UNIVERSITÄT PADERBORN

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INPUT-CONSTRAINED FUNNEL CONTROL

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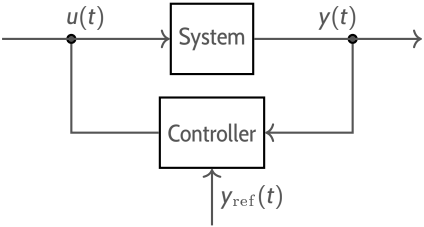
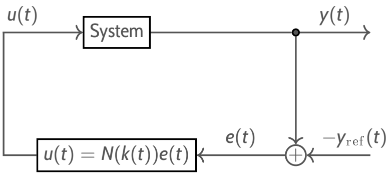
Control objective

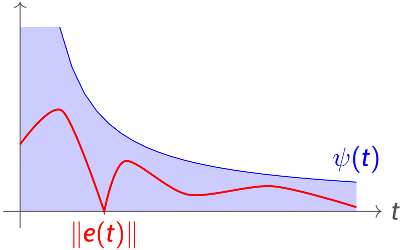


$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), u(t)\big), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

- **Goal:** simple controller, so that "$y(t)$ tracks $y_{\mathrm{ref}}(t)$"
- only uses $y(t)$, no knowledge of x^0 or system parameters
- only: **known relative degree** and **stable internal dynamics**

Funnel control





$$k(t) = \frac{1}{1 - \|e(t)\|^2/\psi(t)^2},$$

$N \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ a surjection (e.g. $N(s) = s \sin s$)

[Ilchmann, Ryan, Sangwin '02]:
Works, if

- relative degree $= 1$
- stable internal dynamics
- high-gain property

Funnel control for systems with arbitrary relative degree

$$
\begin{aligned}
e_1(t) &= e(t) = y(t) - y_{\mathrm{ref}}(t),\\
e_2(t) &= \dot e(t) + k_1(t)e_1(t),\\
e_3(t) &= \ddot e(t) + k_2(t)e_2(t),\\
&\ \vdots\\
e_r(t) &= e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t),
\end{aligned}
$$

$$u(t) = N\big(k_r(t)\big)e_r(t)$$

$$k_i(t) = 1/(1 - \|e_i(t)\|^2/\psi_i(t)^2), \quad i = 1, \ldots, r$$

Theorem [B., Lê, Reis ’18] & [B., Ilchmann, Ryan ’21]

$$y_{\mathrm{ref}} \in W^{r,\infty} \implies u, k_i, y^{(i)} \in L^\infty \text{ and } \|e(t)\| \le \psi(t) - \varepsilon$$

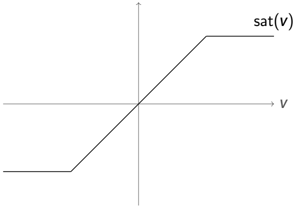
Funnel control under input constraints

$$u(t) = \mathrm{sat}(v(t))$$

such that for some $M > 0$

$$\left.\begin{array}{l} \bullet\ \mathrm{sat} \in C(\mathbb{R}^m, \mathbb{R}^m) \text{ bounded} \\ \bullet\ \forall\, v \in \mathbb{R}^m :\ \|v\| \le M \implies \mathrm{sat}(v) = v \end{array}\right\} (\mathrm{sat}_M)$$

- sat with (sat_M) is assumed arbitrary but *known*
-



Funnel control under input constraints

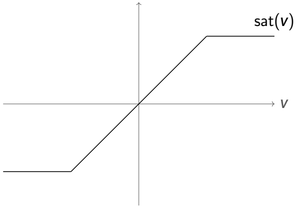
$$u(t) = \mathrm{sat}(v(t))$$

such that for some $M > 0$

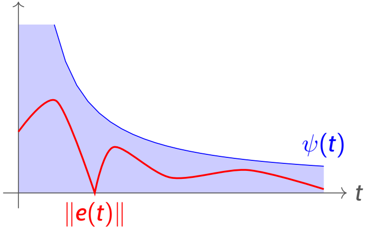
- $\mathrm{sat} \in C(\mathbb{R}^m, \mathbb{R}^m)$ bounded
- $\forall\, v \in \mathbb{R}^m : \; \|v\| \le M \implies \mathrm{sat}(v) = v$

$\Bigg\}$ (sat_M)

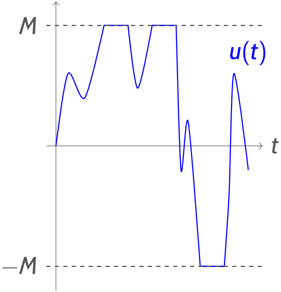
- sat with (sat_M) is assumed arbitrary but *known*
- input constraints and output constraints are conflicting!



