UNIVERSITÄT PADERBORN

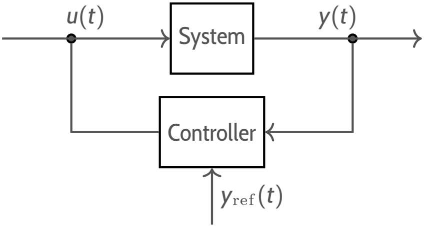
Institute for Mathematics, Paderborn University

ANALYSIS OF SYSTEM-THEORETIC PROPERTIES OF REAL-WORLD PROCESSES

Thomas Berger

Ilmenau, July 2, 2024

Motivation



$$\begin{aligned} \dot{x}(t) &= f\big(t, x(t), u(t)\big), \quad x(t) \in X \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

- feasibility of model-agnostic control requires structural properties
- these properties are independent of the dimension of the state space X

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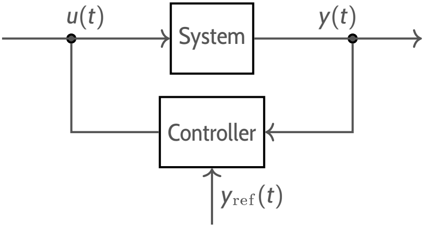

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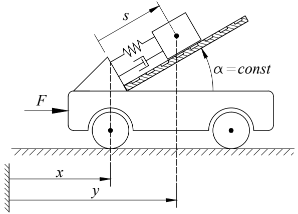
$$

- feasibility of model-agnostic control requires structural properties
- these properties are independent of the dimension of the state space X
  $\rightarrow$ ODEs and PDEs in the same class!

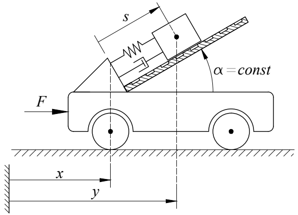


[Seifried, Blajer '13]

angle: $0° \leq \alpha \leq 90°$

spring, damper with nonlinear characteristics: $K(s),\ D(\dot{s})$

$$u(t) = F$$
$$y(t) = x(t) + s(t)\cos\alpha$$



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$$u(t) = F$$
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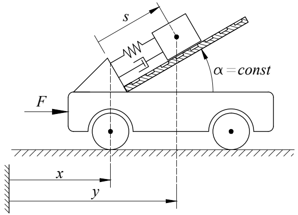
$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\alpha \\ m_2 \cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -K(s) - D(\dot{s}) + m_2 g \sin\alpha \end{pmatrix}$$



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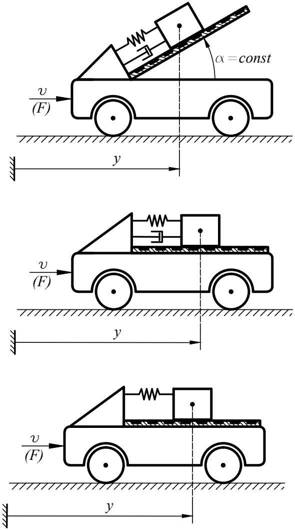
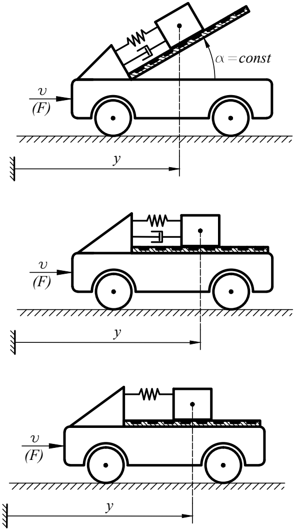
$$\dot{y} = \dot{x} + \dot{s}\cos\alpha$$
$$\ddot{y} = -c_1\left(K(s) + D(\dot{s}) - m_2 g\sin\alpha\right) + \frac{\sin^2\alpha}{m_1 + m_2\sin^2\alpha}u$$



$0° < \alpha \leq 90°$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

relative degree $= 2$



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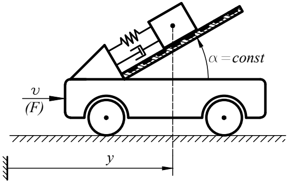
$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

$$y^{(3)} = f_2(s, \dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

relative degree $= 3$



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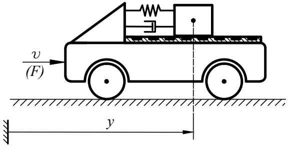

