Controlled invariance for DAEs

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Linear systems

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t} E x(t) = A x(t) + B u(t)} \qquad (E, A, B)$$

$E, A \in \mathbb{R}^{\ell \times n}, B \in \mathbb{R}^{\ell \times m}$

$$\mathfrak{B} = \left\{ \, (x,u) \in \mathcal{C}^1 \times \mathcal{C} \;\middle|\; E\dot{x}(t) = Ax(t) + Bu(t) \, \right\}$$

Def.: $\mathcal{V} \subseteq \mathbb{R}^n$ **is controlled invariant** $:\Longleftrightarrow$

$\forall\, x^0 \in \mathcal{V} \; \exists\, (x,u) \in \mathfrak{B} \;\; \forall\, t \geq 0 : \;\; x(0) = x^0 \;\wedge\; x(t) \in \mathcal{V}$

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Theorem (controlled invariance)

The following is equivalent for (E, A, B) and $\mathcal{V} \subseteq \mathbb{R}^n$:

(1) $\mathcal{V}$ is controlled invariant

(2) $A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B$

(3) $\exists\, F \in \mathbb{R}^{m\times n}: \ (A+BF)\mathcal{V} \subseteq E\mathcal{V}$

Proof: (1)⇒(2):

$$x^0 \in \mathcal{V} \ \Rightarrow \ Ax^0 = Ax(0) = E\dot{x}(0) - Bu(0) \in E\mathcal{V} + \operatorname{im} B$$

(2)⇒(3):

$$\operatorname{im} V = \mathcal{V}, \ AV = EVW + BU \quad \Rightarrow \quad F := -U(V^\top V)^{-1}V^\top$$

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Existence lemma: $E, A \in \mathbb{R}^{\ell \times n}$ such that $\operatorname{im} A \subseteq \operatorname{im} E$, then

$$\forall\, x^0 \in \mathbb{R}^n \;\exists\, x \in \mathcal{C}^\infty \;\forall\, t \in \mathbb{R}:\; x(0) = x^0 \;\wedge\; E\dot{x}(t) = Ax(t)$$

(3)⇒(1):

$$x^0 = Vw^0 \in \mathcal{V}, \quad \operatorname{im}(A+BF)V \subseteq \operatorname{im} EV$$

$$\overset{\text{lemma}}{\Longrightarrow} \quad \exists\, w \in \mathcal{C}^\infty:\; w(0) = w^0 \;\wedge\; EV\dot{w}(t) = (A+BF)Vw(t),$$

$\Longrightarrow$ $x := Vw,\; u := FVw$ satisfy

$(x,u) \in \mathfrak{B},\; x(0) = x^0$ and $x(t) \in \mathcal{V},\; t \geq 0$ □

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Nonlinear systems

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t}E(x(t)) = f(x(t)) + g(x(t))\,u(t)} \qquad (E,f,g)$$

$X \subseteq \mathbb{R}^n$ open, $0 \in X$, $E, f : X \to \mathbb{R}^\ell, g : X \to \mathbb{R}^{\ell\times m}$ diff., $f(0)=0$

$$\mathfrak{B} = \left\{ (x,u) \in \mathcal{C}^1 \times \mathcal{C} \;\middle|\; (x,u) \text{ is a maximal solution of } (E,f,g) \right\}$$

M – connected submanifold of X with $0 \in M$

Def.: M **is locally controlled invariant** $:\Longleftrightarrow$

$\exists$ open neighborhood U of $0 \in X$ such that

$\forall x^0 \in M \cap U\ \exists (x,u) \in \mathfrak{B}\ \exists t_0 \in \operatorname{dom} x,\ x(t_0) = x^0 :$

$\big(\forall t \in \operatorname{dom} x,\ t \ge t_0 :\ x(t) \in M \cap U\big)$

$\vee\ \big(\exists \hat t \in \operatorname{dom} x, \hat t > t_0\ \forall t \in [t_0, \hat t) :\ x(t) \in M \cap U\ \wedge\ x(\hat t) \in \partial(M \cap U)\big)$

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Theorem (local controlled invariance)

$E \in \mathcal{C}^2$, $f, g \in \mathcal{C}^1$, M connected submanifold of X with $0 \in M$ such that, in a neighborhood of $0 \in M$,

$$\dim E'(x)T_xM = \text{const} \quad \wedge \quad \dim\big(E'(x)T_xM + \operatorname{im} g(x)\big) = \text{const}$$

Then the following is equivalent:

(1) M is locally controlled invariant.
(2) $f(x) \in E'(x)T_xM + \operatorname{im} g(x)$ in $M \cap U$.
(3) $\exists\, u \in \mathcal{C}^1(M \cap U \to \mathbb{R}^m):\ f(x) + g(x)u(x) \in E'(x)T_xM$ in $M \cap U$.

