

# Funnel-Regelung über Funnel-Vorkompensatoren

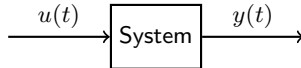
Thomas Berger

Lê Huy Hoàng, Timo Reis

Fachbereich Mathematik, Universität Hamburg

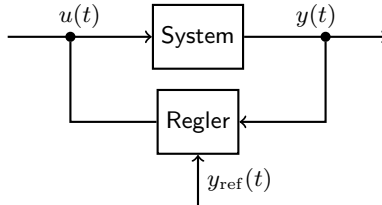
Hamburg, 18. Juli 2017

# Steuerung und Regelung



$$\begin{aligned}\dot{x}(t) &= f(x(t), u(t)), & x(0) &= x^0 \\ y(t) &= h(x(t))\end{aligned}$$

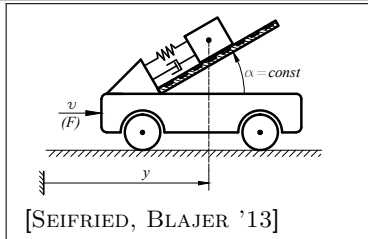
# Steuerung und Regelung



$$\dot{x}(t) = f(x(t), u(t)), \quad x(0) = x^0$$

$$y(t) = h(x(t))$$

- **Ziel:** Reglerentwurf so dass „ $y(t)$  verfolgt  $y_{\text{ref}}(t)$ “
- keine Kenntnis der Systemparameter, nur „strukturelle“ Annahmen an das Modell



Winkel:  $0^\circ \leq \alpha \leq 90^\circ$

Feder, Dämpfer mit nichtlinearer Kennlinie:  $K(z)$ ,  $D(\dot{z})$

$$u(t) = F$$

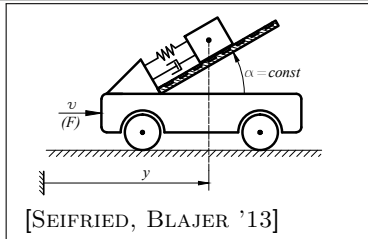
$$y(t) = q(t) + z(t) \cos \alpha$$

$$F = m_1 \ddot{q} + m_2 \ddot{y} = m_1 \ddot{q} + m_2 \ddot{q} + m_2 \ddot{z} \cos \alpha,$$

$$m_2 \ddot{z} = -K(z) - D(\dot{z}) + m_2 g \sin \alpha - m_2 \ddot{q} \cos \alpha$$

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos \alpha \\ m_2 \cos \alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} u \\ -K(z) - D(\dot{z}) + m_2 g \sin \alpha \end{pmatrix}$$

$$\ddot{y} = -c_1 (K(z) + D(\dot{z}) - m_2 g \sin \alpha) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$



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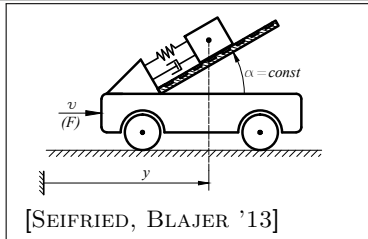
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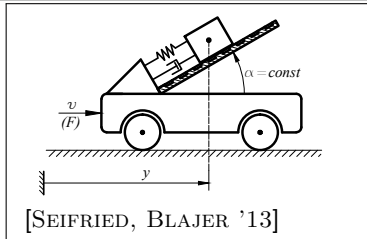
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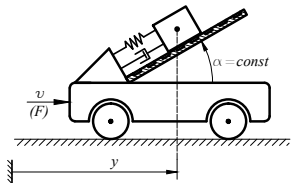
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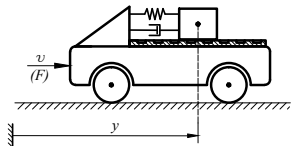
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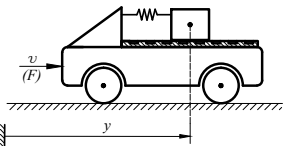
**Relativgrad = 2**



$$\alpha = 0^\circ, \quad D'(\dot{z}) \neq 0$$

$$y^{(3)} = f_2(z, \dot{z}) + \frac{D'(\dot{z})}{m_1 m_2} u$$

**Relativgrad = 3**

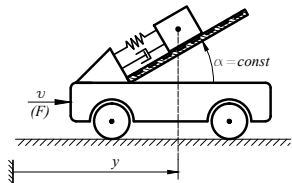


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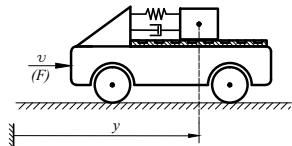




