UNIVERSITÄT PADERBORN

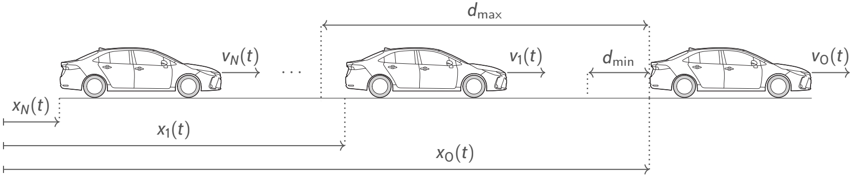
Institute for Mathematics, Paderborn University

STRING STABILITY AND GUARANTEED SAFETY FOR VEHICLE PLATOONS

Thomas Berger and Bart Besselink

Ilmenau, July 3, 2024

Platoons of N vehicles



$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ m_i \dot{v}_i(t) &= u_i(t) - f_i(t, x_i(t), v_i(t)) + d_i(t), \quad i = 1, \ldots, N \end{aligned}$$

- mass m_i, disturbance d_i, $f_i(t,x,v) = F_{i,g}(x) + F_{i,a}(t,x,v) + F_{i,r}(v)$
- $F_{i,g}(x) = m_i\, g \sin\theta_i(x)$
- $F_{i,a}(t,x,v) = \frac{1}{2}\rho_i(t,x) C_{i,d}\, A_i \operatorname{sgn}(v) v^2$
- $F_{i,r}(v) = m_i\, g\, C_{i,r} \operatorname{erf}(\alpha v)$

Control objectives

(O1) guarantee a safety distance between any two vehicles
(O2) ensure a good traffic flow (distances between vehicles don't get too large)

Control objectives

(O1) guarantee a safety distance between any two vehicles
(O2) ensure a good traffic flow (distances between vehicles don't get too large)

$$\forall i = 1, \dots, N \; \forall t \geq 0: \; d_{\min} < x_{i-1}(t) - x_i(t) < d_{\max}.$$

Control objectives

(O1) guarantee a safety distance between any two vehicles
(O2) ensure a good traffic flow (distances between vehicles don't get too large)

$$\forall i = 1, \ldots, N \, \forall t \geq 0 : \; d_{\min} < x_{i-1}(t) - x_i(t) < d_{\max}.$$

(O3) achieve string stability of the controlled platoon
`https://www.youtube.com/watch?v=Rryu85BtALM`

Control objectives

(O1) guarantee a safety distance between any two vehicles
(O2) ensure a good traffic flow (distances between vehicles don't get too large)

$$\forall\, i = 1, \ldots, N \;\forall\, t \geq 0 : \; d_{\min} < x_{i-1}(t) - x_i(t) < d_{\max}.$$

(O3) achieve (practical velocity) string stability of the controlled platoon
https://www.youtube.com/watch?v=Rryu85BtALM

$$\exists\, C_1, C_2 > 0 \;\forall\, N \in \mathbb{N} \;\forall\, i = 1, \ldots, N \;\forall\, v_0 \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}) : \quad \|v_i\|_\infty \leq C_1 + C_2 \|v_0\|_\infty.$$

Control objectives

(O1) guarantee a safety distance between any two vehicles

(O2) ensure a good traffic flow (distances between vehicles don't get too large)

$$\forall\, i = 1, \ldots, N \;\forall\, t \geq 0 : \; d_{\min} < x_{i-1}(t) - x_i(t) < d_{\max}.$$

(O3) achieve (practical velocity) string stability of the controlled platoon
https://www.youtube.com/watch?v=Rryu85BtALM

$$\exists\, C_1, C_2 > 0 \;\forall\, N \in \mathbb{N} \;\forall\, i = 1, \ldots, N \;\forall\, v_0 \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}) : \quad \|v_i\|_\infty \leq C_1 + C_2 \|v_0\|_\infty.$$

(O4) only use decentralized controllers based on local information

Controller design

$$
\begin{aligned}
\xi_i(t) &= x_i(t) - x_{i-1}(t) + d_{\min}, \\
e_i(t) &= \xi_i(t) + \lambda v_i(t), \\
w_i(t) &= v_i(t) - v_{i-1}(t) - \frac{1}{\xi_i(t)} - \frac{1}{M + \xi_i(t)}, \\
k_{i,3}(t) &= \frac{1}{\psi(t) - |w_i(t)|}, \\
u_i(t) &= -k_1(v_i(t) - v_{i-1}(t)) - k_2 e_i(t) - k_{i,3}(t) w_i(t)
\end{aligned}
$$