Proof: (1)⇒(2): $x^0 \in M \cap U$

$$\Rightarrow\ f(x^0) = E'(x(0))\dot{x}(0) - g(x(0))u(0) \in E'(x^0)T_{x^0}M + \operatorname{im} g(x^0)$$

(2)⇒(3): technical, but not difficult

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Existence lemma: $U \subseteq \mathbb{R}^n$ open, $E, f : U \to \mathbb{R}^\ell$ diff. and

$$\forall\, x \in U :\ \operatorname{rk} E'(x) = r \quad \wedge \quad f(x) \in E'(x) T_x U,$$

$$\Longrightarrow\ \forall\, x^0 \in U\, \exists\, x \in \mathcal{C}^1(I \to \mathbb{R}^n)\, \forall\, t \in I :\ x(0) = x^0 \wedge \tfrac{\mathrm{d}}{\mathrm{d}t} E(x(t)) = f(x(t))$$

(3)⇒(1): $x^0 = \psi(w^0) \in M \cap U$, $\psi : G \to M \cap U$ parametrization of M

def. $\tilde{E} := E \circ \psi$, $\tilde{f} := f \circ \psi + (g \circ \psi)(u \circ \psi)$

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and $x(t) \in M \cap U$, $\forall\, t \in I, t \geq 0$

Remains: show that $(x, u \circ x)$ can be extended to a maximal sln.

$\to$ Zorn's Lemma □

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Zero dynamics

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t}E(x(t)) = f(x(t)) + g(x(t))\,u(t), \quad y(t) = h(x(t))} \qquad (E, f, g, h)$$

$E, f : X \to \mathbb{R}^\ell, g : X \to \mathbb{R}^{\ell\times m}, h : X \to \mathbb{R}^p$ diff., $f(0) = 0, h(0) = 0$

Zero dynamics:

$$\mathcal{ZD} = \{\ (x,u) \in \mathfrak{B}\ \mid y(t) = h(x(t)) = 0\ \}$$

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Def.: M **is output zeroing** $:\Longleftrightarrow$

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Example

$$\begin{aligned} \dot{x}_1 &= 0 \\ 0 &= x_1 + u \\ y &= x_1 - x_2^2 \end{aligned} \qquad M := \{\, x \in \mathbb{R}^2 \;|\; x_1 = x_2^2 \,\} \subseteq h^{-1}(0)$$

$x^0 = (x_1^0, x_2^0)^\top \in M \;\Rightarrow\; x(t) := x^0,\; u(t) := -x_1^0,\; t \in \mathbb{R},$ satisfy

$(x, u) \in \mathfrak{B}$ and $x(0) = x^0$, $x(t) \in M$ for $t \in \mathbb{R}$

$\Longrightarrow$ M is output zeroing submanifold

Note: u must satisfy the algebraic constraint $u = -x_1$

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Theorem (zero dynamics algorithm)

$E, f, g, h \in \mathcal{C}^\infty$; def. $M_0 := h^{-1}(0)$ and

$$M_k := \left\{ \ x \in M_{k-1} \ \middle| \ f(x) \in E'(x) T_x M_{k-1} + \operatorname{im} g(x) \ \right\};$$

suppose that M_k is a connected submanifold with $0 \in M_k$

(1) $\exists\, k^* \in \mathbb{N}_0 \ \forall\, j \in \mathbb{N}: \ M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_{k^*} =: Z^* = M_{k^*+j}$

(2) $\dim E'(x) T_x Z^* =$ const
$\wedge \ \dim \big(E'(x) T_x Z^* + \operatorname{im} g(x) \big) =$ const in $Z^* \cap U$,
$\implies$ Z^* is a **locally maximal** output zeroing submanifold

(3) $\exists$ open neighborhood U of $0 \in X$ $\forall$ open $O \subseteq U$ $\forall\, (x,u) \in \mathfrak{B}$ with $x(t) \in O$, $t \in \operatorname{dom} x$:

$$(x,u) \in \mathcal{ZD} \iff x(t) \in Z^* \cap O \ \ \forall\, t \in \operatorname{dom} x.$$

Example revisited

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$$M = M_0 = M_1 = Z^*$$

$\Longrightarrow$ M is locally maximal output zeroing submanifold

BUT!

$$\dim E'(x) T_x Z^* = \dim \operatorname{im} \begin{bmatrix} 2x_2 \\ 0 \end{bmatrix}$$

and $\dim \big(E'(x) T_x Z^* + \operatorname{im} g(x) \big) = \dim \operatorname{im} \begin{bmatrix} 2x_2 & 0 \\ 0 & 1 \end{bmatrix}$

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