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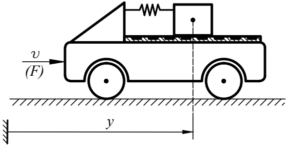
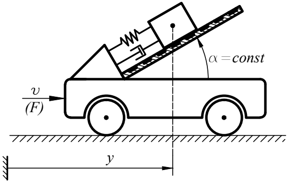


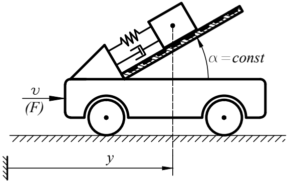
$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

$$y^{(4)} = f_3(s, \dot{s}) + \frac{K'(s)}{m_1 m_2} u$$

relative degree $= 4$

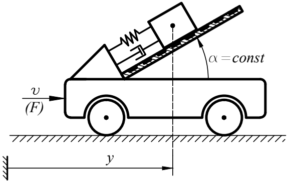


Internal dynamics: remaining dynamics when output is fixed



Internal dynamics: remaining dynamics when output is fixed

$$\ddot{\eta} = -c_3 K \left(\frac{\eta - y \cos \alpha}{\sin^2 \alpha} \right) - c_3 D \left(\frac{\dot{\eta} - \dot{y} \cos \alpha}{\sin^2 \alpha} \right) + c_4 g \sin \alpha$$



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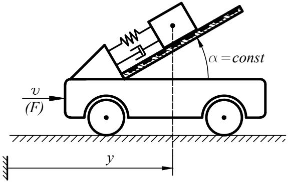
$$\alpha = 90^\circ, m_2 = 1: \qquad \ddot{s} = -K(s) - D(\dot{s}) + g$$



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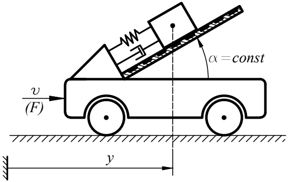
- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s})\,\dot{s} \geq 0$



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- Lyapunov function: kinetic + potential energy
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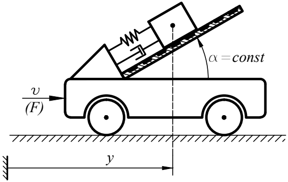
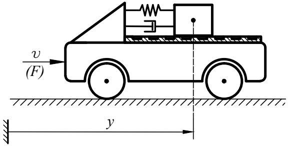
$\Longrightarrow\ s, \dot{s} \in L^\infty$ **(stable internal dynamics)**



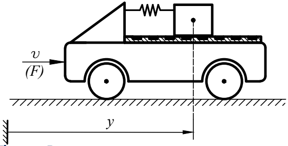
$0^\circ < \alpha \leq 90^\circ$

stable internal dynamics



$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$

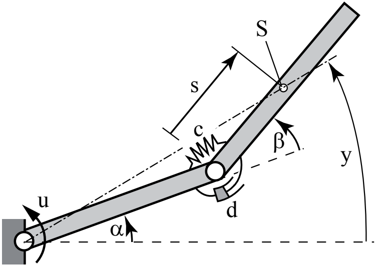
stable internal dynamics



$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$

no internal dynamics

Rotational Manipulator Arm



$$M \begin{pmatrix} \ddot{\alpha}(t) \\ \ddot{\beta}(t) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \alpha(t) + \frac{s}{s+l}\beta(t)$$

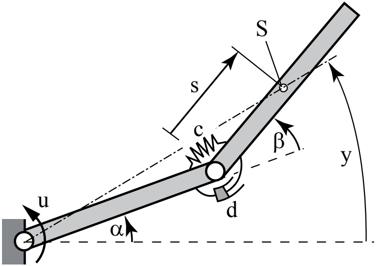
[Seifried, Blajer '13]

$$M = l^2 m \begin{bmatrix} \frac{5}{3} + \cos(\beta(t)) & \frac{1}{3} + \frac{1}{2}\cos(\beta(t)) \\ \frac{1}{3} + \frac{1}{2}\cos(\beta(t)) & \frac{1}{3} \end{bmatrix},$$

$$f_1 = \tfrac{1}{2} l^2 m \dot{\beta}(t)(2\dot{\alpha}(t) + \dot{\beta}(t)) \sin(\beta(t)),$$

$$f_2 = -c\beta(t) - d\dot{\beta}(t) - \tfrac{1}{2} l^2 m \dot{\alpha}(t)^2 \sin(\beta(t))$$