- $M := d_{\max} - d_{\min}$,
- $\psi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$, $\psi(t) > 0$ for all $t \geq 0$, $\liminf_{t\to\infty} \psi(t) > 0$

Controller design

$$
\begin{aligned}
\xi_i(t) &= x_i(t) - x_{i-1}(t) + d_{\min},\\
e_i(t) &= \xi_i(t) + \lambda v_i(t),\\
w_i(t) &= v_i(t) - v_{i-1}(t) - \frac{1}{\xi_i(t)} - \frac{1}{M + \xi_i(t)},\\
k_{i,3}(t) &= \frac{1}{\psi(t) - |w_i(t)|},\\
u_i(t) &= -k_1(v_i(t) - v_{i-1}(t)) - k_2 e_i(t) - k_{i,3}(t) w_i(t)
\end{aligned}
$$

- $M := d_{\max} - d_{\min}$,
- $\psi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})$, $\psi(t) > 0$ for all $t \geq 0$, $\liminf_{t\to\infty} \psi(t) > 0$

$$
e_i \text{ const.} \quad \Longrightarrow \quad \dot{v}_i(t) = -\frac{1}{\lambda} v_i(t) + \frac{1}{\lambda} v_{i-1}(t)
$$

Assumptions

- $|F_{i,g}(x) + F_{i,r}(x) + d_i(t)| \leq \bar{d}$
- $m_i \leq \bar{m}$
- $0 \leq \frac{1}{2}\rho_i(t,x)\, C_{i,d}\, A_i \leq \bar{\rho}$
- $x_0 \in C^2(\mathbb{R}_{\geq 0}, \mathbb{R})$ s.t. $v_0 := \dot{x}_0$ and $\dot{v}_0$ are bounded
- $-M + \delta \leq \xi_i(0) \leq -\delta \quad$ and $\quad |w_i(0)| \leq \psi(0) - \delta$
- $\exists\, N_0\, \forall\, i \geq N_0 : \; |m_i - m_{i-1}| \leq p\, m_i \quad$ and $\quad m_i \leq q\, m_{i-1} \qquad (*)$
  for $p, q \in (0,1)$ with $(1+p)q < 1$

Main result

Theorem [B., Besselink '24]

There exists suff. large $k_2 > 0$ s.t. the CL-system has a global solution with:

(i) v_i and u_i are bounded, independent of i and N

(ii) there exist $\varepsilon_1, \varepsilon_2 > 0$, independent of i and N, s.t.

$$-M + \varepsilon_1 \leq \xi_i(t) \leq -\varepsilon_1 \quad \text{and} \quad |w_i(t)| \leq \psi(t) - \varepsilon_2$$

(iii) $\|v_i\|_\infty \leq \frac{M}{\lambda} + \frac{1}{\lambda k_2}\left(\frac{\|\psi\|_\infty}{\varepsilon_2} + \bar{d}\right) + \left(\frac{k_1}{k_1+\lambda k_2}\right)^i \|v_0\|_\infty$

Sketch of the proof

Step 1: existence of a maximal solution

Sketch of the proof

Step 1: existence of a maximal solution

Step 2: $-M + \varepsilon_1 \leq \xi_i(t) \leq -\varepsilon_1$ for $\varepsilon_1 := \left(\|\psi\|_\infty + \frac{1}{\delta} \right)^{-1}$

Sketch of the proof

Step 1: existence of a maximal solution

Step 2: $-M + \varepsilon_1 \le \xi_i(t) \le -\varepsilon_1$ for $\varepsilon_1 := \left(\|\psi\|_\infty + \frac{1}{\delta}\right)^{-1}$

Step 3: $|v_i(t)| \le \max\left\{ |v_i(0)|, \frac{\bar{d} + k_2 M + k_{i,3}(t)\psi(t) + k_1 |v_{i-1}(t)|}{k_1 + \lambda k_2} \right\}$