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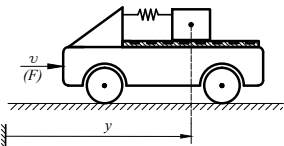
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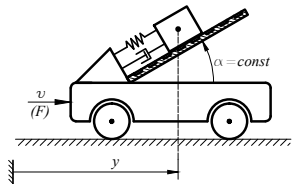
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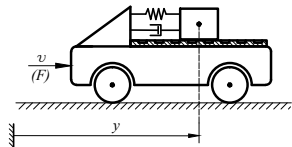
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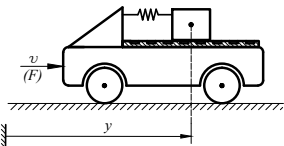
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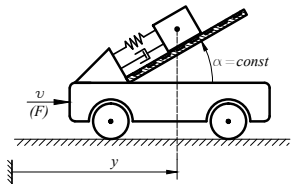
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**Interne Dynamik:** verbleibende Dynamik bei fixiertem Ausgang

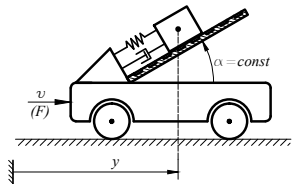
Lyapunov-Funktion: kinetische + potentielle Energie

$$V = \frac{1}{2} \dot{z}^2 + V_{\text{pot}}(z), \quad \frac{d}{dz} V_{\text{pot}}(z) = K(z) - g, \quad \lim_{|z| \rightarrow \infty} V_{\text{pot}}(z) = \infty$$

$$\dot{V} = \dot{z}\ddot{z} + K(z)\dot{z} - g\dot{z} = \underbrace{-D(\dot{z})}_{\geq 0} \dot{z} \leq 0$$

$\implies$  jede Subniveaumenge von  $V$  ist positiv invariant und kompakt

$$\implies z, \dot{z} \in L^\infty \quad (\text{stabile interne Dynamik})$$



**Interne Dynamik:** verbleibende Dynamik bei fixiertem Ausgang

$$\ddot{\eta} = -c_3 K \left( \frac{\eta - y \cos \alpha}{\sin^2 \alpha} \right) - c_3 D \left( \frac{\dot{\eta} - \dot{y} \cos \alpha}{\sin^2 \alpha} \right) + c_4 g \sin \alpha$$

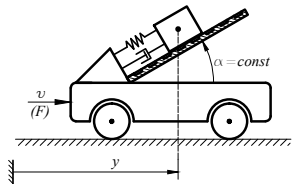
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**Interne Dynamik:** verbleibende Dynamik bei fixiertem Ausgang

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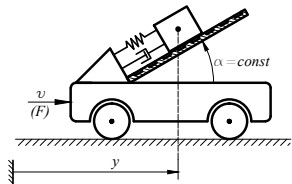
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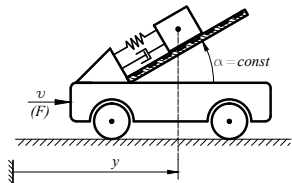
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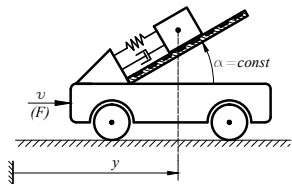
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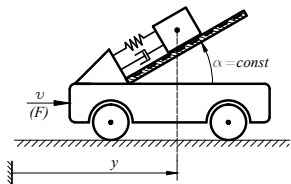
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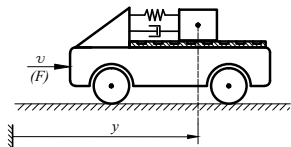
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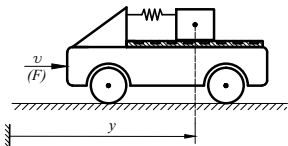
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stabile interne Dynamik



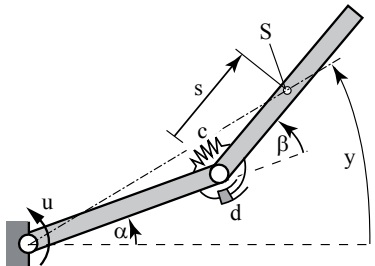
$$\alpha = 0^\circ, \quad D'(\dot{z}) \neq 0$$

stabile interne Dynamik



$$\alpha = 0^\circ, \quad D'(\dot{z}) = 0, \quad K'(z) \neq 0$$

keine interne Dynamik



[SEIFRIED, BLAJER '13]

