UNIVERSITÄT PADERBORN

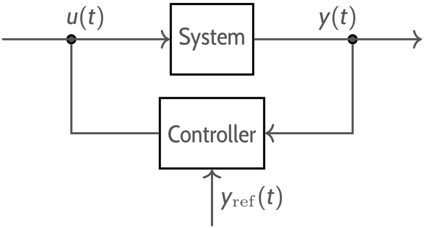
Institute for Mathematics, Paderborn University

FUNNEL MPC – A BRIEF INTRODUCTION

Thomas Berger

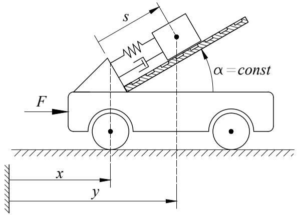
Ilmenau, September 25, 2023

Control objective



$$\begin{aligned}\dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= h(x(t))\end{aligned}$$

Goal: $\|y(t) - y_{\mathrm{ref}}(t)\| < \psi(t)$

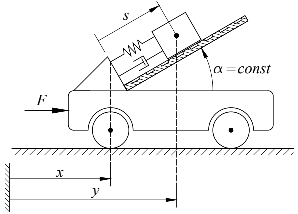


[Seifried, Blajer '13]

angle: $0° \leq \alpha \leq 90°$

spring, damper with nonlinear characteristics: $K(s),\ D(\dot{s})$

$$u(t) = F$$
$$y(t) = x(t) + s(t)\cos\alpha$$



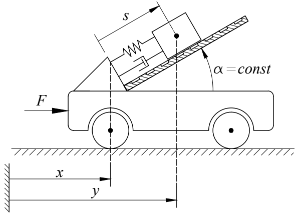
[Seifried, Blajer '13]

angle: $0^\circ \leq \alpha \leq 90^\circ$

spring, damper with nonlinear characteristics: $K(s),\ D(\dot{s})$

$$u(t) = F$$
$$y(t) = x(t) + s(t)\cos\alpha$$

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\alpha \\ m_2 \cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -K(s) - D(\dot{s}) + m_2 g \sin\alpha \end{pmatrix}$$



[Seifried, Blajer '13]

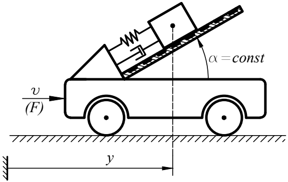
angle: $0^\circ \le \alpha \le 90^\circ$

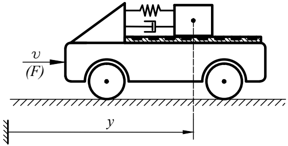
spring, damper with nonlinear characteristics: $K(s),\ D(\dot{s})$

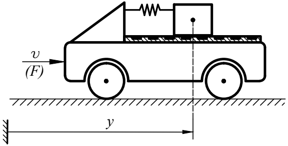
$$u(t) = F$$
$$y(t) = x(t) + s(t)\cos\alpha$$

$$\begin{bmatrix} m_1 + m_2 & m_2\cos\alpha \\ m_2\cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -K(s) - D(\dot{s}) + m_2 g \sin\alpha \end{pmatrix}$$

$$\dot{y} = \dot{x} + \dot{s}\cos\alpha$$
$$\ddot{y} = -c_1\big(K(s) + D(\dot{s}) - m_2 g \sin\alpha\big) + \frac{\sin^2\alpha}{m_1 + m_2\sin^2\alpha} u$$



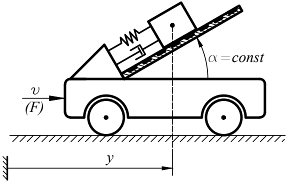




$0^\circ < \alpha \leq 90^\circ$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

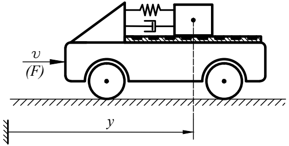
relative degree $= 2$



$0^\circ < \alpha \leq 90^\circ$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