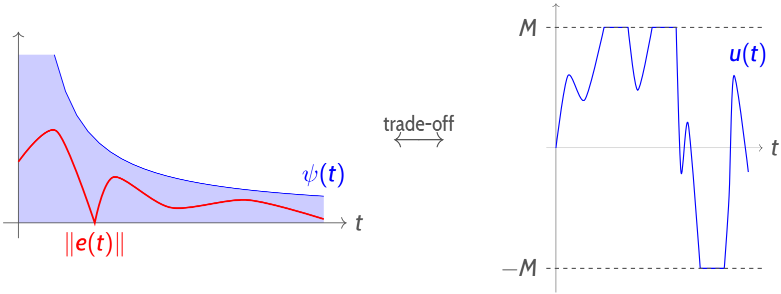
Funnel control under input constraints



trade-off ⟷



Funnel control under input constraints



- design controller which “negotiates” this trade-off
- here: input constraints are strict, output constraints are flexible

Funnel control under input constraints – known results

so far only: output constraints are strict, input constraints are flexible
$\rightarrow$ funnel control works, if M in (sat_M) suff. large

[Hopfe, Ilchmann, Ryan '10 a & b]: relative degree 1

[Hackl, Hopfe, Ilchmann, Müller, Trenn '13]: relative degree 2

[Liberzon & Trenn '13] (bang-bang funnel control): arbitrary relative degree

Funnel control under input constraints – controller design

$$
\begin{aligned}
e_1(t) &= e(t) = y(t) - y_{\mathrm{ref}}(t),\\
e_2(t) &= \dot e(t) + k_1(t)e_1(t),\\
&\;\vdots\\
e_r(t) &= e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t),\\
k_i(t) &= 1/(1 - \|e_i(t)\|^2/\psi_i(t)^2),\\
v(t) &= N\big(k_r(t)\big)e_r(t)
\end{aligned}
$$

$$
\begin{aligned}
\dot\psi_1(t) &= -\alpha_1\psi_1(t) + \beta_1\\
&\quad + p_1\left(\psi_2(t) - \tfrac{\beta_2}{\alpha_2}\right),\\
&\;\vdots\\
\dot\psi_{r-1}(t) &= -\alpha_{r-1}\psi_{r-1}(t) + \beta_{r-1}\\
&\quad + p_{r-1}\left(\psi_r(t) - \tfrac{\beta_r}{\alpha_r}\right),\\
\dot\psi_r(t) &= -\alpha_r\psi_r(t) + \beta_r + \psi_r(t)\tfrac{\kappa(v(t))}{\|e_r(t)\|},\\
\kappa(v(t)) &= \|v(t) - \mathrm{sat}(v(t))\|
\end{aligned}
$$

Funnel control under input constraints – controller design

$$
\begin{aligned}
e_1(t) &= e(t) = y(t) - y_{\text{ref}}(t),\\
e_2(t) &= \dot{e}(t) + k_1(t)e_1(t),\\
&\ \vdots\\
e_r(t) &= e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t),\\
k_i(t) &= 1/(1 - \|e_i(t)\|^2/\psi_i(t)^2),\\
v(t) &= N(k_r(t))e_r(t)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\psi}_1(t) &= -\alpha_1\psi_1(t) + \beta_1\\
&\quad + p_1\left(\psi_2(t) - \tfrac{\beta_2}{\alpha_2}\right),\\
&\ \vdots\\
\dot{\psi}_{r-1}(t) &= -\alpha_{r-1}\psi_{r-1}(t) + \beta_{r-1}\\
&\quad + p_{r-1}\left(\psi_r(t) - \tfrac{\beta_r}{\alpha_r}\right),\\
\dot{\psi}_r(t) &= -\alpha_r\psi_r(t) + \beta_r + \psi_r(t)\tfrac{\kappa(v(t))}{\|e_r(t)\|},\\
\kappa(v(t)) &= \|v(t) - \text{sat}(v(t))\|
\end{aligned}
$$

- $\alpha_1 > \alpha_2 > \ldots > \alpha_r > 0,\ p_i > 1,\ \beta_i > 0,\ \psi_i(0) > \frac{\beta_i}{\alpha_i}$
- Note: $\|e_r(t)\| \ll 1 \implies \kappa(v(t)) = 0$
- $\psi_1, \ldots, \psi_r$ are *not* bounded in general

Lipschitz condition

sat with (sat_M) arbitrary $\rightarrow$ global solutions do not necessarily exist

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Ex.: $\dot{y}(t) = y(t)^2 + u(t), y(0) = 1, u(t) = \mathrm{sat}(v(t))$

Lipschitz condition

sat with (sat_M) arbitrary $\to$ global solutions do not necessarily exist

Ex.: $\dot{y}(t) = y(t)^2 + u(t), y(0) = 1, u(t) = \mathrm{sat}(v(t))$

$$\implies \quad \dot{y}(t) \geq y(t)^2 - M \implies y(t) \geq z(t),$$

$$z(t) = \sqrt{M}\frac{\sqrt{M}+1+(1-\sqrt{M})e^{2\sqrt{M}t}}{\sqrt{M}+1-(1-\sqrt{M})e^{2\sqrt{M}t}}$$