## Rotatorischer Arm

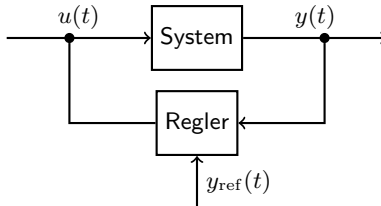
Eingang: Winkelgeschwindigkeit  
des angetriebenen Arms

Ausgang: Lage von  $S$  beschrie-  
ben durch Winkel  $y$

Relativgrad = 1

**instabile interne Dynamik**

# Erinnerung



$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), & x(0) &= x^0 \\ y(t) &= h(x(t)) \end{aligned}$$

- **Ziel:** Reglerentwurf so dass „ $y(t)$  verfolgt  $y_{\text{ref}}(t)$ “
- keine Kenntnis der Systemparameter, nur: **Kenntnis des Relativgrads** und Annahme von **stabiler interner Dynamik**

## Hochverstärkungsbasierte adaptive Regelung

Annahme: Relativgrad = 1, stabile interne Dynamik

**klassischer (nicht-adaptiver) Hochverstärkungsregler**

$$u(t) = -ky(t), \quad k > 0 \text{ hinr. groß} \implies y(t) \rightarrow 0$$

Nachteile:  $k$  evtl. unnötig groß; auf lineare Systeme beschränkt

adaptiver Hochverstärkungsregler (ab ca. 1983)

$$u(t) = -k(t)y(t), \quad \dot{k}(t) = \|y(t)\|^2$$

[BYRNES, ILCHMANN, LOGEMANN, MAREELS, MÅRTENSSON,  
MORSE, NUSSBAUM, OWENS, PRÄTZEL-WOLTERS, WILLEMS, ...]

Nachteile:  $k(t)$  mon. wachsend; auf lineare Systeme beschränkt

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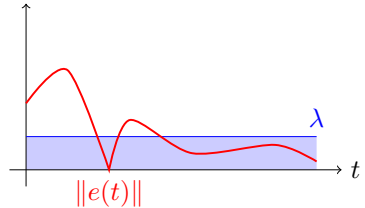
[BYRNES, ILCHMANN, LOGEMANN, MAREELS, MÅRTENSSON, MORSE, NUSSBAUM, OWENS, PRÄTZEL-WOLTERS, WILLEMS, ...]

Nachteile:  $k(t)$  mon. wachsend; auf lineare Systeme beschränkt

## adaptiver $\lambda$ -tracker (ab ca. 1994)

$$u(t) = -k(t) \underbrace{(y(t) - y_{\text{ref}}(t))}_{=: e(t)},$$

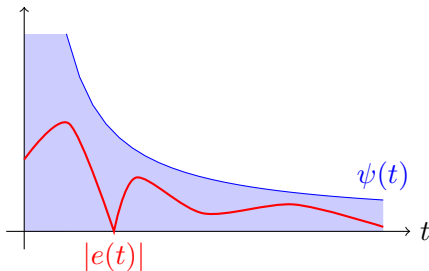
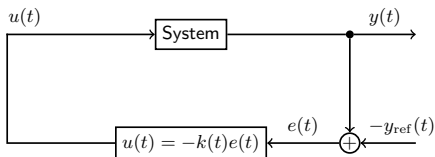
$$\dot{k}(t) = \begin{cases} \frac{\max\{\|e(t)\| - \lambda, 0\}}{\|e(t)\|}, & e(t) \neq 0, \\ 0, & e(t) = 0 \end{cases}$$



[ALLGÖWER, ASHMAN, BULLINGER, ILCHMANN, LOGEMANN, RYAN, SANGWIN, ...]