Rotational Manipulator Arm



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[Seifried, Blajer '13]

- relative degree $= 2$ (for $\cos(\beta) > 2/3$)
- internal dynamics: highly nonlinear, for $s/l > 2/3$ the linearized internal dynamics are unstable
- control: [B., Lanza '21]

Byrnes-Isidori form

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + g\big(x(t)\big)u(t), \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

Byrnes-Isidori form

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + g\big(x(t)\big)u(t), \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

Lie derivative:

- $(L_f h)(z) := h'(z)f(z)$
- $L_f^k h = L_f(L_f^{k-1}h)$ with $L_f^0 h = h$
- for $g(z) = [g_1(z), \ldots, g_m(z)]$ we define

$$\big(L_g h\big)(z) := [(L_{g_1}h)(z), \ldots, (L_{g_m}h)(z)]$$

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$$\big(L_g h\big)\,(z) := [(L_{g_1}h)(z), \ldots, (L_{g_m}h)(z)]$$

Relative degree r on $U \subseteq \mathbb{R}^n$:

- $\forall\, z \in U\, \forall\, k \in \{0, ..., r-2\} :\ (L_g L_f^k h)(z) = 0_{m\times m}$
- $\forall\, z \in U :\ (L_g L_f^{r-1} h)(z) \in \mathbf{Gl}_m(\mathbb{R})$

Byrnes-Isidori form

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + g\big(x(t)\big)u(t), \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

$\exists$ diffeomorphism $\Phi : U \to W$ s.t. $\begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix} = \Phi(x(t)), \xi(t) \in \mathbb{R}^{rm}, \eta(t) \in \mathbb{R}^{n-rm}$ transforms the system into **Byrnes-Isidori form**:

$$\begin{aligned} y(t) &= \xi_1(t), \\ \dot{\xi}_1(t) &= \xi_2(t), \\ &\vdots \\ \dot{\xi}_{r-1}(t) &= \xi_r(t), \\ \dot{\xi}_r(t) &= (L_f^r h)\big(\Phi^{-1}(\xi(t), \eta(t))\big) + \Gamma\big(\Phi^{-1}(\xi(t), \eta(t))\big)u(t), \\ \dot{\eta}(t) &= q(\xi(t), \eta(t)) + p(\xi(t), \eta(t))u(t) \end{aligned}$$

Byrnes-Isidori form

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$\operatorname{im}(g(x))$ **involutive:**

$$\forall\, g_1, g_2 : \mathbb{R}^n \to \mathbb{R}^n \text{ with } g_i(x) \in \operatorname{im}(g(x)) : \;\; [g_1, g_2](x) \in \operatorname{im}(g(x)),$$

where $[g_1, g_2](x) = g_1'(x)g_2(x) - g_2'(x)g_1(x)$

$\Longrightarrow \quad p(\cdot) = 0$ in Byrnes-Isidori form [Isidori '95]

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where $[g_1, g_2](x) = g_1'(x)g_2(x) - g_2'(x)g_1(x)$

$\implies \quad p(\cdot) = 0$ in Byrnes-Isidori form [Isidori '95]

internal dynamics: $\dot{\eta}(t) = q(\xi(t), \eta(t))$

Byrnes-Isidori form

$$\begin{aligned} \dot{x}(t) &= f\big(x(t)\big) + g\big(x(t)\big)u(t), \\ y(t) &= h\big(x(t)\big) \end{aligned}$$

def. $T(y,\ldots,y^{(r-1)})(t) := \begin{pmatrix} y(t) \\ \vdots \\ y^{(r-1)}(t) \\ \eta(t;\eta^0,y,\ldots,y^{(r-1)}) \end{pmatrix}$

