



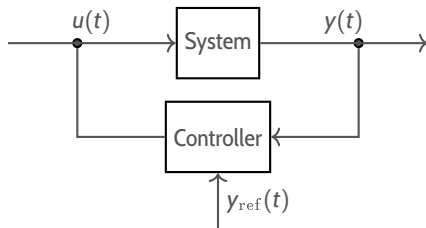
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FUNNEL CONTROL – A BRIEF INTRODUCTION

Thomas Berger

Stockholm, May 2, 2023

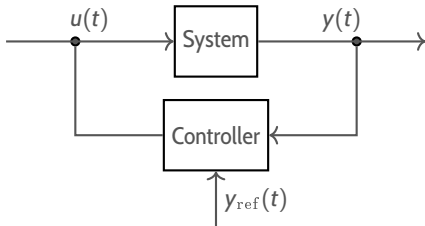
Control objective



$$\begin{aligned}\dot{x}(t) &= f(t, x(t), u(t)), & x(t) &\in X \\ y(t) &= h(x(t))\end{aligned}$$

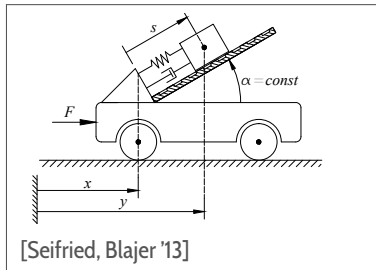
- **Goal:** simple controller, so that “ $y(t)$ tracks $y_{\text{ref}}(t)$ ”
- only uses $y(t)$, no knowledge of $x(t) \in X$ or system parameters

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- **Goal:** simple controller, so that “ $y(t)$ tracks $y_{\text{ref}}(t)$ ”
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→ **ODEs and PDEs in the same class!**

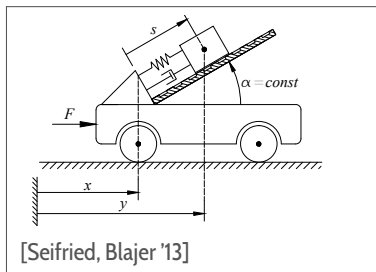


angle: $0^\circ \leq \alpha \leq 90^\circ$

spring, damper with nonlinear characteristics: $K(s)$, $D(\dot{s})$

$$u(t) = F$$

$$y(t) = x(t) + s(t) \cos \alpha$$



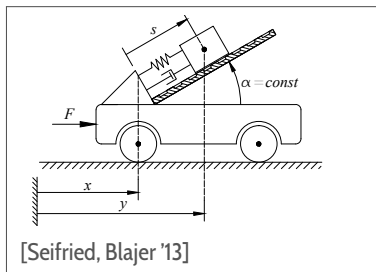
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$$\begin{bmatrix} m_1 + m_2 & m_2 \cos \alpha \\ m_2 \cos \alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{s} \end{pmatrix} = \begin{pmatrix} u \\ -K(s) - D(\dot{s}) + m_2 g \sin \alpha \end{pmatrix}$$



angle: $0^\circ \leq \alpha \leq 90^\circ$

spring, damper with nonlinear characteristics: $K(s)$, $D(\dot{s})$

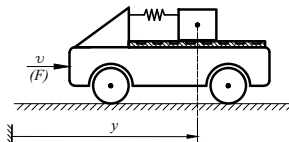
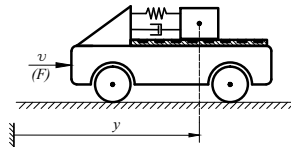
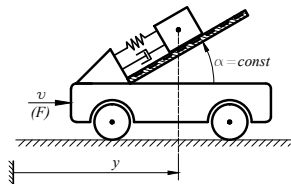
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$$\dot{y} = \dot{x} + \dot{s} \cos \alpha$$

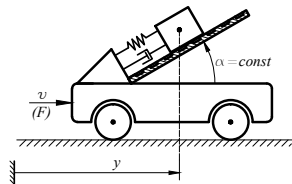
$$\ddot{y} = -c_1(K(s) + D(\dot{s}) - m_2 g \sin \alpha) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$



$$0^\circ < \alpha \leq 90^\circ$$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