Nachteile:  $k(t)$  mon. wachsend; kein transientes Verhalten

# Funnel-Regelung



[ILCHMANN, RYAN, SANGWIN '02]:  
 Funktioniert, wenn

- Relativgrad = 1
- stabile interne Dynamik

$$k(t) = \frac{\psi(t)}{\psi(t) - |e(t)|}$$

## Problem: höherer Relativgrad

Relativgrad 1:

$$\dot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \text{as. stabil für } k \gg 0$$

Relativgrad 2:

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \text{nicht as. stabil}$$

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -k_1y(t) - k_2\dot{y}(t) \implies \text{as. stabil für } k_1, k_2 \gg 0$$



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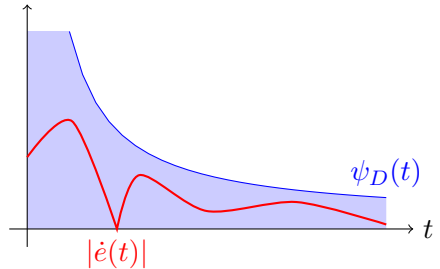
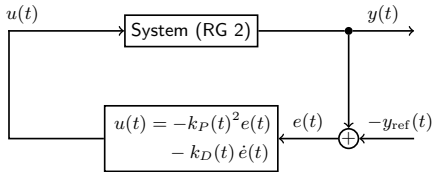
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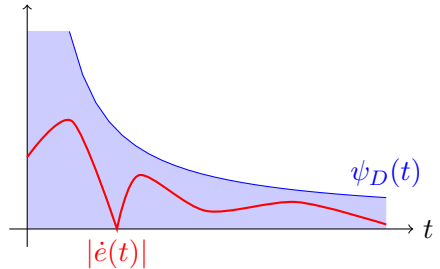
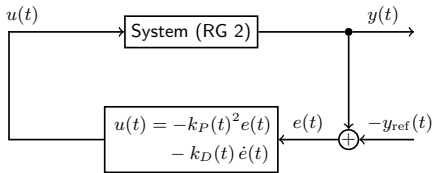
$$k_P(t) = \frac{\psi_P(t)}{\psi_P(t) - |e(t)|},$$

$$k_D(t) = \frac{\psi_D(t)}{\psi_D(t) - |\dot{e}(t)|},$$

$$0 < \delta \leq \frac{d}{dt}\psi_P(t) + \psi_D(t)$$

[HACKL, HOPFE, ILCHMANN,  
MÜLLER, TRENN '13]:  
Funktioniert...

Aber: keine Verallgemeinerung auf höheren Relativgrad!



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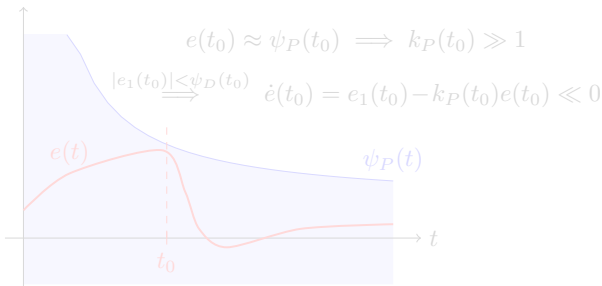
## Ansatz [HACKL ET AL. '13]

$$e(t_0) \approx \psi_P(t_0) \implies \ddot{e}(t) < 0 \text{ auf } [t_0, t_1) \wedge \dot{e}(t) < \dot{\psi}_P(t) \text{ auf } [t_1, t_2)$$

## Alternativer Ansatz [B., HOANG, REIS '16]

$$e(t_0) \approx \psi_P(t_0) \implies \text{zwinge } \dot{e}(t_0) \ll 0$$

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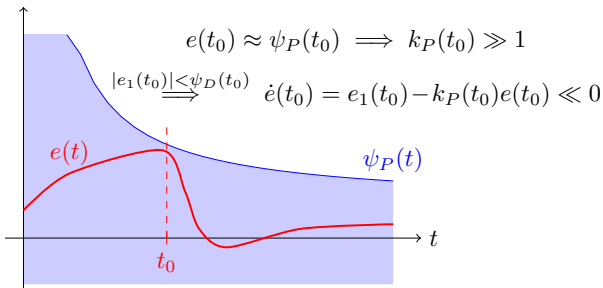
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## Funnel-Regler für Systeme mit bekanntem Relativgrad $r \in \mathbb{N}$