$M \geq 1 \implies z(\cdot)$ globally defined

$M < 1 \implies z(\cdot)$ blows up at $\omega = \frac{1}{2\sqrt{M}} \ln\left(\frac{1+\sqrt{M}}{1-\sqrt{M}}\right)$

$\implies y(\cdot)$ blows up at ω

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$$M \ge 1 \implies \quad z(\cdot) \text{ globally defined}$$

$$M < 1 \implies \quad z(\cdot) \text{ blows up at } \omega = \frac{1}{2\sqrt{M}} \ln\left(\frac{1+\sqrt{M}}{1-\sqrt{M}}\right)$$

$$\implies \quad y(\cdot) \text{ blows up at } \omega$$

$\to$ some kind of Lipschitz condition required

Funnel control under input constraints – results

Theorem 1 [B. '22]

- relative degree r
- Lipschitz condition
- $y_{\mathrm{ref}} \in C^r$

$\implies$ $\exists$ global solution $(y, \psi_1, \ldots, \psi_r)$ with $\|e_i(t)\| \le \varepsilon_i \psi_i(t)$ for $i = 1, \ldots, r-1$ and $\|e_r(t)\| < \psi_r(t)$

$\implies$ if $v(t) = \mathrm{sat}(v(t))$ for $t \in [t_0, t_1]$, then

$$\psi_i(t) \le \frac{\beta_i}{\alpha_i} + \sum_{j=i}^{r} c_{ij}(t_0) e^{-\alpha_j (t - t_0)}$$

Funnel control under input constraints – results

Theorem 2 [B. '22]

- relative degree r
- ~~Lipschitz condition~~ stable internal dynamics and high-gain property
- ~~$y_{\mathrm{ref}} \in C^r$~~ $y_{\mathrm{ref}} \in W^{r,\infty}$

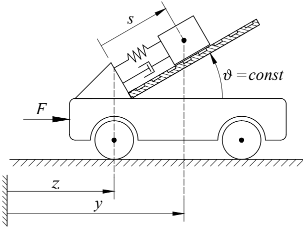
$\Longrightarrow \forall\, \varepsilon \in (0,1)\, \forall\, K > 0\, \exists\, M = M(\varepsilon, K)$ such that

- $\forall$ sat with (sat$_M$),
- $\forall\, y^0$ with $\|e_i(0)\| \le \varepsilon \psi_i(0), i = 1, \ldots, r$,
- $\forall\, y_{\mathrm{ref}} \in W^{r,\infty}$ with $\|y_{\mathrm{ref}}^{(i)}\|_\infty \le K, i = 0, \ldots, r$,

$\exists$ global solution $(y, \psi_1, \ldots, \psi_r)$ with $y \in W^{r,\infty}, \psi_i, k_i \in L^\infty$,

$\limsup_{t\to\infty} \psi_i(t) \le \frac{\beta_i}{\alpha_i}$ and $\|v(t)\| \le M$

Simulation



angle: $\vartheta = \pi/4$

$m_1 = 4,\ m_2 = 1,\ k = 2,\ d = 1$

$z(0) = s(0) = \dot{z}(0) = \dot{s}(0) = 0$

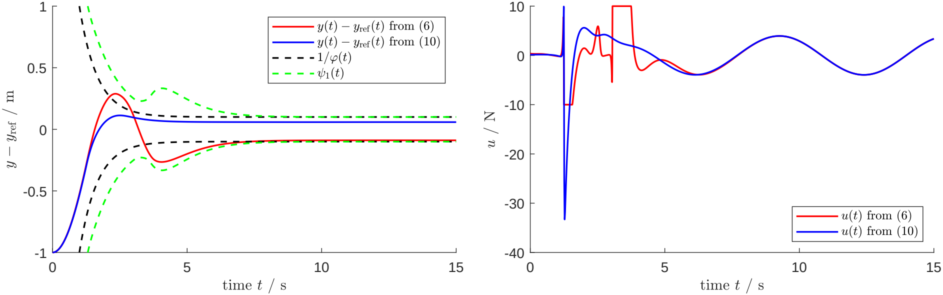
$y_{\rm ref}(t) = \cos t,$

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\vartheta \\ m_2 \cos\vartheta & m_2 \end{bmatrix} \begin{pmatrix} \ddot{z}(t) \\ \ddot{s}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t) + d\dot{s}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix},$$

$$y(t) = z(t) + s(t)\cos\vartheta$$

Controller parameters: $\alpha_1 = 1.5, \alpha_2 = 0.9 \cdot \alpha_1, \beta_1 = 0.15, \beta_2 = 0.5 \cdot \alpha_2,$
$p_1 = 1.1, \psi_1^0 = 4.1, \psi_2^0 = 2,\ N(s) = -s^2 \cos s$

Simulation



blue curve: funnel controller from [B., Ilchmann, Ryan ’21] with

$$\varphi(t) = (4e^{-3t/2} + 0.1)^{-1}$$

red curve: input-constrained funnel controller from [B. ’22]