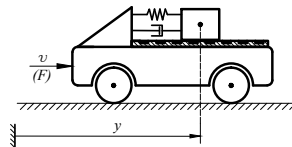
relative degree = 2



$$0^\circ < \alpha \leq 90^\circ$$

$$\ddot{y} = f_1(s, \dot{s}) + \frac{\sin^2 \alpha}{m_1 + m_2 \sin^2 \alpha} u$$

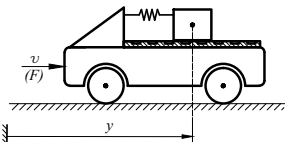
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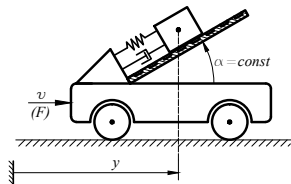


$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

$$y^{(3)} = f_2(s, \dot{s}) + \frac{D'(\dot{s})}{m_1 m_2} u$$

relative degree = 3

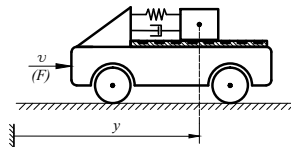




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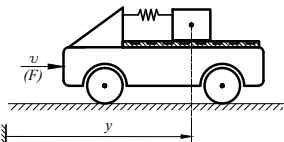
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$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

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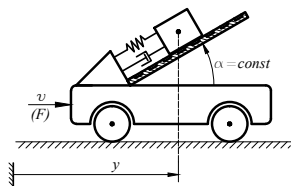
relative degree = 3



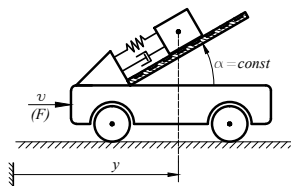
$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

$$y^{(4)} = f_3(s, \dot{s}) + \frac{K'(s)}{m_1 m_2} u$$

relative degree = 4

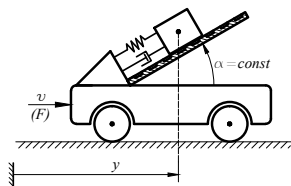


Internal dynamics: remaining dynamics
when output is fixed



Internal dynamics: remaining dynamics when output is fixed

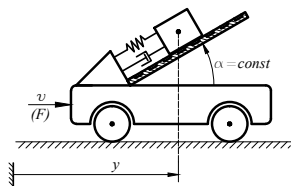
$$\ddot{\eta} = -c_3 K \left(\frac{\eta - y \cos \alpha}{\sin^2 \alpha} \right) - c_3 D \left(\frac{\dot{\eta} - \dot{y} \cos \alpha}{\sin^2 \alpha} \right) + c_4 g \sin \alpha$$



Internal dynamics: remaining dynamics when output is fixed

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$$\alpha = 90^\circ, m_2 = 1: \quad \ddot{s} = -K(s) - D(\dot{s}) + g$$

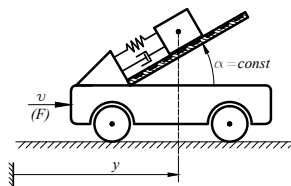


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- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s}) \dot{s} \geq 0$



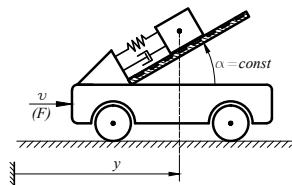
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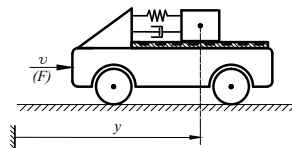
- Lyapunov function: kinetic + potential energy
- dissipativity: $D(\dot{s}) \dot{s} \geq 0$