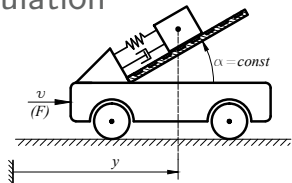
$$\begin{aligned}
 e_0(t) &= e(t) &= y(t) - y_{\text{ref}}(t), \\
 e_1(t) &= \dot{e}_0(t) &+ k_0(t)e_0(t), \\
 e_2(t) &= \dot{e}_1(t) &+ k_1(t)e_1(t), \\
 &\vdots \\
 e_{r-1}(t) &= \dot{e}_{r-2}(t) &+ k_{r-2}(t)e_{r-2}(t), \\
 u(t) &= -k_{r-1}(t)e_{r-1}(t) \\
 k_i(t) &= \frac{\psi_i(t)}{\psi_i(t) - \|e_i(t)\|}, \quad i = 0, \dots, r-1
 \end{aligned}$$

### Theorem [B., HOANG, REIS '16]

$y_{\text{ref}} \in \mathcal{W}^{r,\infty} \wedge \|e_i(0)\| < \psi_i(0) \implies u, k_i, y^{(i)} \in L^\infty$  und

$$\|e_i(t)\| \leq \psi_i(t) - \varepsilon_i$$

# Simulation

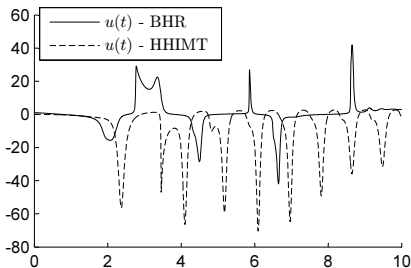
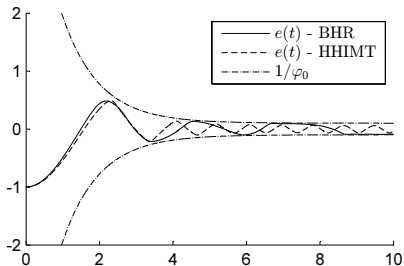


Winkel:  $\alpha = 45^\circ$

$m_1 = 4, m_2 = 1, g = 0$

$K(z) = 2z, D(z) = z$

$y_{\text{ref}}(t) = \cos t, \quad \psi_P(t) = 5e^{-t} + 0.1, \quad \psi_D(t) = 5e^{-t} + 0.5$





## Funnel-Regler ohne Ausgangs-Ableitungen – Backstepping

[ILCHMANN, RYAN, TOWNSEND '06, '07]

$$\text{Filter: } \begin{aligned} \dot{\xi}_i(t) &= -\xi_i(t) + \xi_{i+1}(t), \quad i = 1, \dots, r-2, \\ \dot{\xi}_{r-1}(t) &= -\xi_{r-1}(t) + u(t) \end{aligned}$$

$$r = 2: \quad u = -ke - (\|e\|^2 + k^2) \cdot k^4(1 + \|\xi\|^2)(\xi + ke)$$

$$\begin{aligned} r = 3: \quad u &= -ke - k^4(e^2 + k^2)(1 + \xi_1^2)(\xi_1 + ke) - \left\{ \left[ +e + (1 + \xi_1^2) \right. \right. \\ &\quad \cdot \left. \left[ 2k^5(\xi_1 + ke) + 4k^3(e^2 + k^2)(\xi_1 + ke) + k^4(e^2 + k^2)e \right] \right]^2 \\ &\quad + \left[ +k + k^4(1 + \xi_1^2) \left[ 2e(\xi_1 + ke) + k(e^2 + k^2) \right] \right]^2 \\ &\quad + \left[ k^4(e^2 + k^2) \left[ 2\xi_1(\xi_1 + ke) + (1 + \xi_1^2) \right] \right]^2 \left. \right\} k^4(1 + \xi_1^2 + \xi_2^2) \\ &\quad \cdot \left[ \xi_2 + ke + k^4(e^2 + k^2)(1 + \xi_1^2)(\xi_1 + ke) \right] \end{aligned}$$

In Anwendungen nicht relevant, vgl. [HACKL '12]

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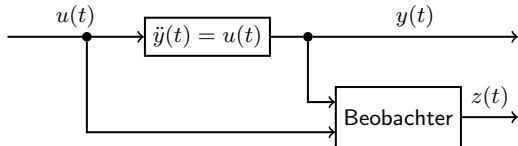
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In Anwendungen nicht relevant, vgl. [HACKL '12]