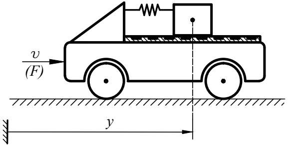
relative degree $= 2$

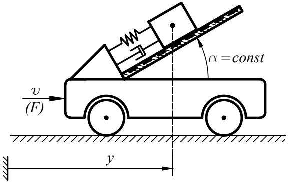


$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$

$$y^{(3)} = f_2(s, \dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

relative degree $= 3$

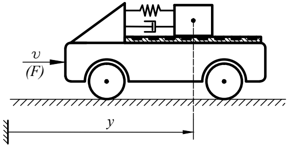




$$0^\circ < \alpha \leq 90^\circ$$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

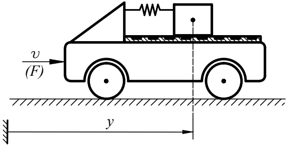
relative degree $= 2$



$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

$$y^{(3)} = f_2(s, \dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

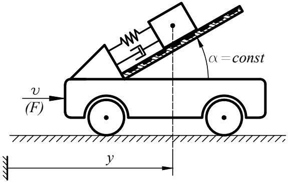
relative degree $= 3$



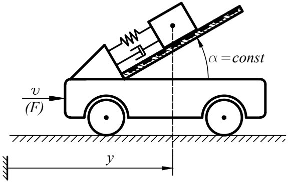
$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

$$y^{(4)} = f_3(s, \dot{s}) + \frac{K'(s)}{m_1 m_2} u$$

relative degree $= 4$



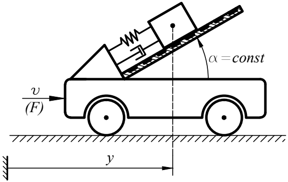
Internal dynamics: remaining dynamics when output is fixed



Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K \left(\frac{\eta - y \cos\alpha}{\sin^2 \alpha} \right) - c_3 D \left(\frac{\dot{\eta} - \dot{y} \cos\alpha}{\sin^2 \alpha} \right) + c_4 g \sin\alpha$$

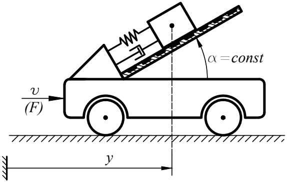
UNIVERSITÄT PADERBORN



Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K\left(\frac{\eta - y\cos\alpha}{\sin^2\alpha}\right) - c_3 D\left(\frac{\dot{\eta} - \dot{y}\cos\alpha}{\sin^2\alpha}\right) + c_4 g \sin\alpha$$

$$\alpha = 90^\circ, m_2 = 1: \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$

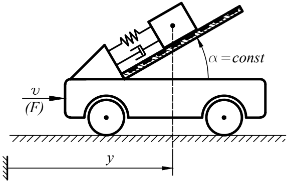


Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K\left(\frac{\eta - y\cos\alpha}{\sin^2\alpha}\right) - c_3 D\left(\frac{\dot{\eta} - \dot{y}\cos\alpha}{\sin^2\alpha}\right) + c_4 g\sin\alpha$$

$$\alpha = 90^\circ, m_2 = 1: \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$

- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\,\dot{s} \geq 0$



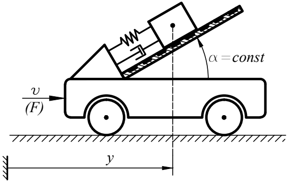
Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K\left(\frac{\eta - y\cos\alpha}{\sin^2\alpha}\right) - c_3 D\left(\frac{\dot{\eta} - \dot{y}\cos\alpha}{\sin^2\alpha}\right) + c_4 g \sin\alpha$$

$$\alpha = 90^\circ, m_2 = 1: \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$

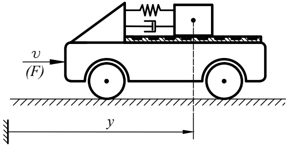
- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\,\dot{s} \geq 0$

$\Longrightarrow \quad s, \dot{s} \in L^\infty$ **(stable internal dynamics)**



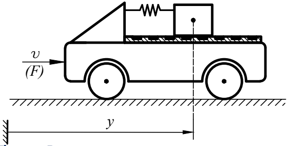
$0^\circ < \alpha \leq 90^\circ$

stable internal dynamics



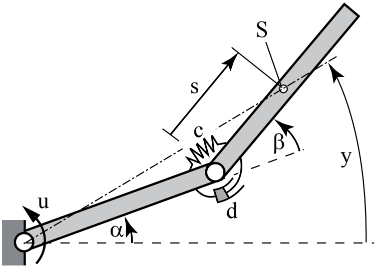
$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$

stable internal dynamics



$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$

no internal dynamics



[Seifried, Blajer '13]