Sketch of the proof

Step 1: existence of a maximal solution

Step 2: $-M + \varepsilon_1 \le \xi_i(t) \le -\varepsilon_1$ for $\varepsilon_1 := \left(\|\psi\|_\infty + \frac{1}{\delta}\right)^{-1}$

Step 3: $|v_i(t)| \le \max\left\{ |v_i(0)|, \frac{\bar{d} + k_2 M + k_{i,3}(t)\psi(t) + k_1 |v_{i-1}(t)|}{k_1 + \lambda k_2} \right\}$

Step 4: $|w_i(t)| \le \psi(t) - \varepsilon_2$ for $\varepsilon_2 > 0$ independent of i and N

Sketch of the proof

Step 4a: $|w_1(t)| \leq \psi(t) - \hat{\varepsilon}_1$ for $\hat{\varepsilon}_1 = \hat{\varepsilon}_1(k_2)$ with k_2 suff. large s.t.
$k_1 + \lambda k_2 \geq 2m_1/\varepsilon_1^2$

Sketch of the proof

Step 4a: $|w_1(t)| \leq \psi(t) - \hat{\varepsilon}_1$ for $\hat{\varepsilon}_1 = \hat{\varepsilon}_1(k_2)$ with k_2 suff. large s.t. $k_1 + \lambda k_2 \geq 2m_1/\varepsilon_1^2$

Step 4b: $|w_i(t)| \leq \psi(t) - \hat{\varepsilon}_i$ for $\hat{\varepsilon}_i = \hat{\varepsilon}_i(k_2, \hat{\varepsilon}_{i-1}) \leq \hat{\varepsilon}_{i-1}$:

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t} w_i(t)^2}{2|w_i(t)|} \leq -\frac{\psi(t)-\hat{\varepsilon}_i}{m_i \hat{\varepsilon}_i} + \frac{\psi(t)}{m_{i-1}\hat{\varepsilon}_{i-1}} + \frac{\psi(t)}{\hat{\varepsilon}_{i-1}} \left| \frac{1}{m_i} - \frac{1}{m_{i-1}} \right| + \frac{c_1}{m_i k_2 \hat{\varepsilon}_{i-1}} + \frac{c_2}{m_i k_2^2 \hat{\varepsilon}_{i-1}^2} + \frac{k_2}{m_i} c_3 + \frac{c_4}{m_i}.$$

Sketch of the proof

Step 4a: $|w_1(t)| \leq \psi(t) - \hat{\varepsilon}_1$ for $\hat{\varepsilon}_1 = \hat{\varepsilon}_1(k_2)$ with k_2 suff. large s.t.
$k_1 + \lambda k_2 \geq 2m_1/\varepsilon_1^2$

Step 4b: $|w_i(t)| \leq \psi(t) - \hat{\varepsilon}_i$ for $\hat{\varepsilon}_i = \hat{\varepsilon}_i(k_2, \hat{\varepsilon}_{i-1}) \leq \hat{\varepsilon}_{i-1}$:

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t} w_i(t)^2}{2|w_i(t)|} \leq -\frac{\psi(t) - \hat{\varepsilon}_i}{m_i \hat{\varepsilon}_i} + \frac{\psi(t)}{m_{i-1}\hat{\varepsilon}_{i-1}} + \frac{\psi(t)}{\hat{\varepsilon}_{i-1}} \left| \frac{1}{m_i} - \frac{1}{m_{i-1}} \right| + \frac{c_1}{m_i k_2 \hat{\varepsilon}_{i-1}} + \frac{c_2}{m_i k_2^2 \hat{\varepsilon}_{i-1}^2} + \frac{k_2}{m_i} c_3 + \frac{c_4}{m_i}.$$

Step 4c: $\forall i : \hat{\varepsilon}_i \geq \varepsilon_2$; by $(*)$: $\left| \frac{1}{m_i} - \frac{1}{m_{i-1}} \right| \leq \frac{p}{m_{i-1}} \leq \frac{pq}{m_i}$ for $i \geq N_0$, hence

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t} w_i(t)^2}{2|w_i(t)|} \leq -\frac{\psi(t)}{m_i \hat{\varepsilon}_i} + \frac{\psi(t)}{m_i \hat{\varepsilon}_{i-1}}(1+p)q + \frac{c_1}{m_i k_2 \hat{\varepsilon}_{i-1}} + \frac{c_2}{m_i k_2^2 \hat{\varepsilon}_{i-1}^2} + \frac{k_2}{m_i} c_3 + \frac{c_4 + 1}{m_i}$$
$$\overset{!}{\leq} -\|\dot{\psi}\|_\infty$$