$$\Rightarrow s, \dot{s} \in L^\infty \quad (\text{stable internal dynamics})$$



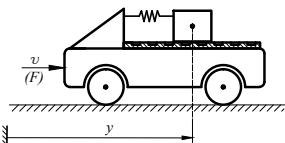
$$0^\circ < \alpha \leq 90^\circ$$

stable internal dynamics



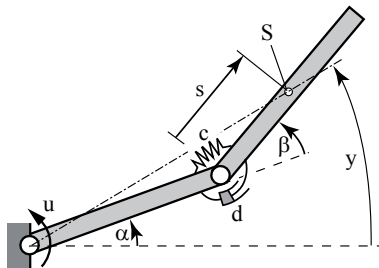
$$\alpha = 0^\circ, \quad D'(\dot{s}) \neq 0$$

stable internal dynamics



$$\alpha = 0^\circ, \quad D'(\dot{s}) = 0, \quad K'(s) \neq 0$$

no internal dynamics



[Seifried, Blajer '13]

Rotational Manipulator Arm

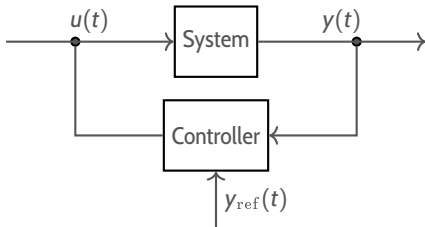
Input: angular velocity of first link

Output: position of S described by angle y

relative degree = 1

unstable internal dynamics

Reminder



$$\begin{aligned} \dot{x}(t) &= f(t, x(t), u(t)), & x(t) &\in X \\ y(t) &= h(x(t)) \end{aligned}$$

- no knowledge of system parameters, only: **known relative degree** and assumption of **stable internal dynamics**
- Goal:** design simple controller such that “ $y(t)$ tracks $y_{\text{ref}}(t)$ ”

High-gain based adaptive control

assumption: relative degree = 1, stable internal dynamics

classical (non-adaptive) high-gain controller

$$u(t) = -ky(t), \quad k > 0 \text{ suff. large} \implies y(t) \rightarrow 0$$

drawbacks: k unnecessary large; restricted to linear systems

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adaptive high-gain controller (since 1983)

$$u(t) = -k(t)y(t), \quad \dot{k}(t) = \|y(t)\|^2$$

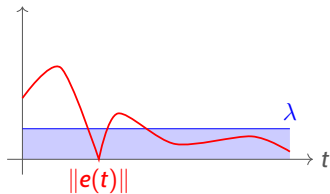
[Byrnes, Ilchmann, Logemann, Mareels, Mårtensson, Morse, Nussbaum, Owens, Prätzel-Wolters, Willems, ...]

drawbacks: $k(t)$ mon. increasing; restricted to linear systems

adaptive λ -tracker (since 1994)

$$u(t) = -k(t) \underbrace{(y(t) - y_{\text{ref}}(t))}_{=:e(t)},$$

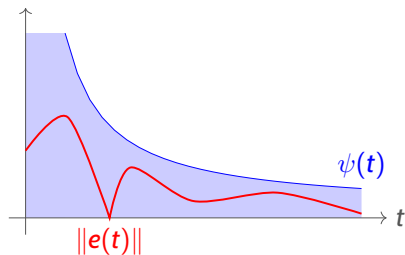
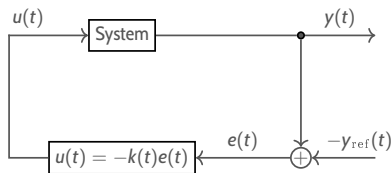
$$\dot{k}(t) = \begin{cases} \frac{\max\{\|e(t)\| - \lambda, 0\}}{\|e(t)\|}, & e(t) \neq 0, \\ 0, & e(t) = 0 \end{cases}$$



[Allgöwer, Ashman, Bullinger, Ilchmann, Logemann, Ryan, Sangwin, ...]

drawbacks: $k(t)$ mon. increasing; no transient behavior

Funnel control



[Ilchmann, Ryan, Sangwin '02]:

Works, if

- relative degree = 1
- stable internal dynamics

Funnel control for systems with higher relative degree

Funnel control via backstepping: [Ilchmann, Ryan, Townsend '06 & '07]

drawbacks: escalating controller complexity for relative degree ≥ 2 , hence a typically bad controller performance

relative degree = 2: [Hackl, Hopfe, Ilchmann, Müller, Trenn '13]

drawbacks: no generalization to arbitrary relative degree

Bang-bang funnel controller: [Liberzon & Trenn '13]

drawbacks: restricted to SISO systems, strong compatibility assumptions

“Prescribed-Performance Control”: [Bechlioulis & Rovithakis '14]

drawbacks: restricted to systems with trivial internal dynamics

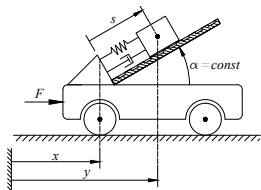
Funnel control for systems with arbitrary relative degree $r \in \mathbb{N}$

$$\begin{aligned}e_1(t) &= e(t), & e(t) &= y(t) - y_{\text{ref}}(t), \\e_2(t) &= \dot{e}(t) + k_1(t)e_1(t), \\&\vdots \\e_r(t) &= e^{(r-1)}(t) + k_{r-1}(t)e_{r-1}(t), \\u(t) &= -k_r(t)e_r(t) \\k_i(t) &= 1/(\psi_i(t) - \|e_i(t)\|), & i &= 1, \dots, r\end{aligned}$$

Theorem [B., Lê, Reis '18] & [B., Ilchmann, Ryan '21]

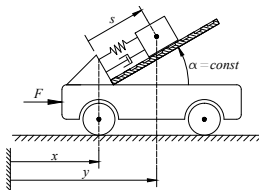
$$y_{\text{ref}} \in W^{r, \infty} \implies u, k_i, y^{(i)} \in L^\infty \text{ and } \|e_i(t)\| \leq \psi_i(t) - \varepsilon_i$$

Control of multibody systems – jointly with R. Seifried (TU Hamburg, Germany)

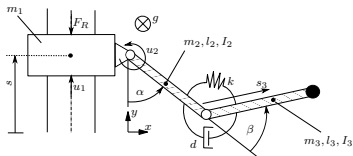


[B., Lê, Reis '18]: $u(t) = u_{\text{FC}}(t)$

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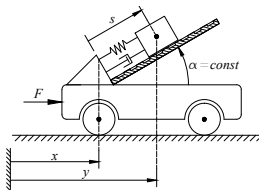


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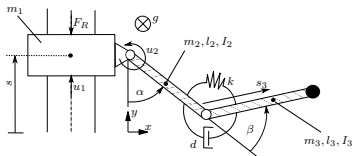


[B., Otto, Reis, Seifried '19]:
 $u(t) = u_{FC}(t) + u_{FF}(t)$

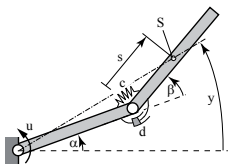
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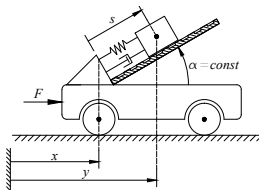


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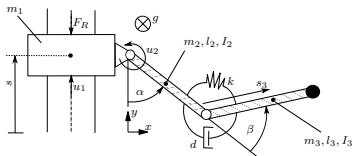


[B., Lanza '20]: $u(t) = u_{FC}(t)$
unstable internal dynamics

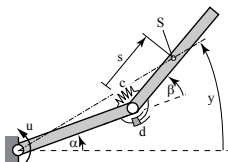
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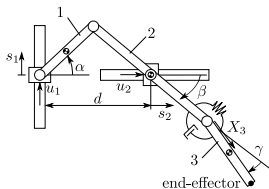
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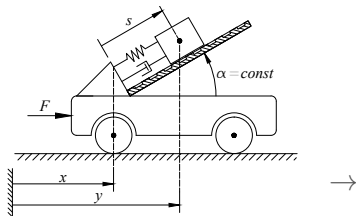
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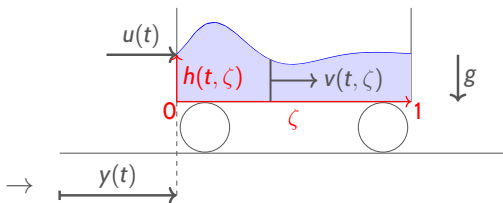
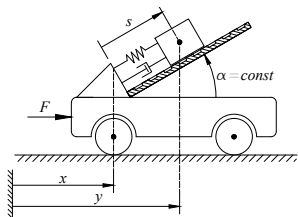
[B., Lanza '20]: $u(t) = u_{FC}(t)$
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[B., Drücker, Lanza, Reis, Seifried '21]
 $u(t) = u_{FC}(t) + u_{FF}(t)$
unstable internal dynamics, DAE formulation