Rotational Manipulator Arm

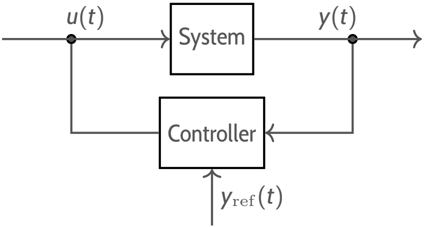
Input: angular velocity of first link

Output: position of S described by angle y

relative degree $= 1$

unstable internal dynamics

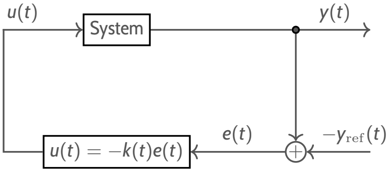
Reminder

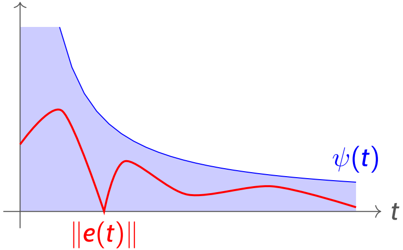


$$\begin{aligned} \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \quad x(0) = x^0 \in \mathbb{R}^n \\ y(t) &= h(x(t)) \end{aligned}$$

- **relative degree** is well-defined and assumption of **stable internal dynamics**
- **Goal:** design controller such that $\|y(t) - y_{\mathrm{ref}}(t)\| < \psi(t)$

Funnel control





$$k(t) = \frac{1}{\psi(t) - \|e(t)\|}$$

[Ilchmann, Ryan, Sangwin '02]:
Works, if

- relative degree $= 1$
- stable internal dynamics

Funnel control

Advantages:

- Tracking of reference signal within predefined boundaries
- No system model required; only structural assumptions

Drawbacks:

- Large input signal with peaks possible
- High sampling rate required

Model Predictive Control (MPC)

Algorithm: Set time shift $\delta > 0$, prediction horizon $T \geq \delta$, maximal control $u_{\max} > 0$, time $\hat{t} := 0$

(a) Obtain state measurement $\hat{x} := x(\hat{t})$

Model Predictive Control (MPC)

Algorithm: Set time shift $\delta > 0$, prediction horizon $T \geq \delta$, maximal control $u_{\max} > 0$, time $\hat{t} := 0$

(a) Obtain state measurement $\hat{x} := x(\hat{t})$

(b) Compute a solution $u^\star$ of the optimal control problem (OCP)

$$\underset{u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)}{\text{minimize}} \int_{\hat{t}}^{\hat{t}+T} \ell(t, x(t), u(t)) \mathrm{d}t$$

$$\text{subject to} \quad \dot{x}(t) = f(x(t)) + g(x(t))u(t), \qquad x(\hat{t}) = \hat{x},$$

$$y(t) = h(x(t)),$$

$$\|u\|_\infty \leq u_{\max},$$

Model Predictive Control (MPC)

Algorithm: Set time shift $\delta > 0$, prediction horizon $T \geq \delta$, maximal control $u_{\max} > 0$, time $\hat{t} := 0$

(a) Obtain state measurement $\hat{x} := x(\hat{t})$

(b) Compute a solution $u^\star$ of the optimal control problem (OCP)

$$\underset{u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)}{\text{minimize}} \int_{\hat{t}}^{\hat{t}+T} \ell(t, x(t), u(t)) \mathrm{d}t$$

$$\begin{aligned} \text{subject to} \quad \dot{x}(t) &= f(x(t)) + g(x(t))u(t), \qquad x(\hat{t}) = \hat{x}, \\ y(t) &= h(x(t)), \\ \|u\|_\infty &\leq u_{\max}, \end{aligned}$$

(c) Implement the feedback law $\mu : [\hat{t}, \hat{t} + \delta) \times \mathbb{R}^n \to \mathbb{R}^m$, $\mu(t, \hat{x}) := u^\star(t)$, increase $\hat{t}$ by δ and go to step (a)

Model Predictive Control (MPC)