## Hochverstärkungsbeobachter [ATASSI, ESFANDIARI, KHALIL, SABERI, SANNUTI, TORNAMBÈ, ...]



$$\dot{z}_1 = z_2 + p_1 k (y - z_1),$$

$$\dot{z}_2 = u + p_2 k^2 (y - z_1),$$

$$p_1, p_2 > 0$$

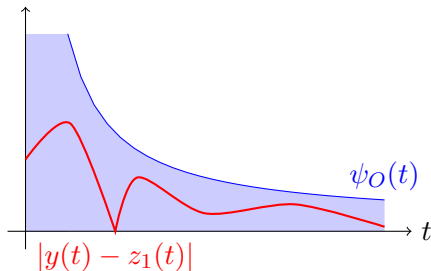
$$k > 0 \text{ hinr. groß} \implies z_1 \approx y \wedge z_2 \approx \dot{y}$$

## „Funnel-Beobachter“

$$\dot{z}_1 = z_2 + (q_1 + p_1 k_O(t))(y - z_1),$$

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$$k_O(t) = \frac{\psi_O(t)}{\psi_O(t) - |y(t) - z_1(t)|}$$



$$A := \begin{bmatrix} -q_1 & 1 \\ -q_2 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \quad \text{s.d.} \quad \boxed{A^\top P + PA = -I}$$

$$p_1 = 1, \quad p_2 = -\frac{p_{12}}{p_{22}}$$

### Theorem [B., REIS '16]

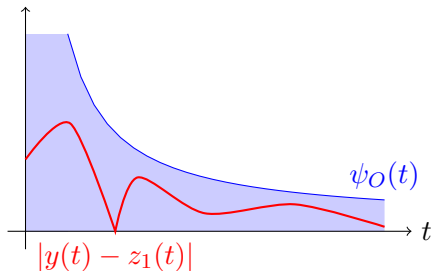
$$\dot{y} \in L^\infty \implies z_1, z_2, k_O \in L^\infty \wedge |y(t) - z_1(t)| < \psi_O(t) - \varepsilon$$

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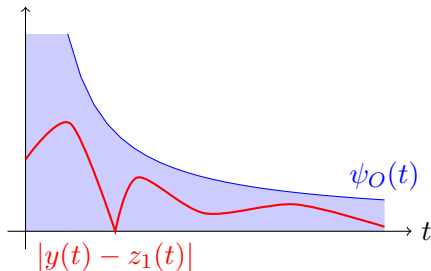
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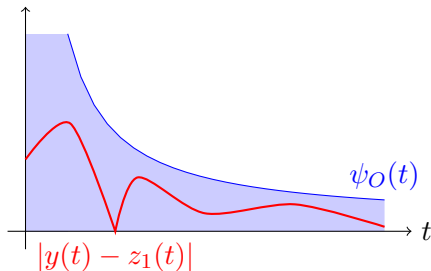
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## → Funnel-Vorkompensator

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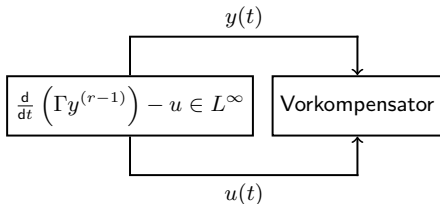


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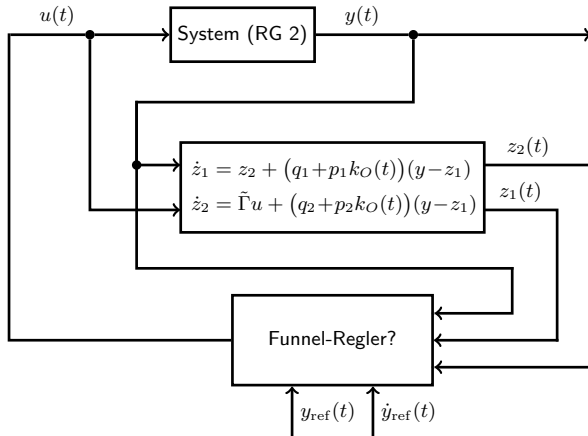
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 \end{aligned}$$





- für geg.  $\psi(t)$  wollen wir  $|y(t) - y_{\text{ref}}(t)| < \psi(t)$  für alle  $t > 0$
- der Regler soll nur  $y, z_1, z_2$  benötigen

## Bisheriger Regler:

$$u(t) = -k_D(t)(\dot{y}(t) - \dot{y}_{\text{ref}}(t) + k_P(t)(y(t) - y_{\text{ref}}(t)))$$

Neuer Ansatz: Wir stecken  $z_1 - y_{\text{ref}}$  in den Funnel!

