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$$\iff \quad x_1 = -u_1, \quad x_2 = -\dot{u}_1, \quad u_2 = 0, \quad x_3 \text{ free}$$

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$$\mathfrak{B} = \left\{ \ (x, u) \in \mathcal{C}^1 \times \mathcal{C} \ \middle| \ E\dot{x}(t) = Ax(t) + Bu(t) \ \right\}$$

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Def.: $\mathcal{V} \subseteq \mathbb{R}^n$ **is controlled invariant** $:\Longleftrightarrow$

$$\forall\, x^0 \in \mathcal{V}\; \exists\, (x,u) \in \mathfrak{B}\; \forall\, t \ge 0: \; x(0) = x^0 \;\wedge\; x(t) \in \mathcal{V}$$

Theorem (controlled invariance)

The following is equivalent for (E, A, B) and $\mathcal{V} \subseteq \mathbb{R}^n$:

(1) $\mathcal{V}$ is controlled invariant

(2) $A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B$

(3) $\exists\, F \in \mathbb{R}^{m\times n}: \ (A+BF)\mathcal{V} \subseteq E\mathcal{V}$

Proof: (1)⇒(2):

$$x^0 \in \mathcal{V} \ \Rightarrow \ Ax^0 = Ax(0) = E\dot{x}(0) - Bu(0) \in E\mathcal{V} + \operatorname{im} B$$

(2)⇒(3):

$$\operatorname{im} V = \mathcal{V}, \ AV = EVW + BU \quad \Rightarrow \quad F := -U(V^\top V)^{-1} V^\top$$

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Existence lemma: $E, A \in \mathbb{R}^{\ell \times n}$ such that $\operatorname{im} A \subseteq \operatorname{im} E$, then

$$\forall\, x^0 \in \mathbb{R}^n\ \exists\, x \in \mathcal{C}^\infty\ \forall\, t \in \mathbb{R}:\ x(0) = x^0\ \wedge\ E\dot{x}(t) = Ax(t)$$

(3)⇒(1):

$$x^0 = Vw^0 \in \mathcal{V}, \quad \operatorname{im}(A+BF)V \subseteq \operatorname{im} EV$$

$$\overset{\text{lemma}}{\Longrightarrow} \quad \exists\, w \in \mathcal{C}^\infty:\ w(0) = w^0\ \wedge\ EV\dot{w}(t) = (A+BF)Vw(t),$$

$\Longrightarrow$ $x := Vw,\ u := FVw$ satisfy

$(x,u) \in \mathfrak{B},\ x(0) = x^0$ and $x(t) \in \mathcal{V},\ t \geq 0$ □

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Nonlinear systems

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t} E(x(t)) = f(x(t)) + g(x(t))\, u(t)} \qquad (E, f, g)$$

$X \subseteq \mathbb{R}^n$ open, $0 \in X$, $E, f : X \to \mathbb{R}^\ell$, $g : X \to \mathbb{R}^{\ell \times m}$ diff., $f(0) = 0$

$$\mathfrak{B} = \left\{ \ (x,u) \in \mathcal{C}^1 \times \mathcal{C} \ \middle| \ (x,u) \text{ is a maximal solution of } (E,f,g) \ \right\}$$

M – connected submanifold of X with $0 \in M$

Def.: M **is locally controlled invariant** $:\Longleftrightarrow$

$\exists$ open neighborhood U of $0 \in X$ such that

$\forall x^0 \in M \cap U \ \exists (x,u) \in \mathfrak{B} \ \exists t_0 \in \operatorname{dom} x,\ x(t_0) = x^0 :$

$\big(\forall t \in \operatorname{dom} x,\ t \ge t_0 : \ x(t) \in M \cap U\big)$

$\vee \ \big(\exists \hat{t} \in \operatorname{dom} x, \hat{t} > t_0 \ \forall t \in [t_0, \hat{t}) : \ x(t) \in M \cap U \ \wedge \ x(\hat{t}) \in \partial(M \cap U)\big)$

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Theorem (local controlled invariance)

$E \in \mathcal{C}^2$, $f, g \in \mathcal{C}^1$, M connected submanifold of X with $0 \in M$ such that, in a neighborhood of $0 \in M$,

$$\dim E'(x)T_xM = \text{const} \quad \wedge \quad \dim\big(E'(x)T_xM + \operatorname{im} g(x)\big) = \text{const}$$

Then the following is equivalent:

(1) M is locally controlled invariant.

(2) $f(x) \in E'(x)T_xM + \operatorname{im} g(x)$ in $M \cap U$.

(3) $\exists\, u \in \mathcal{C}^1(M \cap U \to \mathbb{R}^m):\ f(x) + g(x)u(x) \in E'(x)T_xM$ in $M \cap U$.

Proof: (1)⇒(2): $x^0 \in M \cap U$

$$\Rightarrow\ f(x^0) = E'(x(0))\dot{x}(0) - g(x(0))u(0) \in E'(x^0)T_{x^0}M + \operatorname{im} g(x^0)$$

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Existence lemma: $U \subseteq \mathbb{R}^n$ open, $E, f : U \to \mathbb{R}^\ell$ diff. and

$$\forall\, x \in U :\ \operatorname{rk} E'(x) = r \quad \wedge \quad f(x) \in E'(x) T_x U,$$

$$\Longrightarrow\ \forall\, x^0 \in U\, \exists\, x \in \mathcal{C}^1(I \to \mathbb{R}^n)\, \forall\, t \in I :\ x(0) = x^0 \wedge \tfrac{\mathrm{d}}{\mathrm{d}t} E(x(t)) = f(x(t))$$

(3)⇒(1): $x^0 = \psi(w^0) \in M \cap U$, $\psi : G \to M \cap U$ parametrization of M

def. $\tilde{E} := E \circ \psi$, $\tilde{f} := f \circ \psi + (g \circ \psi)(u \circ \psi)$

$\Longrightarrow$ $\operatorname{rk} \tilde{E}'(x) = \operatorname{rk} E'(\psi(x))\psi'(x) = \dim E'(\psi(x)) T_{\psi(x)} M =$ const,

$\tilde{f}(x) \in E'(\psi(x)) T_{\psi(x)} M = E'(\psi(x))\psi'(x) T_x G = \tilde{E}'(x) T_x G$

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and $x(t) \in M \cap U$, $\forall\, t \in I, t \geq 0$

Remains: show that $(x, u \circ x)$ can be extended to a maximal sln.

$\rightarrow$ Zorn's Lemma □

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