Sketch of the proof

Step 4c: set $z_i := 1/\hat{\varepsilon}_i$, then

$$-\frac{\psi(t)}{m_i} z_i + \frac{\psi(t)}{m_i}(1+p) q z_{i-1} + \frac{c_1}{m_i k_2} z_{i-1} + \frac{c_2}{m_i k_2^2} z_{i-1}^2 + \frac{k_2}{m_i} c_3 + \frac{c_4+1}{m_i} \overset{!}{\leq} -\|\dot{\psi}\|_\infty$$

Sketch of the proof

Step 4c: set $z_i := 1/\hat{\varepsilon}_i$, then

$$-\frac{\psi(t)}{m_i}z_i + \frac{\psi(t)}{m_i}(1+p)qz_{i-1} + \frac{c_1}{m_i k_2}z_{i-1} + \frac{c_2}{m_i k_2^2}z_{i-1}^2 + \frac{k_2}{m_i}c_3 + \frac{c_4+1}{m_i} \overset{!}{\leq} -\|\dot{\psi}\|_\infty$$

define sequence (z_i) by

$$z_i = (1+p)qz_{i-1} + \frac{\tilde{c}_1}{k_2}z_{i-1} + \frac{\tilde{c}_2}{k_2^2}z_{i-1}^2 + k_2\tilde{c}_3 + \tilde{c}_4, \quad i \geq N_0$$

Sketch of the proof

Step 4c: set $z_i := 1/\hat{\varepsilon}_i$, then

$$-\frac{\psi(t)}{m_i} z_i + \frac{\psi(t)}{m_i}(1+p)q z_{i-1} + \frac{c_1}{m_i k_2} z_{i-1} + \frac{c_2}{m_i k_2^2} z_{i-1}^2 + \frac{k_2}{m_i} c_3 + \frac{c_4+1}{m_i} \overset{!}{\leq} -\|\dot{\psi}\|_\infty$$

define sequence (z_i) by

$$z_i = (1+p)q z_{i-1} + \frac{\tilde{c}_1}{k_2} z_{i-1} + \frac{\tilde{c}_2}{k_2^2} z_{i-1}^2 + k_2 \tilde{c}_3 + \tilde{c}_4, \quad i \geq N_0$$

(z_i) is bounded: choose k_2 suff. large s.t. $\alpha := (1+p)q + \frac{\tilde{c}_1}{k_2} < 1$ and

$$\frac{k_2^2(1-\alpha)^2}{4\tilde{c}_2} \geq k_2 \tilde{c}_3 + \tilde{c}_4,$$

Sketch of the proof

Step 4c: set $z_i := 1/\hat{\varepsilon}_i$, then

$$-\frac{\psi(t)}{m_i} z_i + \frac{\psi(t)}{m_i}(1+p)q z_{i-1} + \frac{c_1}{m_i k_2} z_{i-1} + \frac{c_2}{m_i k_2^2} z_{i-1}^2 + \frac{k_2}{m_i} c_3 + \frac{c_4+1}{m_i} \overset{!}{\leq} -\|\dot{\psi}\|_\infty$$

define sequence (z_i) by

$$z_i = (1+p)q z_{i-1} + \frac{\tilde{c}_1}{k_2} z_{i-1} + \frac{\tilde{c}_2}{k_2^2} z_{i-1}^2 + k_2 \tilde{c}_3 + \tilde{c}_4, \quad i \geq N_0$$

(z_i) is bounded: choose k_2 suff. large s.t. $\alpha := (1+p)q + \frac{\tilde{c}_1}{k_2} < 1$ and

$$\frac{k_2^2(1-\alpha)^2}{4\tilde{c}_2} \geq k_2 \tilde{c}_3 + \tilde{c}_4,$$

then: $\forall\, i \geq N_0 :\ z_{i-1} \leq \hat{z} \quad \Longrightarrow \quad z_i \leq \hat{z}$, where

$$\hat{z} := \frac{k_2^2(1-\alpha)}{2\tilde{c}_2} + \sqrt{\frac{k_2^4(1-\alpha)^2}{4\tilde{c}_2^2} - \frac{k_2^2}{\tilde{c}_2}\left(k_2\tilde{c}_3 + \tilde{c}_4\right)}$$