$\Longrightarrow$ Byrnes-Isidori form is equivalent to

$$\boxed{y^{(r)}(t) = F\big(T(y,\ldots,y^{(r-1)})(t)\big) + G\big(T(y,\ldots,y^{(r-1)})(t)\big)u(t)}$$

with $F = (L_f^r h) \circ \Phi^{-1}, \quad G = \Gamma \circ \Phi^{-1}$

Byrnes-Isidori form

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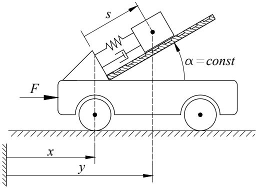
with $F = (L_f^r h) \circ \Phi^{-1}, \quad G = \Gamma \circ \Phi^{-1}$

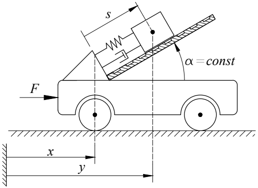
internal dynamics stable $\iff$ T is BIBO

Control of multibody systems – jointly with T. Reis and R. Seifried



[B., Lê, Reis '18]

Control of multibody systems – jointly with T. Reis and R. Seifried



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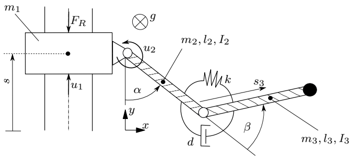
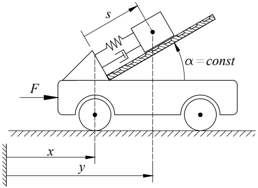


[B., Otto, Reis, Seifried '19]

Control of multibody systems – jointly with T. Reis and R. Seifried



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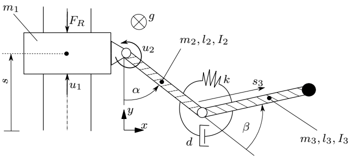
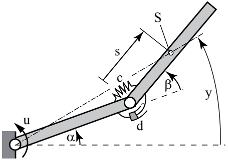
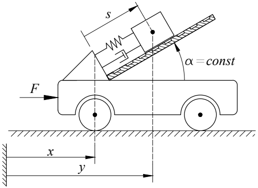


[B., Otto, Reis, Seifried '19]



[B., Lanza '20]: unstable internal dynamics

Control of multibody systems – jointly with T. Reis and R. Seifried



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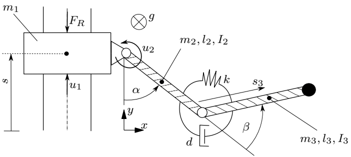
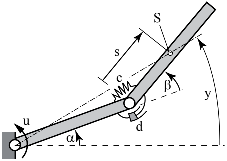
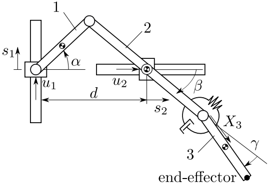


[B., Otto, Reis, Seifried '19]



[B., Lanza '20]: unstable internal dynamics



[B., Drücker, Lanza, Reis, Seifried '21]
unstable internal dynamics,
DAE formulation

Multibody systems in DAE formulation

$$\begin{aligned} \dot{q}(t) &= v(t), \\ M(q(t))\dot{v}(t) &= f\big(q(t), v(t)\big) + J(q(t))^\top \mu(t) + G(q(t))^\top \lambda(t) + B(q(t))\, u(t), \\ 0 &= J(q(t))v(t) + j(q(t)), \\ 0 &= g(q(t)), \\ y(t) &= h\big(q(t), v(t)\big), \end{aligned}$$

with

- generalized mass matrix $M : \mathbb{R}^n \to \mathbb{R}^{n\times n}$,
- generalized forces $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$,
- holonomic constraints $g : \mathbb{R}^n \to \mathbb{R}^\ell$ and $G : \mathbb{R}^n \to \mathbb{R}^{\ell\times n}$,
- nonholonomic constraints $J : \mathbb{R}^n \to \mathbb{R}^{p\times n}$ and $j : \mathbb{R}^n \to \mathbb{R}^p$,
- input distribution matrix $B : \mathbb{R}^n \to \mathbb{R}^{n\times m}$,
- output measurement function $h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$