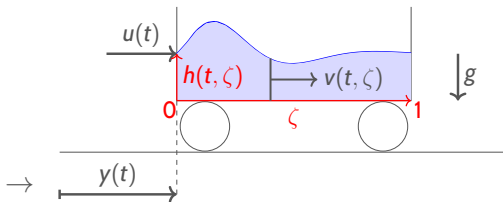
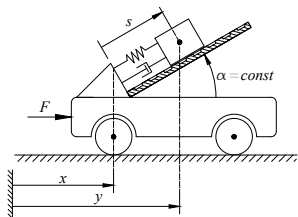
$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma u(t)$$



$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma u(t)$$

$$\ddot{y}(t) = \hat{T}(y, \dot{y})(t) + \hat{\gamma} u(t)$$

[B., Puche, Schwenninger '22]



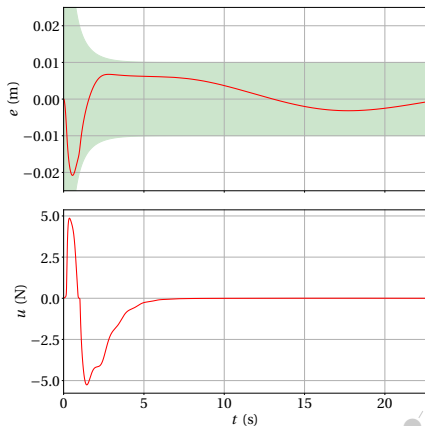
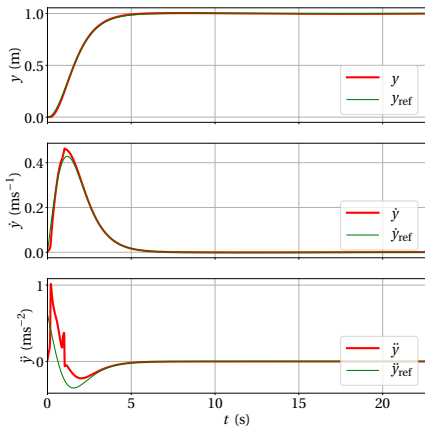
$$\ddot{y}(t) = T(y, \dot{y})(t) + \gamma u(t)$$

$$\ddot{y}(t) = \hat{T}(y, \dot{y})(t) + \hat{\gamma} u(t)$$

[B., Puche, Schwenninger '22]

Finite and infinite dimensional systems in the same class!

Simulation



Funnel control for ∞ -dimensional systems

“Simple” ∞ -dimensional systems – there is a concept of relative degree

- [Ilchmann, Ryan, Sangwin '02], [Ilchmann, Selig, Trunk '16], [B., Puche, Schwenninger '20]

Funnel control for ∞ -dimensional systems

“Simple” ∞ -dimensional systems – there is a concept of relative degree

- [Ilchmann, Ryan, Sangwin '02], [Ilchmann, Selig, Trunk '16], [B., Puche, Schwenninger '20]

“Hard” ∞ -dimensional systems – there is NO concept of relative degree

- boundary controlled heat equation [Reis, Selig '15]

$$\partial_t x(t) = \Delta x(t), \quad u(t) = (\nu^\top \cdot \nabla x(t))|_{\partial\Omega},$$

$$y(t) = \int_{\partial\Omega} x(t)(\zeta) d\zeta$$

- general class of boundary control problems based on m -dissipative operators [Puche, Reis, Schwenninger '21], [Puche '19]
- Fokker-Planck equation [B. 21'] \rightarrow *video clip*