- $\dot{z}_1 = z_2 + (q_1 + p_1 k_O(t))(y - z_1)$
- $|y(t) - y_{\text{ref}}(t)| \leq |y(t) - z_1(t)| + |z_1(t) - y_{\text{ref}}(t)| < \psi_O(t) + \psi_P(t)$

Für geg.  $\psi(t)$  sei  $\psi_P(t) = \psi_O(t) = \frac{1}{2}\psi(t)$

Problem:  $r > 2 \implies \ddot{z}_1$  nicht bekannt!

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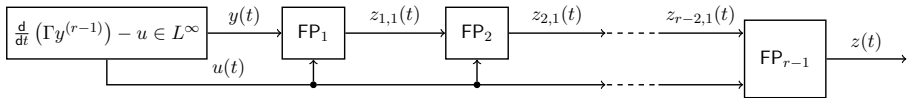
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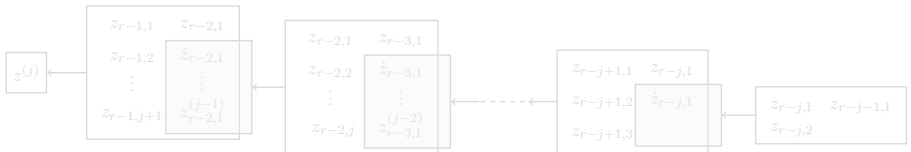
# Kaskade von Funnel-Vorkompensatoren



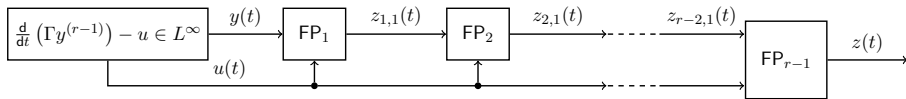
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$\Gamma y^{(r-1)}, \frac{d}{dt}(\Gamma y^{(r-1)}) - u \in L^\infty \implies z_{i,j}, k_{O,i} \in L^\infty$  und

- $\|y(t) - z(t)\| < \sum_{i=1}^{r-1} \psi_{O,i}(t) - \varepsilon$
- $z, \dot{z}, \dots, z^{(r-1)}$  explizit bekannt



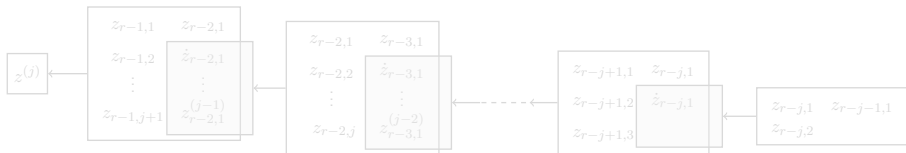
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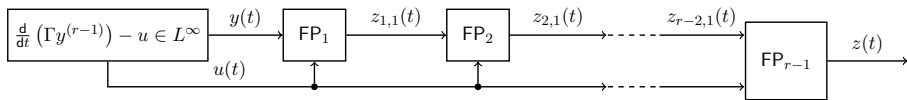
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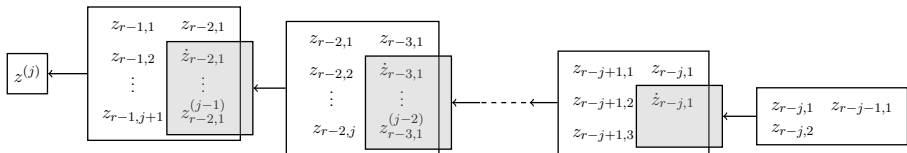
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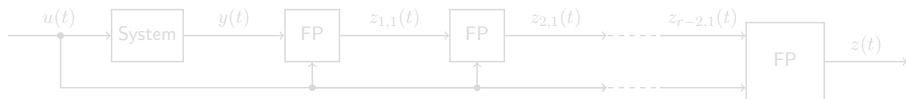
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## Erhaltung der stabilen internen Dynamik

$$y^{(r)}(t) = \sum_{i=1}^r R_i y^{(i-1)}(t) + f(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)) + \Gamma u(t)$$

$\Gamma > 0$ ,  $T : \mathcal{C} \rightarrow \mathcal{L}_{\text{loc}}^{\infty}$  kausal, lok. Lip.,  $y \in L^{\infty} \Rightarrow T(y, \dots, y^{(r-1)}) \in L^{\infty}$