Multibody systems in DAE formulation

$$\dot{x} = \begin{pmatrix} x_2 \\ M(x_1)^{-1} f(x_1, x_2) \end{pmatrix} + \begin{bmatrix} I_n & 0 \\ & M(x_1)^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ J(x_1)^\top & G(x_1)^\top & B(x_1) \end{bmatrix} u_{\mathrm{aux}},$$

$$y_{\mathrm{aux}} = H(x),$$

with $x_1 = q, x_2 = v$ and auxiliary inputs and outputs

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Multibody systems in DAE formulation

$$\dot{x} = \begin{pmatrix} x_2 \\ M(x_1)^{-1} f(x_1, x_2) \end{pmatrix} + \begin{bmatrix} I_n & 0 \\ & M(x_1)^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ J(x_1)^\top & G(x_1)^\top & B(x_1) \end{bmatrix} u_{\mathrm{aux}},$$

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Idea:

- transform auxiliary ODE system into Byrnes-Isidori form
- add the constraints $y_{\mathrm{aux},1} = 0$ and $y_{\mathrm{aux},2} = 0$ afterwards to derive the internal dynamics of the original MBS

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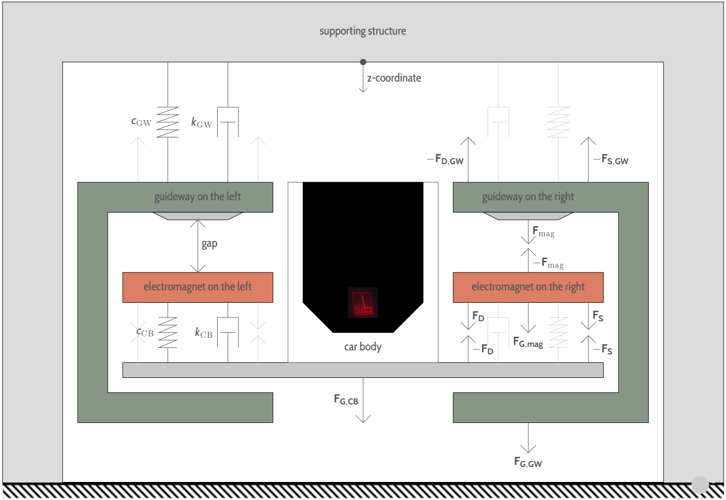
stability analysis without Byrnes-Isidori form: [Lanza '21]

Magnetic levitation system – cooperation with TU Ilmenau and Max Bögl

Bildquelle: https://transportsystemboegl.com, Pressemitteilung vom 26.04.2024

Magnetic levitation system – cooperation with TU Ilmenau and Max Bögl



Magnetic levitation system – cooperation with TU Ilmenau and Max Bögl

$$\dot{x}(t) = \begin{pmatrix} x_4(t) \\ x_5(t) \\ x_6(t) \\ \frac{1}{m_{\mathrm{mag}}}\left(F_{\mathrm{G,mag}} + F_{\mathrm{S,D}}(x(t)) - F_{\mathrm{mag}}(x(t))\right) \\ \frac{1}{m_{\mathrm{CB}}}\left(F_{\mathrm{G,CB}} - F_{\mathrm{S,D}}(x(t))\right) \\ \frac{1}{m_{\mathrm{GW}}}\left(F_{\mathrm{G,GW}} - \tilde{F}_{\mathrm{S,D}}(x(t)) + F_{\mathrm{mag}}(x(t))\right) \\ -2RI_{\mathrm{mag}}(x(t)) \end{pmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = x_1(t) - x_3(t)$$

Input: voltage at the electromagnet

[Oppeneiger, Lanza, Schell, Dennstädt, Schaller, Zamzow, B., Worthmann ’24]: relative degree $= 3$ on $\{x_7 \neq 0\}$ and internal dynamics are stable

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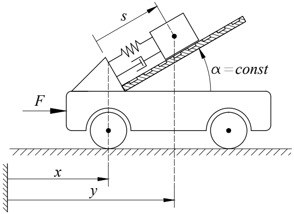
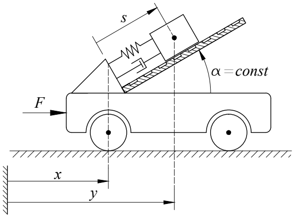
Open problem: relative degree not well-defined everywhere