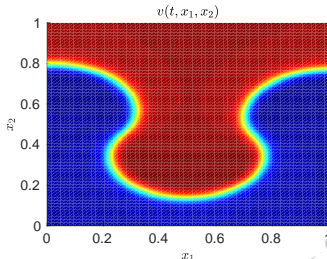
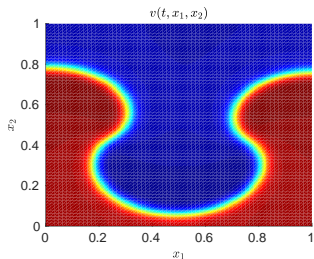
Monodomain equations [B., Breiten, Puche, Reis '21] – (simple) model for the electric activity of the human heart to describe defibrillation processes

$$\begin{aligned}\partial_t v(t) &= \nabla \cdot (D \nabla v(t)) + p_3(v)(t) - w(t) + I_{s,i}(t) + B I_{s,e}(t), \\ \partial_t w(t) &= cv(t) - dw(t), \quad y(t) = B'v(t)\end{aligned}$$

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Control objective: “reentry waves”, which can be interpreted as fibrillation processes, should be terminated



Recent research: Funnel MPC [B., Kästner, Worthmann '20]

OCP: minimize $\int_{\hat{t}}^{\hat{t}+T} \ell(t, x(t), u(t)) dt$
 $u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)$

subject to $\dot{x}(t) = f(t, x(t), u(t)),$
 $x(\hat{t}) = \hat{x}, \quad \|u\|_\infty \leq M$

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 $u \in L^\infty([\hat{t}, \hat{t}+T], \mathbb{R}^m)$

subject to $\dot{x}(t) = f(t, x(t), u(t)),$
 $x(\hat{t}) = \hat{x}, \quad \|u\|_\infty \leq M$

Idea:

$$\ell(t, x, u) = \|h(x) - y_{\text{ref}}(t)\|^2 + \lambda \|u\|^2$$

$$\rightarrow \ell(t, x, u) = \frac{1}{\psi(t) - \|h(x) - y_{\text{ref}}(t)\|} + \lambda \|u\|^2$$

Recent research: Funnel MPC

Funnel MPC Algorithm

Choose time shift $\delta > 0$, prediction horizon $T \geq \delta$, initialize $\hat{t} := t_0$

1. Obtain a measurement of the state at \hat{t} , set $\hat{x} = x(\hat{t})$
2. Compute a solution $u^* \in L^\infty([\hat{t}, \hat{t} + T], \mathbb{R}^m)$ of the OCP
3. Apply the feedback law $\mu(t, \hat{x}) = u^*(t), t \in [\hat{t}, \hat{t} + \delta)$
4. Set $\hat{t} := \hat{t} + \delta$ and go to 1.

Recent research: Funnel MPC

Theorem [B., Dennstädt, Ilchmann, Worthmann '22]

relative degree = 1, stable internal dynamics $\implies \exists M > 0$: Funnel MPC is initially and recursively feasible such that $\|u(t)\| \leq M$ and $\|e(t)\| \leq \psi(t)$

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Theorem [B., Dennstädt, Ilchmann, Worthmann '22]

relative degree = 1, stable internal dynamics $\implies \exists M > 0$: Funnel MPC is initially and recursively feasible such that $\|u(t)\| \leq M$ and $\|e(t)\| \leq \psi(t)$

1. no terminal costs or conditions required
2. no additional state constraints in the OCP
3. independent of the length of the prediction horizon $T > 0$

Recent research: Funnel MPC

Future goals:

- extension to arbitrary relative degree (first results in [B., Dennstädt '22], however requiring feasibility constraints in the OCP)
- extend control scheme so that robustness is inherited from funnel control (results for relative degree one systems submitted [B., Dennstädt, Lanza, Worthmann '23])
- combine with machine learning for continuous model adaptation and improvement of controller performance (results for relative degree one systems submitted [Lanza, Dennstädt, B., Worthmann '23])