Sketch of the proof

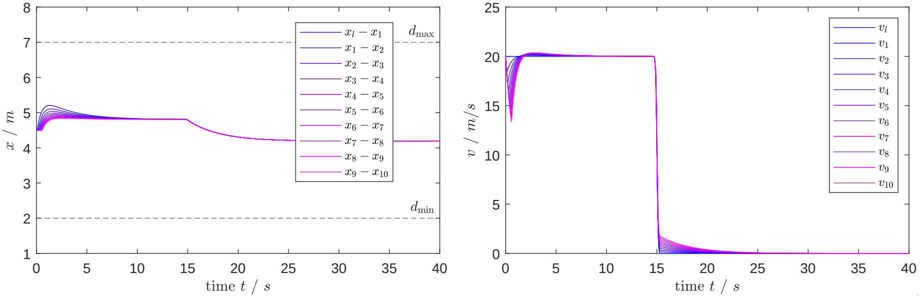
Step 5: by Step 3: $|v_i(t)| \leq \frac{M}{\lambda} + \frac{\bar{d}}{\lambda k_2} + \frac{\psi(t)}{\lambda k_2 \varepsilon_2} + \left(\frac{k_1}{k_1+\lambda k_2}\right)^i |v_0(t)|$, where $\varepsilon_2 := \frac{1}{\hat{z}}$.

Sketch of the proof

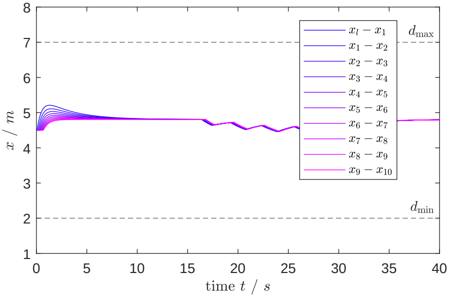
Step 5: by Step 3: $|v_i(t)| \leq \frac{M}{\lambda} + \frac{\bar{d}}{\lambda k_2} + \frac{\psi(t)}{\lambda k_2 \varepsilon_2} + \left(\frac{k_1}{k_1+\lambda k_2}\right)^i |v_0(t)|$, where $\varepsilon_2 := \frac{1}{\hat{z}}$.

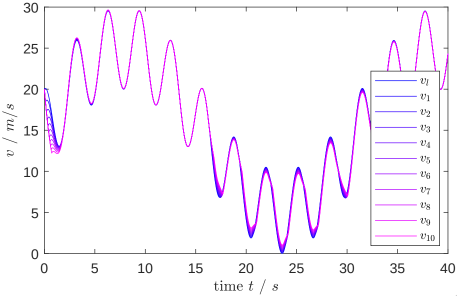
Step 6: uniform boundedness of v_i and u_i $\square$

Simulation – platoon with 10 vehicles

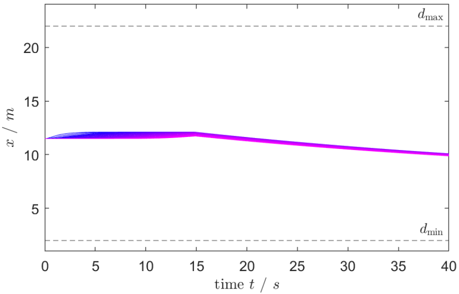


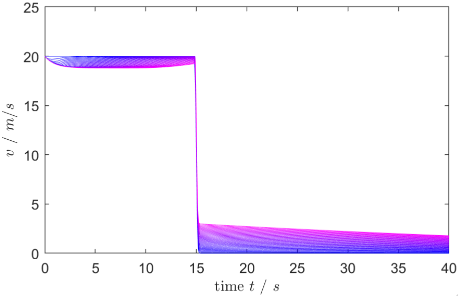
Simulation – platoon with 10 vehicles





Simulation – platoon with 30 vehicles





Open questions

- simulations exhibit synchronization $\to$ proof?
- conditions on the parameters s.t. it works with input constraints?
- incorporate filter/pre-compensator to avoid measurements of the velocities
- sampling?
- How to treat general interconnected systems?