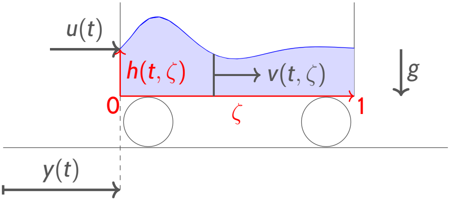


→

$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma\, u(t)$$



$\rightarrow$



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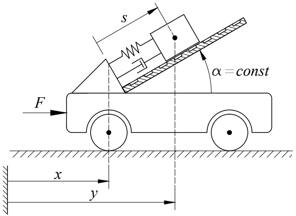
$$\ddot{y}(t) = \hat{T}(y, \dot{y})(t) + \hat{\gamma}\, u(t)$$

[B., Puche, Schwenninger '22]



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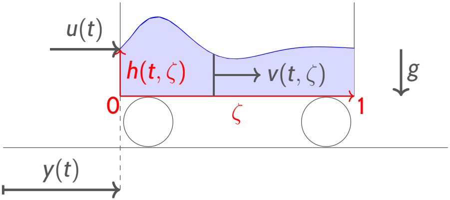

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Finite and infinite dimensional systems in the same class!

Class of ∞-dimensional systems

$$y^{(r)}(t) = F\big(T(y,\dots,y^{(r-1)})(t)\big) + G\big(T(y,\dots,y^{(r-1)})(t)\big)u(t)$$

allows for a “simple” class of ∞-dimensional systems
$\to$ internal dynamics described by PDE

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$r = 1$: [Ilchmann, Ryan, Sangwin '02, etc.]

$$(Ty)(t) = A_1 y(t) + A_2 \int_0^t \mathcal{T}(t-s) A_3 y(s)\,\mathrm{d}s$$

- $(\mathcal{T}(t))_{t\geq 0}$ exp. stable C^0-semigroup on real Hilbert space X with generator $A_4 : \mathcal{D}(A_4) \subseteq X \rightarrow X$ *(finite dimensional: $\mathcal{T}(t) = e^{A_4 t}$)*
- (A_4, A_3, A_2) "regular well-posed", $A_1 \in \mathbb{R}^{m\times m}$

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$r \in \mathbb{N}$: [Ilchmann, Selig, Trunk '16]
Byrnes-Isidori form for linear ∞-dimensional systems

$$\begin{aligned} \dot\eta(t) &= A_4\eta(t) + A_3 y(t), \\ y^{(r)}(t) &= R_1 y(t) + \dots R_r y^{(r-1)}(t) + A_2\eta(t) + \gamma u(t) \end{aligned}$$

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with $T(y,\dots,y^{(r-1)})(t)$

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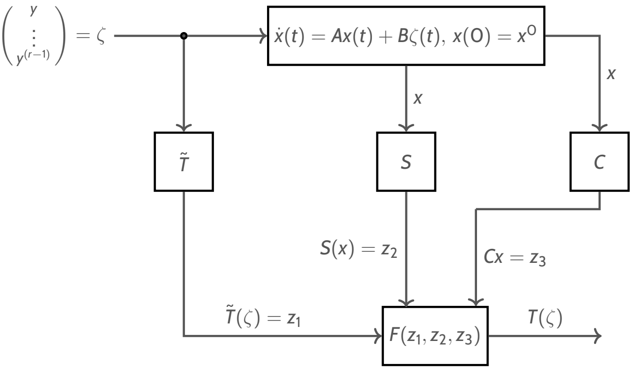
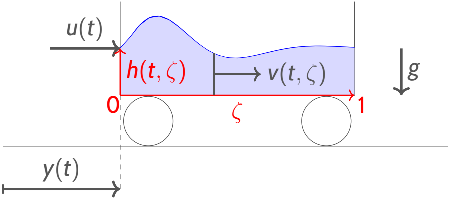


- (A, B, C) regular well-posed
- $f \mapsto \mathcal{L}^{-1}(G) * f$ bounded
- S nonlinear, $\mathcal{D}(S) = X$, (A, B, S) BIBO stable

Rolling water tank – [B., Puche, Schwenninger '22]

$$\partial_t h + \partial_\zeta (hv) = 0,$$

$$\partial_t v + \partial_\zeta \left(\frac{v^2}{2} + gh \right) + hS\left(\frac{v}{h}\right) = -\ddot{y}$$