Algorithm: Set time shift $\delta > 0$, prediction horizon $T \geq \delta$, maximal control $u_{\max} > 0$, time $\hat{t} := 0$

(a) Obtain state measurement $\hat{x} := x(\hat{t})$

(b) Compute a solution $u^\star$ of the optimal control problem (OCP)

$$\underset{u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)}{\text{minimize}} \int_{\hat{t}}^{\hat{t}+T} \|h(x(t)) - y_{\text{ref}}(t)\|^2 + \lambda_u \|u(t)\|^2 \text{d}t$$

$$\text{subject to} \quad \dot{x}(t) = f(x(t)) + g(x(t))u(t), \qquad x(\hat{t}) = \hat{x},$$
$$y(t) = h(x(t)),$$
$$\|u\|_\infty \leq u_{\max}, \qquad \|y(t) - y_{\text{ref}}(t)\| \leq \psi(t).$$

(c) Implement the feedback law $\mu : [\hat{t}, \hat{t} + \delta) \times \mathbb{R}^n \to \mathbb{R}^m$, $\mu(t, \hat{x}) := u^\star(t)$, increase $\hat{t}$ by δ and go to step (a)

Model Predictive Control (MPC)

Drawbacks:

- Accurate model is required
- Initial and recursive feasibility have to be guaranteed
- Time varying output constraints

Funnel MPC

Idea: Use cost function inspired by funnel control:

$$\ell(t,x,u) = \begin{cases} \dfrac{1}{\psi(t) - \|h(x) - y_{\mathrm{ref}}(t)\|} + \lambda_u \|u\|^2, & \|h(x) - y_{\mathrm{ref}}(t)\| \neq \psi(t) \\ \infty, & \text{else.} \end{cases}$$

Funnel MPC

Idea: Use cost function inspired by funnel control:

$$\ell(t,x,u) = \begin{cases} \dfrac{1}{\psi(t) - \|h(x) - y_{\text{ref}}(t)\|} + \lambda_u \|u\|^2, & \|h(x) - y_{\text{ref}}(t)\| \neq \psi(t) \\ \infty, & \text{else.} \end{cases}$$

Theorem [B., Dennstädt, Ilchmann, Worthmann '22]

- relative degree $= 1$,
- stable internal dynamics

$\Longrightarrow\ \exists\, u_{\max} > 0$: Funnel MPC is initially & recursively feasible.

Funnel MPC

Remarks:

- no terminal costs or conditions required
- no additional state constraints in the OCP
- independent of the length of the prediction horizon $T > 0$

Funnel MPC

Remarks:

- no terminal costs or conditions required
- no additional state constraints in the OCP
- independent of the length of the prediction horizon $T > 0$

How to treat higher relative degree?

Funnel MPC

Remarks:

- no terminal costs or conditions required
- no additional state constraints in the OCP
- independent of the length of the prediction horizon $T > 0$

How to treat higher relative degree?

- funnel control $\rightarrow$ [B., Lê, Reis '18] & [B., Ilchmann, Ryan '21]

Funnel MPC

Remarks:

- no terminal costs or conditions required
- no additional state constraints in the OCP
- independent of the length of the prediction horizon $T > 0$

How to treat higher relative degree?

- funnel control $\rightarrow$ [B., Lê, Reis '18] & [B., Ilchmann, Ryan '21]
- funnel MPC $\rightarrow$ [B., Dennstädt '22] (but: feasibility constraints!)

Funnel MPC for systems with relative degree r

Idea: For $k_1, \ldots, k_{r-1} > 0$ define a function e_r such that

$$e_r(t, x(t)) := \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + k_1\right) \cdots \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + k_{r-1}\right)\big(h(x(t)) - y_{\mathrm{ref}}(t)\big)$$

and use the cost function:

$$\ell(t, x, u) = \begin{cases} \dfrac{1}{\psi_r(t) - \|e_r(t, x)\|} + \lambda_u \|u\|^2, & \|e_r(t, x)\| \neq \psi_r(t) \\ \infty, & \text{else.} \end{cases}$$