### Theorem [B., REIS '16]

$r \in \{2, 3\}$ ,  $\tilde{\Gamma} > 0$  und (falls  $r = 3$ )  $I - \Gamma \tilde{\Gamma}^{-1} > 0$

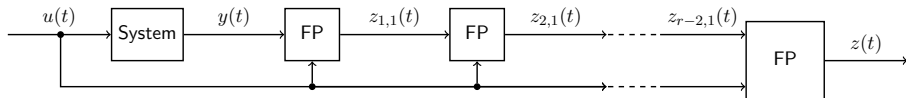
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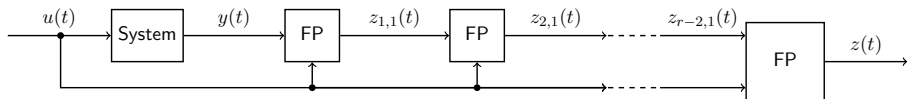
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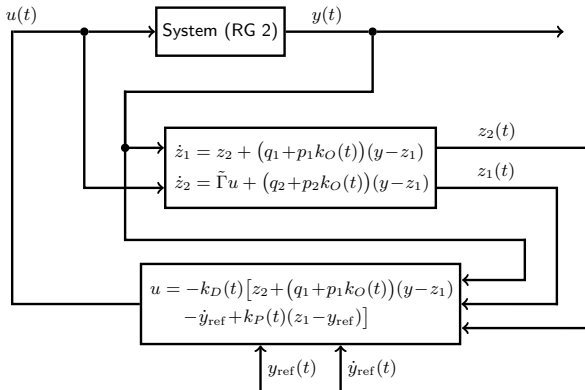
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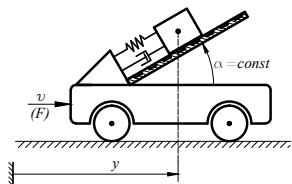
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### Theorem [B., REIS '16]

$$\begin{aligned}
 |y(t) - z_1(t)| &< \psi_O(t), & |z_1(t) - y_{\text{ref}}(t)| &< \psi_P(t), \\
 |\dot{z}_1(t) - \dot{y}_{\text{ref}}(t)| &< \psi_D(t), & |y(t) - y_{\text{ref}}(t)| &< \psi_P(t) + \psi_O(t)
 \end{aligned}$$

# Simulation



Winkel:  $\alpha = 45^\circ$

$$m_1 = 4, \quad m_2 = 1, \quad g = 0$$

$$K(z) = 2z^3 + z, \quad D(\dot{z}) = \dot{z}^3$$

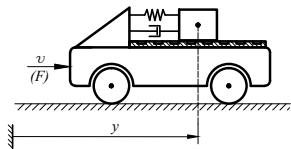
$$y_{\text{ref}}(t) = \sin t,$$

$$\tilde{\Gamma} = \frac{1}{4} > \frac{1}{9} = \Gamma$$

$$(s + 5)^2 \rightarrow q_1 = 10, \quad q_2 = 25, \quad p_1 = 1, \quad p_2 = \frac{125}{63}$$

$$\psi_P(t) = \psi_O(t) = \frac{1}{2} (10e^{-t} + 0.1), \quad \psi_D(t) = 0.5e^{-5t} + 0.01$$

# Simulation



Winkel:  $\alpha = 0^\circ$  (RG 3)

$$m_1 = 4, \quad m_2 = 1$$

$$K(z) = 2z, \quad D(\dot{z}) = \dot{z}$$

$$y_{\text{ref}}(t) = \sin t,$$

$$\tilde{\Gamma} = \frac{3}{4} > \frac{1}{4} = \Gamma$$

$$(s+5)^3 \rightarrow q_1 = 15, \quad q_2 = 75, \quad q_3 = 125, \quad p_1 = 1, \quad p_2 = \frac{1383}{391}, \quad p_3 = \frac{2230}{333}$$

$$\psi_P(t) = \psi_{O,1}(t) = \psi_{O,2}(t) = \frac{1}{3} (10e^{-t} + 0.1)$$

$$\psi_D(t) = 15e^{-t} + 0.01, \quad \psi_{DD}(t) = 2.5e^{-8t} + 0.01$$

# Ausblick

- Kaskade von Vorkompensatoren bei Relativgrad  $\geq 4$ ?
- instabile interne Dynamik  $\rightarrow$  Vorsteuerung
- optimale Wahl der Entwurfparameter
- Experimente