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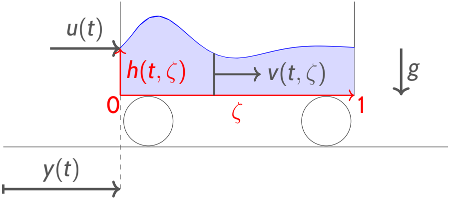

Linearized model:

$$\partial_t z_1 = -h_0 \partial_\zeta z_2, \qquad \partial_t z_2 = -g\partial_\zeta z_1 - \mu z_2 - \ddot{y}, \qquad z_2(t,0) = z_2(t,1) = 0$$

$$\ddot{y}(t) = \frac{g}{2m_T}\left(z_1(t,1) - z_1(t,0)\right)\left(2h_0 + z_1(t,1) + z_1(t,0)\right) + \frac{\mu h_0}{m_T}\int_0^1 z_2(t,\zeta)\,\mathrm{d}\zeta + \frac{\mu}{m_T}\int_0^1 z_1(t,\zeta) z_2(t,\zeta)\,\mathrm{d}\zeta + \frac{u(t)}{m_T}$$

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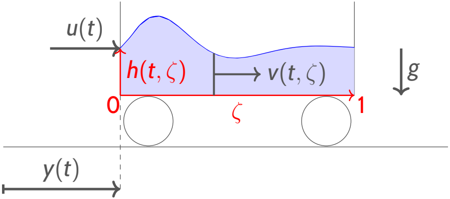

Linearized model:

$$\dot{z}(t) = Az(t) + Ab\dot{y}(t), \quad Az = -\begin{pmatrix} h_0 \partial_\zeta z_2 \\ g\partial_\zeta z_1 + \mu z_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$z_1(t,1) - z_1(t,0) = Cz(t) = C\mathcal{T}(t)z(0) + C\int_0^t \mathcal{T}(t-s)Ab\dot{y}(s)\,\mathrm{d}s$$
$$= c(t) + \big((\mathfrak{h}_{L^1} + \mathfrak{h}_\delta) * \dot{y}\big)(t)$$

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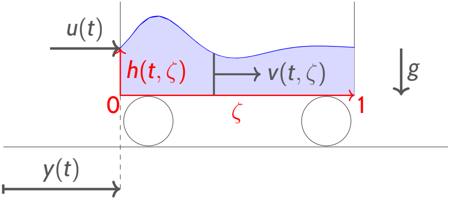

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$$\ddot{y}(t) = T(\dot{y})(t) + \frac{u(t)}{m_T}$$

Lemma: $\mathfrak{h}_\delta = \delta_0 + 2\sum_{k\in\mathbb{N}} (-1)^k e^{-k\mu/\sqrt{h_0 g}} \delta_{k/\sqrt{h_0 g}} \quad \Longrightarrow \quad T$ is BIBO

“Hard” class of ∞-dimensional systems

$\rightarrow$ **there is NO concept of relative degree**

- boundary controlled heat equation [Reis, Selig ’15]

$$\partial_t x(t) = \Delta x(t), \qquad u(t) = \left(\nu^\top \cdot \nabla x(t)\right)|_{\partial\Omega},$$
$$y(t) = \int_{\partial\Omega} x(t)(\zeta)\,\mathrm{d}\zeta$$

- general class of boundary control problems based on m-dissipative operators [Puche, Reis, Schwenninger ’21, Puche ’19]

$$\dot{x}(t) = \mathfrak{A}x(t), \qquad x(0) = x_0 \in \mathcal{D}(\mathfrak{A}) \subseteq X,$$
$$u(t) = \mathfrak{B}x(t), \qquad y(t) = \mathfrak{C}x(t)$$

  e.g. lossy transmission line, wave equation, diffusion equation
- Fokker-Planck equation [B. ’21]

Monodomain equations [B., Breiten, Puche, Reis '21] – (simple) model for the electric activity of the human heart to describe defibrillation processes

$$\partial_t v(t) = \nabla \cdot \big(D\nabla v(t)\big) + p_3(v)(t) - w(t) + I_{s,i}(t) + BI_{s,e}(t),$$
$$\partial_t w(t) = cv(t) - dw(t), \quad y(t) = B'v(t)$$

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Control objective: “reentry waves”, which can be interpreted as fibrillation processes, should be terminated