Funnel MPC for systems with relative degree r

Idea: For $k_1, \ldots, k_{r-1} > 0$ define a function e_r such that

$$e_r(t, x(t)) := \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + k_1\right) \cdots \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + k_{r-1}\right)\big(h(x(t)) - y_{\mathrm{ref}}(t)\big)$$

and use the cost function:

$$\ell(t, x, u) = \begin{cases} \dfrac{1}{\psi_r(t) - \|e_r(t,x)\|} + \lambda_u \|u\|^2, & \|e_r(t,x)\| \neq \psi_r(t) \\ \infty, & \text{else.} \end{cases}$$

Theorem [B., Dennstädt '23]

- relative degree $= r$,
- stable internal dynamics

$\implies \exists\, u_{\max} > 0\ \exists\, k_1, \ldots, k_{r-1} > 0\ \exists\, \psi_r$: Funnel MPC is initially & recursively feasible and $\|y(t) - y_{\mathrm{ref}}(t)\| \leq \psi(t)$

Funnel MPC for systems with relative degree r

Recipe: Construction of $k_1, \ldots, k_{r-1}$ and ψ_r

Assume $\dot{\psi}(t) \geq -\alpha\psi(t) + \beta$ and $\|e_1(0)\| \leq \gamma^r \psi(0)$, then choose

$$k_1 \geq \frac{2\,\|\dot{e}_1(0)\|}{\gamma^{r-1}(1-\gamma)\psi(0)} + \frac{2\left(\alpha + \frac{1}{\gamma^{r-1}}\right)}{1-\gamma},$$

$$k_i \geq \frac{2\gamma\,\|\dot{e}_i(0)\|}{(1-\gamma)\left(\,\|e_i(0)\| + \frac{\beta}{\alpha\gamma^{i-2}}\right)} + \frac{2(1+\alpha)}{1-\gamma},$$

$$e_i(t) = \left(\tfrac{\mathrm{d}}{\mathrm{d}t} + k_1\right)\cdots\left(\tfrac{\mathrm{d}}{\mathrm{d}t} + k_{i-1}\right)e(t), \quad e_1(t) = y(t) - y_{\mathrm{ref}}(t)$$

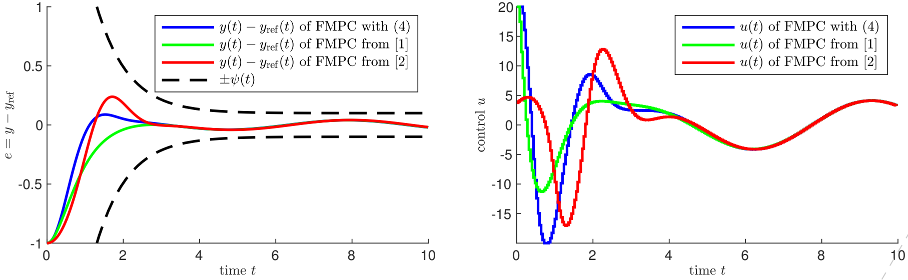
and

$$\psi_r(t) := \frac{1}{\gamma}\left(\,\|\dot{e}_{r-1}(0)\| + k_{r-1}\,\|e_{r-1}(0)\|\,\right)e^{-\alpha t} + \frac{\beta}{\alpha\gamma^{r-1}}$$

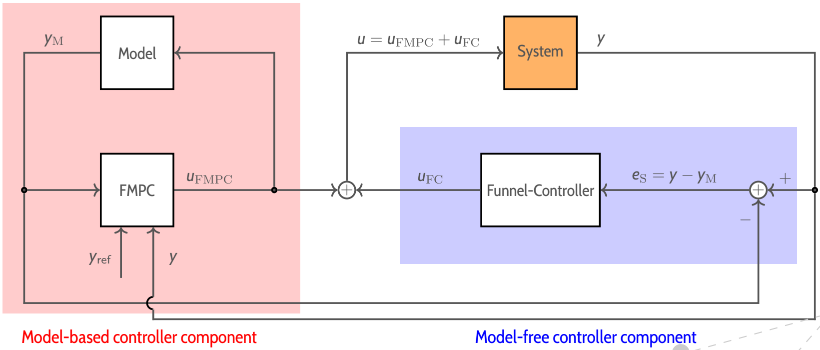
UNIVERSITÄT PADERBORN

Simulation

Mass-on-car system: relative degree $r = 2$, $u_{\max} = 20$



Learning-based Robust Funnel MPC



Learning-based Robust Funnel MPC